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A SIMPLE ANALYTIC PROOF OF AN INEQUALITY BY P. BUSER

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ABSTRACT. We present a simple analytic proof of the inequality of P. Buser showing the equivalence of the first eigenvalue of a compact Riemannian manifold without boundary and Cheeger's isoperimetric constant under a lower bound on the Ricci curvature. Our tools are the Li-Yau inequality and ideas of Varopoulos in his functional approach to isoperimetric inequalities and heat kernel estimates on groups and manifolds. The method is easily modified to yield a logarithmic isoperimetric inequality involving the hypercontractivity constant of the manifold.

1. BUSER'S INEQUALITY

Throughout this paper, M will denote a compact Riemannian manifold without boundary of dimension n. We denote by μ the normalised Riemannian measure on M, by Δ the Laplace operator, and by ∇f the gradient of a smooth function f on M with Riemannian length $|\nabla f|$.

The first nontrivial eigenvalue λ_1 of the Laplacian is characterised via the min-max theorem by the Poincaré type inequality

$$\lambda_1 \int f^2 \, d\mu \leq \int |\nabla f|^2 \, d\mu$$

holding for all smooth functions f on M with $\int f \, d\mu = 0$. Alternatively, by the spectral theorem (or simply by differentiation),

$$\|P_t f\|_2 \leq e^{-\lambda_1 t} \|f\|_2, \qquad t \geq 0, \tag{1}$$

for all f with $\int f \, d\mu = 0$, where $\|\cdot\|_p$ is the L^p-norm ($1 \leq p \leq \infty$) with respect to μ and where $P_t = e^{t\Delta}$, $t \geq 0$, is the heat semigroup on M.

In 1970, Cheeger [C] introduced an isoperimetric constant to bound below the first eigenvalue λ_1. Set

$$h = \inf \frac{a(\partial A)}{\mu(A)},$$

where the infimum runs over all open subsets A with $\mu(A) \leq \frac{1}{2}$ and smooth boundary ∂A, and where $a(\cdot)$ denotes the $(n-1)$-dimensional measure.

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Cheeger's result is that

$$\lambda_1 \geq \frac{h^2}{4}. \tag{2}$$

One simple argument to derive (2) may be sketched as follows (see [Y, GHL]). First note that the definition of h together with the coarea formula [F, C] leads to

$$h\int_0^\infty \min(\mu(g \geq s),\, 1-\mu(g\geq s))\,ds \leq \int |\nabla g|\,d\mu \tag{3}$$

for every positive smooth g on M. Now, let f be a smooth function on M and denote by m a median of f for μ, i.e., $\mu(f \geq m) \geq \frac{1}{2}$ and $\mu(f \leq m) \geq \frac{1}{2}$. Set $f^+ = \max(f-m, 0)$, $f^- = -\min(f-m, 0)$ so that $f - m = f^+ - f^-$. By the definition of the median, for every $s > 0$,

$$\mu((f^+)^2 \geq s) \leq \tfrac{1}{2} \quad \text{and} \quad \mu((f^-)^2 \geq s) \leq \tfrac{1}{2}.$$

Hence, (3) applied to $g = (f^+)^2$ and $g = (f^-)^2$ together with integration by parts yields

$$\begin{aligned} h\int |f-m|^2\,d\mu &= h\int (f^+)^2\,d\mu + h\int (f^-)^2\,d\mu \\ &= h\int_0^\infty \mu((f^+)^2 \geq s)\,ds + h\int_0^\infty \mu((f^-)^2\geq s)\,ds \\ &\leq \int |\nabla (f^+)^2|\,d\mu + \int |\nabla (f^-)^2|\,d\mu. \end{aligned}$$

By the Cauchy-Schwarz inequality, the right-hand side of this inequality is less than

$$2\left(\int |f-m|^2\,d\mu\right)^{1/2}\left(\int |\nabla f|^2\,d\mu\right)^{1/2}.$$

Therefore, for every median m of f,

$$\frac{h^2}{4}\int |f-m|^2\,d\mu \leq \int |\nabla f|^2\,d\mu.$$

Since the mean $\int f\,d\mu$ minimises $\int |f-c|^2\,d\mu$, $c \in \mathbb{R}$, inequality (2) immediately follows. If M has a boundary, then Cheeger's inequality still holds if λ_1 is subject to the Neumann boundary condition.

Cheeger's inequality $\lambda_1 \geq h^2/4$ proved extremely useful in finding geometrical lower bounds on λ_1 via the isoperimetric constant h. It was therefore an important observation by Buser [B] that this inequality is sharp in the sense that λ_1 and h are actually equivalent, with constants depending only on the dimension and the Ricci curvature of M. More precisely, Buser obtained the following result.

