PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 121, Number 3, July 1994

THE HAUSDORFF DIMENSION OF ELLIPTIC AND ELLIPTIC-CALORIC MEASURE IN $\mathbf{R}^n$, $n \geq 3$

CAROLINE SWEEZY

(Communicated by J. Marshall Ash)

ABSTRACT. The existence of an L-caloric measure with parabolic Hausdorff dimension $4-\varepsilon$ in $\mathbf{R}^2 \times \mathbf{R}^1$ is demonstrated. The method is to use a specially constructed quasi-disk Q whose boundary has Hausdorff dim $= 2-\varepsilon$. There is an elliptic measure supported on the entire boundary of Q. Then the L-caloric measure on $\partial_p Q \times [0, T]$ is compared with the corresponding elliptic measure. The same method gives the existence of an elliptic measure in $\mathbf{R}^n$ whose support has Hausdorff dim $n-\varepsilon$ for $n \geq 3$, and an L-caloric measure in $\mathbf{R}^n \times \mathbf{R}^1$ supported on a set of parabolic Hausdorff dimension $n+2-\varepsilon$.

The purpose of this paper is to demonstrate the existence of a domain in $\mathbf{R}^2 \times \mathbf{R}^1$ and a strictly elliptic operator L so that the L-caloric measure associated with $\partial/\partial t - L$ has support of parabolic Hausdorff dimension $4-\varepsilon$. Our example is the product domain $[0, T] \times Q$ where Q is the quasi-disk constructed in [8] and the operator $\partial/\partial t - L$, where L is the operator in [8]. Our proof is based on Lemmas 1 and 2. In Lemma 2 we show that the product measure, $dm = dt \times dw_l$, where w_l is the elliptic measure of L on ∂Q, is absolutely continuous with L-caloric measure. It follows easily from Lemma 1 that dm has support of parabolic Hausdorff dimension $4-\varepsilon$.

Using basically the same argument one can show there are sets in $\mathbf{R}^n$, $n > 2$, such that a particular elliptic measure has support of Hausdorff dimension $n-\varepsilon$ for any $\varepsilon > 0$. The domain in $\mathbf{R}^3$ would be $[-R, R] \times Q$ and the operator $\partial^2/\partial z^2 + L$ where L and Q are as in [8]. The analogues of Lemmas 1 and 2 for NTA domains and elliptic operators are well known (see [4]). So, as above, one can deduce the existence of L-caloric measure in $\mathbf{R}^{n+1}$ whose support has parabolic Hausdorff dimension $n+2-\varepsilon$, for $n > 2$.

We note that caloric measure $(\partial/\partial t - \Delta)$ has support of $p - H - \dim \leq 3$ in $\mathbf{R}^3$ on any cylinder set $D \times [0, T]$ where D is an NTA domain in $\mathbf{R}^2$, as follows from Lemmas 1, 2 and the fact that harmonic measure has support of $H - \dim \leq 1$ in $\mathbf{R}^2$ (Jones and Wolff [5]).

Let $D_T = Q \times [0, T]$ where Q is the quasi-disk in $\mathbf{R}^2$ constructed in [8],

Received by the editors June 23, 1992 and, in revised form, October 10, 1992; the contents of this paper were presented in part at the AMS meeting in Los Angeles at UCLA, November 19, 1989, and in full at a workshop on P.D.E. at the University of Chicago, March 20–25, 1990.

1991 *Mathematics Subject Classification.* Primary 35K20, 35J25.

Key words and phrases. L-caloric measure, parabolic Hausdorff dimension, NTA domains.

so that ∂Q has $H-\dim(2-\varepsilon)$ and there is a strictly elliptic L-operator in divergence form whose associated elliptic measure w_l has support on $H-\dim(2-\varepsilon)$.

Theorem 1. *If L and D_T are as above and w_L is the L-caloric measure of $\partial/\partial t - L$ on D_T, then the parabolic Hausdorff dimension of* $\operatorname{supp} w_L$ *in $\partial_p D_T$ is $4-\varepsilon$.*

The proof of Theorem 1 depends on the following two lemmas. Fix $x_0 \in Q$ such that $d(x_0, \partial Q) \geq r_0$ and let $dm_{x_0} = dt \times dw_l^{x_0}$ be a product measure on $\partial_p D_T$. Then dm_{x_0} is a Borel measure on $\partial_p D_T \cap \{t > 0\}$, since $dt =$ Lebesgue measure on $\mathbf{R}^1$, $dw_l^{x_0} =$ elliptic measure (of L) on ∂Q are both Borel measures and $\partial_p D_T \cap \{t > 0\} = \partial Q \times (0, T]$. dm_{x_0} is supported on $\operatorname{supp} w_l^{x_0} \times [0, T]$. By Lemma 1 the support of dm_{x_0} will have parabolic Hausdorff dimension $4-\varepsilon$ since $H-\dim(\operatorname{supp} dw_l) = 2-\varepsilon$ [8].