Theorem 1. *Let M be a compact Riemannian manifold without boundary whose Ricci curvature is bounded below by $-K$, $K \geq 0$. Then*

$$\lambda_1 \leq C(\sqrt{K}h + h^2),$$

where $C > 0$ is a constant which depends only on the dimension of M.

Proof of Theorem 1. While the proof of Buser is geometric, the aim of this paper is to provide a simple analytic proof of this inequality using some semigroup techniques inspired from the work of Varopoulos [V1, V2] in his functional approach to isoperimetric inequalities and heat kernel estimates on groups and manifolds.

We present the basic idea of the proof first. Recall the heat semigroup $(P_t)_{t\geq 0}$. We start from the Li-Yau inequality [LY]: for every f positive and smooth, and every $\alpha > 1$, $t > 0$, at each point of M,

$$\frac{|\nabla P_t f|^2}{(P_t f)^2} - \alpha \frac{\Delta P_t f}{P_t f} \leq \frac{n\alpha^2}{2t}\left(1 + \frac{Kt}{\alpha - 1}\right). \tag{4}$$

We will use this inequality with simply, say, $\alpha = 2$. Following [V2], this inequality implies that, for every $t_0 > 0$, $0 < t \leq t_0$, and every f positive and smooth,

$$\| |\nabla P_t f| \|_\infty \leq \frac{C}{\sqrt{t}} \|f\|_\infty, \tag{5}$$

where $C = [3n(1 + Kt_0)]^{1/2}$.

For the sake of completeness, we briefly recall below the proofs of (4) and (5), but first, we would like to describe how inequality (5) may be used to establish the theorem.

We assume in the following that $K > 0$. When $K = 0$, (4) actually holds with $\alpha = 1$ and (5) for every $t > 0$, so that the argument below is trivially modified to this case. Let us choose therefore $t_0 = 1/K$ in (5) (hence $C = (6n)^{1/2}$). Integrating (5) yields, by duality, for every f positive and smooth and every $0 < t \leq 1/K$,

$$\|f - P_t f\|_1 \leq 2C\sqrt{t} \| |\nabla f| \|_1. \tag{6}$$

Indeed, for every g smooth with $\|g\|_\infty \leq 1$,

$$\begin{aligned} \int g(f - P_t f)\, d\mu &= -\int_0^t \left(\int g \Delta P_s f\, d\mu\right) ds = \int_0^t \left(\int \nabla P_s g \cdot \nabla f\, d\mu\right) ds \\ &\leq \| |\nabla f| \|_1 \int_0^t \| |\nabla P_s g| \|_\infty\, ds \leq 2C\sqrt{t} \| |\nabla f| \|_1, \end{aligned}$$

where we used (5) in the last step. Now, we simply apply inequality (6) to smooth functions approximating the characteristic function χ_A of an open set A in M with smooth boundary ∂A. It yields , for every $0 < t \leq 1/K$,

$$\begin{aligned} 2C\sqrt{t} a(\partial A) &\geq \int_A [1 - P_t(\chi_A)]\, d\mu + \int_{A^c} P_t(\chi_A)\, d\mu \\ &= 2\left(\mu(A) - \int_A P_t(\chi_A)\, d\mu\right) = 2\left(\mu(A) - \|P_{t/2}(\chi_A)\|_2^2\right). \end{aligned} \tag{7}$$

Now, by (1),

$$\|P_{t/2}(\chi_A)\|_2^2 = \mu(A)^2 + \|P_{t/2}(\chi_A - \mu(A))\|_2^2 \leq \mu(A)^2 + e^{-\lambda_1 t}\|\chi_A - \mu(A)\|_2^2 \tag{8}$$

so that, with the preceding,

$$2C\sqrt{t} a(\partial A) \geq 2\mu(A)(1 - \mu(A))(1 - e^{-\lambda_1 t})$$

for every $0 < t \leq 1/K$. Therefore,

$$h \geq \frac{1}{2C} \sup_{0 < t \leq 1/K} \left(\frac{1 - e^{-\lambda_1 t}}{\sqrt{t}}\right). \tag{9}$$

The proof is complete. Indeed, if $\lambda_1 \geq K$, we can choose $t = 1/\lambda_1$ in the supremum of (9) to get

$$h \geq \frac{1}{2C}\left(1 - \frac{1}{e}\right)\sqrt{\lambda_1},$$

while if $\lambda_1 \leq K$, we simply take $t = 1/K$ and then

$$h \geq \frac{1}{2C}\sqrt{K}(1 - e^{-\lambda_1/K}) \geq \frac{1}{4C} \cdot \frac{\lambda_1}{\sqrt{K}}.$$

In any case,

$$\lambda_1 \leq 4C\sqrt{K}h + 16C^2h^2$$

which is the result.