A Hausdorff measure which is suitable for solutions of the heat equation in cylinder sets in $\mathbf{R}^{n+1}$ can be defined as in Taylor and Watson [9]:

$$\Lambda_p^\alpha(E) = \liminf_{\delta\to 0}\left\{\sum_{i=1}^\infty r_i^\alpha : E \subseteq \bigcup_{i=1}^\infty P_{r_i},\ P_{r_i}(Q, s) = \{(x, t) \mid |x_j - Q_j| < r_i,\ j = 1, 2, \ldots, n;\ |t-s| < r_i^2\} \text{ and } \operatorname{diam} P_{r_i} \sim r_i < \delta\right\}.$$

The parabolic Hausdorff dimension of a set E is the $\inf\{\beta : \Lambda_p^\beta(E) = 0\}$. So parabolic Hausdorff dimension β and classical Hausdorff dimension α are related by $\beta - 1 \leq \alpha \leq (\beta+n)/2$ since every cube of side length r contains $1/r$ parabolic boxes of dimensions $r \times r^2$, and each parabolic box of dimensions $r\times r^2$ contains $1/r^n$ cubes of side length r^2, and Hausdorff measure using cubes compares with ordinary Hausdorff measure. Both extremes are possible, as simple examples show.

Lemma 1 [6]. *If $D_T = E \times [0, T]$, E is a set in $\mathbf{R}^n$, then*

$$\Lambda_p^{\alpha+2}(D_T) \geq k\Lambda^\alpha(E)\Lambda^1([0, T])$$

for some k independent of E. $\Lambda_p^{\alpha+2}(D_T)$ is the parabolic Hausdorff measure on D_T; $\Lambda^\alpha(E)$ is the usual Hausdorff measure of the set E.

Then Lemma 2 can be used to prove Theorem 1.

Lemma 2. *For $D_T = D \times [0, T]$ where D is an NTA domain in $\mathbf{R}^n$, with NTA constant M,*

$$\Delta_r(Q, s) = \partial_p D_T \cap P_r(Q, s), \qquad r < \delta(M) r_0,\ T_0 - r_0^2 > s > \tfrac{1}{2} r_0^2,$$

and $T \geq T_0$, there are constants c_4 and c_5 depending only on M, X_0, T_0, r_0, λ, n so that for (X_0, T_0) fixed and $d(X_0, \partial D) > r_0$,

$$c_4 \leq \frac{w_L^{(X_0, T_0)}(\Delta_r(Q, s))}{m_{X_0}(\Delta_r(Q, s))} \leq c_5.$$

Here r_0 is a fixed constant, λ is the ellipticity constant of L and w_L is L-caloric measure on D_T.

By Lemma 2, $dw_L^{(X_0, T_0)}$ and dm_{X_0} are mutually absolutely continuous and $p - H - \dim(\operatorname{supp} m_{X_0}) = 4 - \varepsilon$. So Lemmas 1 and 2 prove Theorem 1.

To prove Lemma 2, one needs some standard results for comparing caloric measure with the Green's function and a local comparison theorem for solutions vanishing at the boundary for $D_T = \text{NTA} \times [0, T]$. (See Theorems 2 and 3 in §2.) These theorems are proved in Fabes, Garofalo, and Salsa for $D \times [0, T]$, D is a Lipschitz domain (Theorems 1.4 and 1.6 in [3]). To prove these results, it is necessary to prove versions of Hölder continuity at the boundary and the Carleson box condition for L-caloric functions vanishing on a Δ_{4r} disk in $\partial_p D_T \cap \{t > 0\}$, on an NTA $\times$ time domain. The proofs of these results will be outlined in §3.