As announced, and for the sake of completeness, we briefly recall, to conclude this proof, the steps (4) and (5) due, respectively, to Li and Yau [LY] and Varopoulos [V2].

Our exposition of the Li-Yau inequality follows [D]. The starting point is the Bochner formula (see [BGM, GHL]) which leads to the inequality

$$\frac{1}{2}\Delta(|\nabla g|^2) - \nabla g \cdot \nabla(\Delta g) \geq \frac{1}{n}(\Delta g)^2 - K|\nabla g|^2 \tag{10}$$

for all smooth functions g on M. Now, let f be positive and smooth and, on $M \times [0, T]$, $T > 0$, set $g = \log P_t f$. We observe that

$$\Delta g + |\nabla g|^2 = g_t, \tag{11}$$

where g_t is differentiation with respect to time. By (10), it follows that

$$\begin{aligned}\frac{2}{n}(\Delta g)^2 - 2K|\nabla g|^2 &\leq \Delta(|\nabla g|^2) - 2\nabla g \cdot \nabla g_t + 2\nabla g \cdot \nabla(|\nabla g|^2) \\ &= \Delta(|\nabla g|^2) - \frac{d}{dt}|\nabla g|^2 + 2\nabla g \cdot \nabla(|\nabla g|^2).\end{aligned}$$

Multiply this inequality by t and set $H = t|\nabla g|^2$ so that

$$t\left[\frac{2}{n}(\Delta g)^2 - 2K|\nabla g|^2\right] \leq \Delta H - H_t + \frac{H}{t} + 2\nabla g \cdot \nabla H.$$

Now, if we let $I = tg_t$, it is elementary from (11) that

$$\Delta I - I_t + \frac{I}{t} + 2\nabla g \cdot \nabla I = 0.$$

Therefore, if, for any real number α, we let $G = H - \alpha I = t(|\nabla g)|^2 - \alpha g_t)$, we have obtained that, on $M \times [0, T]$,

$$t\left[\frac{2}{n}(\Delta g)^2 - 2K|\nabla g|^2\right] \leq \Delta G - G_t + \frac{G}{t} + 2\nabla g \cdot \nabla G. \tag{12}$$

Let (x, t) be a point in $M \times [0, T]$ at which G takes its maximum value. Assume first that $G(x, t) > 0$. Then $t > 0$ and, at (x, t), $\nabla G = 0$, $\Delta G \leq 0$, $G_t \geq 0$. Therefore, at (x, t), (12) yields

$$\frac{2}{n}(\Delta g)^2 \leq \frac{G}{t^2} + 2K|\nabla g|^2.$$

Assume $\alpha > 1$. Recall that $\Delta g = g_t - |\nabla g|^2$ and $G = t(|\nabla g|^2 - \alpha g_t)$. Thus

$$\frac{2}{n}\left[\left(1-\frac{1}{\alpha}\right)|\nabla g|^2 + \frac{G}{\alpha t}\right]^2 \leq \frac{G}{t^2} + 2K|\nabla g|^2 .$$

Multiply both sides of this inequality by $\alpha^2 t^2$ and set $J = |\nabla g|^2/G$. Simplifying by G then yields

$$\frac{2}{n}(1+(\alpha-1)tJ)^2 G \leq \alpha^2(1+2Kt^2J).$$

Hence, at (x, t),

$$G \leq \frac{n\alpha^2}{2}\cdot\frac{1+2Kt^2J}{[1+(\alpha-1)tJ]^2} \leq \frac{n\alpha^2}{2}\left(1+\frac{Kt}{\alpha-1}\right).$$

This inequality also holds when $G(x, t) \leq 0$, and, recalling that $g = \log P_t f$, is exactly (4).

To deduce (5) from (4), let thus $\alpha = 2$ and note that (4) implies that, for every $0 < t \leq t_0$ and every f positive and smooth,

$$(\Delta P_t f)^- \leq n(1+Kt_0)\frac{1}{t}P_t f,$$

where $(\cdot)^-$ is the negative part. Since $\int \Delta P_t f\, d\mu = 0$ and $\int P_t f\, d\mu = \int f\, d\mu$,

$$\|\Delta P_t f\|_1 \leq \frac{2C'}{t}\|f\|_1$$

with $C' = n(1+Kt_0)$. By duality, for every f and $t > 0$,

$$\|\Delta P_t f\|_\infty \leq \frac{2C'}{t}\|f\|_\infty .$$

Coming back to the Li-Yau inequality (4),

$$\|\,|\nabla P_t f|\,\|_\infty^2 \leq \frac{3C'}{t}\|f\|_\infty^2$$

which is (5).