M will denote the NTA constant of D, and d is parabolic distance

$$d(x, t; y, s) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2 + \sqrt{|t - s|}}.$$

1

The proof of Lemma 1 depends on the following version of Lemma 4 from Marstrand's paper:

Lemma 3 [6]. *Suppose a linear set X is contained in a finite set $\bigcup_{j=1}^N I_j$ of dyadic intervals, each of length $< \delta$. Suppose also there is a positive number p such that for every $x \in X$, $\sum_{j, x \in I_j} f(j) > p$ where f is a function from I_j to the positive real numbers. Then*

$$\sum_{j=1}^{N} f(j) |I_j|^s > p L_{\delta, d}^s(X)$$

where

$$L_{\delta, d}^s(X) = \inf \left\{ \sum_{j=1}^{\infty} r_j^s : X \subseteq \bigcup_{j=1}^{\infty} Q(x_j, r_j); \right.$$
$$\left. Q(x_j, r_j) \text{ are dyadic intervals of length } r_j,\ r_j < \delta \right\}.$$

Proof of Lemma 1. Let $\bigcup_{j=1}^\infty P_j$ be a cover of $E \times [0, T]$ by dyadic parabolic boxes of dimension r_j such that $r_j < \delta$ and

$$\sum_{j=1}^{\infty} r_j^{\alpha+2} \le (1 + \varepsilon) L_{P, \delta, d}^{\alpha+2}(E \times [0, T]).$$

$L_{P, \delta, d}^{\alpha+2}$ is the dyadic parabolic Hausdorff measure. For each t,

$$\sum_{\substack{j \\ (x, t) \in \{E \times [0, T]\} \cap P_j}} r_{j, t}^{\alpha} \ge L_{\delta, d}^{\alpha}(E),$$

where $r_{j,t}$ is the side length of $P_{j,t}$; and $P_{j,t}$ is the projection of P_j onto $\mathbf{R}^2$ if $(x, t) \in P_j$. The $\bigcup_j P_{j,t}$ forms a dyadic cover of E in $\mathbf{R}^2$.

Now for $f(j) = r_j^\alpha$ take $\bigcup_{i=1}^m P_j$ so that

$$\sum_{j=1}^{m} r_j^\alpha \geq (1-\varepsilon) L_{\delta,d}^\alpha(E)$$

so $m = m(\varepsilon)$, $X = [0, T]$, and $I_j = \operatorname{Proj} P_j$ into $[0, T]$, so $[0, T] \subseteq \bigcup_{j=1}^N I_j$ some $N > m > 0$ since $[0, T]$ is compact. For $p = (1-\varepsilon)L_{\delta,d}^\alpha(E)$, one can apply Lemma 3 to obtain

$$\begin{aligned}(1+\varepsilon)L_{\delta,d}^{\alpha+2}(D_T) &\geq \sum_{j=1}^{\infty} r_j^{\alpha+2} > pL_{\delta,d}^1([0, T]) \\ &> (1-\varepsilon)L_{\delta,d}^\alpha(E)T \quad \text{since } |I_j| = r_j^2.\end{aligned}$$

Dyadic Hausdorff measure compares with usual Hausdorff measure in both the parabolic and nonparabolic case and $\varepsilon > 0$ is arbitrary so

$$\Lambda_p^{\alpha+2}(D_T) \geq k\Lambda^\alpha(E)\Lambda^1([0, T]).$$

Lemma 1 shows that the parabolic Hausdorff dim of $\partial_p D_T \geq H-\dim(\partial D)+2$ where $D_T = D \times [0, T]$ so $\partial_p D_T \cap \{t > 0\} = \partial D \times (0, T]$. It is easy to show the reverse inequality. □

2

Let

$$\begin{gathered} P_r(Q, s) = \{(x, t) \| X - Q| < r \text{ and } |t-s| < r^2\}, \\ \Delta_r(Q, s) = P_r(Q, s) \cap \partial_p D_T \quad \text{and} \quad \overline{A}_r(Q, s) = (A_r(Q), s + 2r^2), \\ \underline{A}_r(Q, s) = (A_r(Q), s - 2r^2), \end{gathered}$$

where $A_r(Q)$ is the nontangential point in condition (1) of [4] for $D =$ NTA domain.

Theorem 2 [3]. *For $(Q, s) \in \partial_p D_T$, $s > 0$, if $(\chi, t) \in D_T$, $s + 4r^2 < t$, $r \leq \min(\frac{1}{2}\sqrt{s}, \frac{1}{2}r_0, \frac{1}{2}\sqrt{T-s})$, then there are constants c_1 and c_2 so that*

$$c_1 r^n G(\chi, t; \overline{A}_r(Q, s)) \leq w_L^{(\chi, t)}(\Delta_r(Q, s)) \leq c_2 r^n G(\chi, t; \underline{A}_r(Q, s)),$$

where c_1, c_2 depend only on λ, M, r_0, n, T.