The preceding proof of Theorem 1 extends to complete noncompact manifolds. Define the bottom of the spectrum λ_1 as the infimum of $\int |\nabla f|^2 d\mu / \int f^2 d\mu$, where f runs over sufficiently smooth functions with compact support (assume the volume of M is infinite). Let h be as before but with the additional condition that $A \cup \partial A$ be compact. Since (4) and (5) hold similarly in the noncompact case, one gets in the same way from (6), (7), and (8) that for every open subset A such that $A \cup \partial A$ is compact and all $0 < t \leq 1/K$,

$$2C\sqrt{t}a(\partial A) \geq 2(\mu(A) - \|P_{t/2}(\chi_A)\|_2^2) \geq 2\mu(A)(1-e^{-\lambda_1 t}).$$

The proof is then completed in the same way. Since, as mentioned in [B], $h \leq \sqrt{(n-1)K}$ in the noncompact case, the final result here is that $\lambda_1 \leq C\sqrt{K}h$, where C only depends on the dimension.

2. On a logarithmic isoperimetric inequality

It is a simple matter to modify the preceding approach by using other geometric invariants of M instead of the spectral gap λ_1. One may use, as in [V1, V2, Co], Sobolev constants via heat kernel decays. One may also use the so-called hypercontractivity constant ρ_0 of the Laplace operator on M defined as the least $\rho > 0$ such that whenever $1 < p < q < \infty$ and $e^{\rho t} \geq [(q-1)/(p-1)]^{1/2}$,

$$\|P_t f\|_q \leq \|f\|_p \tag{13}$$

for every f on M (in $L^p(\mu)$). It is known [R1] that on every compact Riemannian manifold $\lambda_1 \geq \rho_0 > 0$, and that [Gr] (13) may be expressed equivalently as a logarithmic Sobolev inequality

$$\rho_0 \left[\int f^2 \log|f| \, d\mu - \int f^2 \, d\mu \log \left(\int f^2 \, d\mu \right)^{1/2} \right] \leq \int |\nabla f|^2 \, d\mu \tag{14}$$

for all smooth functions f on M.

Now, if we follow the proof of Theorem 1 and replace, in (7), (8), the spectral estimate by the hypercontractivity inequality (13), we simply get that, for all open subsets A of M with smooth boundary ∂A and for all $0 < t \leq 1/K$,

$$2C\sqrt{t}a(\partial A) \geq 2(\mu(A) - \|P_{t/2}(\chi_A)\|_2^2) \geq 2(\mu(A) - \mu(A)^{2/p(t)}),$$

where $p(t) = 1 + e^{-\rho_0 t}$ (and $C = (12n)^{1/2}$). Set $t_0 = \min(1/K, 1/\rho_0)$. Since $1 - e^{-x} \geq x/2$ for $0 \leq x \leq 1$, it follows that for every $0 < t \leq t_0$,

$$C\sqrt{t}a(\partial A) \geq \mu(A)(1 - \mu(A)^{\rho_0 t/4}) = \mu(A) \left[1 - \exp\left(-\frac{\rho_0 t}{4} \log \frac{1}{\mu(A)} \right) \right]. \tag{15}$$

Assume $\mu(A) > 0$. Choose then $0 < t \leq t_0$ such that

$$t = 4t_0 \left(\log \frac{1}{\mu(A)} \right)^{-1}$$

provided $\mu(A)$ is small enough so that $\mu(A) \leq e^{-4}$. For this value of t, (15) reads as

$$\begin{aligned} a(\partial A) &\geq \frac{1}{2C\sqrt{t_0}}(1 - e^{-\rho_0 t_0})\mu(A) \left(\log \frac{1}{\mu(A)} \right)^{1/2} \\ &\geq \frac{1}{4C} \rho_0 \sqrt{t_0} \mu(A) \left(\log \frac{1}{\mu(A)} \right)^{1/2} \end{aligned}$$

since $\rho_0 t_0 \leq 1$.