Theorem 3 [3]. *If u and v are solutions of $(\partial/\partial t - L)u = 0$ in D_T such that u and v vanish continuously on $\Delta_{4r}(Q, s)$ where $(Q, s) \in \partial_p D_T$ and $r < \min(\frac{1}{2}r_0, \frac{1}{2}\sqrt{s}, \frac{1}{2}\sqrt{T-s})$, then there are constants $\delta(M)$, c_3, and c_4 so that for $(\chi, t) \in P_{\delta(M)r}(Q, s) \cap D_T$*

$$c_4 \frac{u(\underline{A}_r(Q, s))}{v(\overline{A}_r(Q, s))} \leq \frac{u(\chi, t)}{v(\chi, t)} \leq c_3 \frac{u(\overline{A}_r(Q, s))}{v(\underline{A}_r(Q, s))}$$

where c_3 and c_4 depend only on λ, n, M, T, r_0.

Proof of Lemma 2. Let $g_l(x, y)$ be the Green's function of L in D. Then $g_l(x, y)$ is a solution to $(\partial/\partial t + L)u = 0$ in $D_T \setminus \{B_\varepsilon(x) \times [0, T]\}$.

$$(1)\quad \frac{w_L^{(X_0, T_0)}(\Delta_r(Q, s))}{m_{X_0}(\Delta_r(Q, s))} = \frac{w_L^{(X_0, T_0)}(\Delta_r(Q, s))}{r^2 w_l^{X_0}(\Delta_r(Q))}$$

$$(2)\quad \le c' \frac{r^n G_L(X_0, T_0; \underline{A}_r(Q, s))}{r^2 r^{n-2} g_l(X_0; A_r(Q))} \le c'' \frac{G_L(X_0, T_0; \overline{A}_{r_{0/2}}(Q, s))}{g_l(X_0; A_{r_{0/2}}(Q))} \le c,$$

$$c = \sup_{\substack{(Q, s) \in \partial_p D_T \\ T_0 - r_0^2 > s > 0}} c'' \frac{G_L(X_0, T_0; \overline{A}_{r_{0/2}}(Q, s))}{g_l(X_0; A_{r_{0/2}}(Q))}.$$

Equality (1) is by Theorem 2 and the corresponding result for elliptic measure in [2] and [4], and (2) is by Theorem 3 in the adjoint variable for $u(y, w) = G_L(X_0, T_0; y, w)$ and $v(y, w) = g_l(X_0; y)$ for all $w \in [0, T]$.

Reversing the roles of m_{X_0} and $w_L^{(X_0, T_0)}$ gives the lower bound $1/c_4'$ in Lemma 2,

$$c_4' = \sup_{\substack{(Q, s) \in \partial_p D_T \\ T_0 > s > r_0^2}} c'' \frac{g_l(X_0; A_{r_{0/2}}(Q))}{G_l(X_0, T_0; \underline{A}_{r_{0/2}}(Q, s))}. \quad \square$$

3

If $D_T = D \times [0, T]$, D is an NTA domain in $\mathbf{R}^n$, the following three conditions hold.

(1) For any $(Q, s) \in \partial_p D_T^+$ and any $r < \min(r_0, s)$, there is a point $A_r(Q, s) \in D_T$ such that $r/M < d(A_r(Q, s), (Q, s)) < r$ and $d(A_r(Q, s), \partial_p D_T^+) > r/M$. There is a parabolic cylinder

$$B\left(A_r(Q), \frac{1}{2M} r\right) \times \left[s - \frac{1}{4M^2} r^2, s + \frac{1}{4M^2} r^2\right]$$

around $A_r(Q, s)$ whose diameter compares with its distance from $\partial_p^+ D_T$. $\partial_p^+ D_T = \partial_p D_T \cap \{t > 0\}$.

(2) D_T^c satisfies condition (1).

(3) Harnack Chain Condition: If (y_1, s_1) and $(y_2, s_2) \in D_T$ such that

$$\min d((y_i, s_i), \partial_p D_T) > \varepsilon, \qquad d((y_1, s_1), (y_2, s_2)) < c_1 \varepsilon,$$

and $s_2 - s_1 \ge C_2 \varepsilon^2$, then there is a Harnack chain of parabolic cylinders $p_1, p_2, \ldots, p_m$, where $(y_1, s_1) \in p_1$, $(y_2, s_2) \in p_m$, $p_{k-1} \cap p_k \ne \varnothing$, $p_k = B_k \times [t_k - \delta r_k^2, t_k + \delta r_k^2]$, and the t_k are times: $t_0 = s_1 < t_1 < t_2 < \cdots < t_m = s_2$. δ and m depend on c_1, c_2, and M but not on ε. (A Harnack chain is a sequence of parabolic cylinders $p_1, p_2, \ldots, p_m$ such that $(y_1, s_1) \in p_1$, $(y_2, s_2) \in p_m$, $\operatorname{int} p_k \cap p_{k+1} \ne \varnothing$, and $d(p_k, \partial_p^+ D_T) \sim$ diam of p_k.)

Conditions (1)–(3) follow from the corresponding conditions for the NTA domain D in $\mathbf{R}^n$ (Jerison and Kenig [4]).

If (y_1, s_1), (y_2, s_2) are as in condition (3) and $u \ge 0$, $(\partial/\partial t - L)u = 0$ in D_T, then there is a constant $C = C(M, T, n, \lambda, c_1, c_2)$ such that $u(y_1, s_1) \le C u(y_2, s_2)$.

One can prove the following versions of the continuity lemma and the Carleson box lemma for D_T when $s > 0$ and $r < \min(r_0, \frac{1}{2}\sqrt{s}, \frac{1}{2}\sqrt{T-s})$.

Continuity Lemma [4]. *If* u *is a positive function on* D_T *so that* $(\partial/\partial t - L)u = 0$ *on* D_T, u *vanishes continuously on* $P_r(Q, s) \cap \partial_p D_T = \Delta_r(Q, s)$, *then there is a constant* k *such that* $1/M \le k \le 1$, *and for all* $(\chi, t) \in P_{kr}(Q, s) \cap \{t > s\} \cap D_T$, *there is* $\beta = \beta(M)$ *so that*

$$u(\chi, t) \le C(M) \sup_{(y, v) \in D_T \cap \partial_p P_r} u(y, v) \cdot \left[\frac{d((\chi, t), (Q, s))}{r}\right]^{\beta}.$$

Proof. The argument of Salsa in the proof of Lemma 4.2 [7] can be adapted to prove this result.

The Continuity Lemma gives the boundary estimate

$$w^{\overline{A}_r(Q, s)}(\Delta_r(Q, s)) \ge c, \qquad \text{where } \overline{A}_r(Q, s) = (A_r(Q), s + 2r^2).$$

Carleson Box Lemma. *If* $u \ge 0$ *on* D, $(\partial/\partial t - L)u = 0$ *in* D *and* u *vanishes continuously on* $\Delta_{3r}(Q, s)$, $s > 9r^2$, *then there is a constant* $C = C(M)$ *such that*

$$u(\chi, t) \le C u(\overline{A}_r(Q, s))$$

for all $(\chi, t) \in P_r(Q, s)$ *where* $\overline{A}_r(Q, s) = (A_r(Q), s + 2r^2)$,

$$P_r(Q, s) = \{(\chi, t) | |\chi - Q| < r, \ |t - s| < r^2\},$$

$\Delta_r(Q, s) = P_r(Q, s) \cap \partial_p D_T$ *and* $\Delta_{3r}(Q, s) = P_{3r}(Q, s) \cap \partial_p D_T$, *where*

$$P_{3r}(Q, s) = \{(y, t) | |Q - y| < 3r, \ |t - s| < 9r^2\}.$$

Proof [4]. By the Continuity Lemma one can find M_1 depending only on M such that

$$\sup\{u(x, t) | (x, t) | \in P_{r/M_1}(Q, s) \cap D_T\} \le \tfrac{1}{2} \sup\{u(x, t) | (x, t) \in P_r(Q, s)\};$$

without loss of generality $u(\overline{A}_r(Q, s)) = 1$. By the Harnack chain condition there is M_2 depending on M such that if $u(y, t) \ge M_2^h$ then $d((y, t), \partial_p D_T) < [1/M_1^h] r$ for $(y, t) \in P_r(Q, s)$. (h is a fixed constant.)

Now by a standard argument one can obtain a sequence of points in D_T, $(y_k, t_k) \to \partial_p D_T \cap \Delta_{3r}(Q, s)$ and $u(y_k, t_k) \to \infty$, which contradicts u vanishing continuously on $\Delta_{3r}(Q, s)$.

Theorem 2 [3]. *Let* $(Q, s) \in \partial_p D_T$. *For* $r < \min(\frac{1}{2} r_0, \frac{1}{2}\sqrt{s}, \frac{1}{2}\sqrt{T-s})$ *and* $(\chi, t) \in D_T$ *so that* $s + 4r^2 \le t \le T$, *there are constants* c_1, c_2 *depending only on* λ, r_0, M, T, n *such that*

$$c_1 r^n G(\chi, t : \overline{A}_r(Q, s)) \le w^{(\chi, t)}(\Delta_r(Q, s)) \le c_2 r^n G(\chi, t : \underline{A}_r(Q, s)).$$

Proof [3, 4]. The theorem is proved by using Aronson's estimates on the Green's function and the method of proof of the analogous result in Theorem 1.4 in Fabes, Garofalo, and Salsa [3].

Theorem 3 (Local Comparison) [3]. *Let* $(Q, s) \in \partial_p D_T$, $s > 0$, *and* u, v *be solutions in* D_T *of* $(\partial/\partial t - L)w = 0$ *such that*

$$u|_{\Delta_{Mr}(Q, s)} = 0 = v|_{\Delta_{Mr}(Q, s)}.$$

Then there is a constant $c = c(\lambda, r_0, M, n, T)$ *such that for* $r \le \min(\frac{1}{2}r_0, \frac{1}{2}\sqrt{s}, \frac{1}{2}r_0\sqrt{T-s})$ *and* $(\chi, t) \in P_{r/M^3}(Q, s) \cap D_T$, *then*

$$\frac{1}{c}\frac{u(\underline{A}_r(Q, s))}{v(\overline{A}_r(Q, s))} \le \frac{u(\chi, t)}{v(\chi, t)} \le c\frac{u(\overline{A}_r(Q, s))}{v(\underline{A}_r(Q, s))}$$

where

$$\overline{A}_r(Q, s) = (A_r(Q), s + 2r^2, \underline{A}_r(Q, s)) = (A_r(Q), s - 2r^2).$$

Proof [3, 4]. The theorem can be proved by adapting the argument in Jerison and Kenig [4] for local comparison on NTA domains to the caloric setting.

References

1. D. Aronson, *Non-negative solutions of linear parabolic equations*, Ann. Scuola Norm. Sup. Pisa **22** (1968), 607–694.
2. L. Caffarelli, E. Fabes, S. Mortola, and S. Salsa, *Boundary behavior of nonnegative solutions of elliptic operators in divergence form*, Indiana Univ. Math. J. **30** (1981), 621–640.
3. E. Fabes, N. Garofalo, and S. Salsa, *A backward Harnack inequality and Fatou theorem for nonnegative solutions of parabolic equations*, Illinois J. Math. **30** (1986), 536–565.
4. D. Jerison and C. Kenig, *Boundary behavior of harmonic functions in non-tangentially accessible domains*, Adv. Math. **46** (1982), 80–147.
5. P. Jones and T. Wolff, *Hausdorff dimension of harmonic measures in the plane*, Acta Math. **161** (1988), 131–144.
6. J. M. Marstrand, *The dimension of Cartesian product sets*, Math. Proc. Cambridge Philos. Soc. **50** (1954), 198–202.
7. S. Salsa, *Some properties of nonnegative solutions of parabolic differential operators*, Ann. Mat. Pura Appl. **128** (1981), 193–206.
8. C. Sweezy, *The Hausdorff dimension of elliptic measure—a counterexample to the Oksendahl conjecture in* $\mathbf{R}^2$, Proc. Amer. Math. Soc. **116** (1992), 361–368.
9. S. J. Taylor and N. A. Watson, *A Hausdorff measure classification of polar sets for the heat equation*, Math. Proc. Cambridge Philos. Soc. **97** (1985), 325–344.

Department of Mathematical Sciences, New Mexico State University, Las Cruces, New Mexico 88003

Current address: Department of Mathematics, University of Chicago, 5734 University Ave., Chicago, Illinois 60637

E-mail address: `csweezy@cs.uchicago.edu`

E-mail address: `csweezy@nmsu.edu`