The preceding inequality holds for $\mu(A) \leq e^{-4}$. In general however, when $0 \leq \mu(A) \leq \frac{1}{2}$, we can always apply (15) with $t = t_0$ to get

$$a(\partial A) \geq \frac{1}{C\sqrt{t_0}} \mu(A) \left[1 - \exp\left(-\frac{\rho_0 t_0}{4} \log 2 \right) \right] \geq \frac{1}{16C} \rho_0 \sqrt{t_0} \mu(A),$$

so that, combined with the preceding, for every A with $0 \leq \mu(A) \leq \frac{1}{2}$,

$$a(\partial A) \geq \frac{1}{32C} \rho_0 \sqrt{t_0} \mu(A) \left(\log \frac{1}{\mu(A)} \right)^{1/2}.$$

Hence, we have established the following theorem.

Theorem 2. *Let M be a compact Riemannian manifold without boundary whose Ricci curvature is bounded below by $-K$, $K \geq 0$. Then, if ρ_0 denotes the hypercontractivity constant of M, for every open subset A of M with $0 \leq \mu(A) \leq \frac{1}{2}$ and smooth boundary ∂A,*

$$a(\partial A) \geq \frac{1}{C} \min\left(\frac{\rho_0}{\sqrt{K}}, \sqrt{\rho_0}\right) \mu(A) \left(\log \frac{1}{\mu(A)}\right)^{1/2},$$

where C only depends on the dimension of M.

While only of logarithmic type with respect to the results of [G, B-B-G], this isoperimetric inequality, on the other hand, involves ρ_0 rather than the diameter of the manifold. The isoperimetric function in Theorem 2 is precisely (a form of) the isoperimetric function in Gaussian space (cf. [L]). On the basis of Theorem 2, one may thus conjecture some "infinite dimensional" extension of the Lévy-Gromov isoperimetric inequality of [G] which would compare, independently of the dimension, the isoperimetric property of a diffusion generator (with positive curvature) to the Gaussian isoperimetric inequality. In the context of Theorem 2, this would amount to showing that the constant C may actually be chosen independent of the dimension of the manifold. Going back to the proof, we would need the constant in (6) or (5) to be independent of n.

Theorem 2 may actually be stated in an equivalent formulation close to Buser's inequality. Define the logarithmic isoperimetric constant k of M as the infimum of

$$\frac{a(\partial A)}{\mu(A)(\log \frac{1}{\mu(A)})^{1/2}}$$

over all open subsets A with not more than half of the volume and smooth boundary ∂A. Note that clearly $h \geq k/2$. One may compare k to ρ_0 as in Cheeger's inequality. Namely, let g be positive and smooth on M. Then, by the coarea formula,

$$k \int_{s_0}^{\infty} \mu(g \geq s) \left(\log \frac{1}{\mu(g \geq s)}\right)^{1/2} ds \leq \int |\nabla g| \, d\mu,$$

where s_0 is such that $\mu(g \geq s_0) \leq \frac{1}{2}$. Let now f be smooth with $\int f^2 \, d\mu = 1$ and apply the preceding inequality to $g = f^2(\log(3 + f^2))^{1/2}$. After some elementary, but tedious, computations, we get

$$\int f^2 \log(3 + f^2) \, d\mu \leq \frac{\alpha}{k^2} \int |\nabla f|^2 \, d\mu + \alpha$$

for some numerical constant $\alpha > 0$. For any function f on M, set

$$E(f) = \int f^2 \log|f| \, d\mu - \int f^2 \, d\mu \log\left(\int f^2 \, d\mu\right)^{1/2}.$$

So far, we have obtained by homogeneity that, for some numerical constant α (not necessarily the same at each occurrence) and every smooth function f on M,

$$E(f) \leq \alpha\left(\frac{1}{k^2} \int |\nabla f|^2 \, d\mu + \int f^2 \, d\mu\right). \tag{16}$$

This is not yet (14), and to get rid of the extra factor, we may use the spectral gap λ_1. Namely, by [DS, p. 246] or [R2], we know that for every f,

$$E(f) \leq E(f - \textstyle\int f\,d\mu) + \displaystyle\int |f - \textstyle\int f\,d\mu|^2\,d\mu.$$

Hence, (16) applied to $f - \int f\,d\mu$ combined with this inequality yields

$$\begin{aligned} E(f) &\leq \frac{\alpha}{k^2}\int |\nabla f|^2\,d\mu + (\alpha+1)\int |f - \textstyle\int f\,d\mu|^2\,d\mu \\ &\leq \left(\frac{\alpha}{k^2} + \frac{\alpha+1}{\lambda_1}\right)\int |\nabla f|^2\,d\mu \end{aligned}$$

for every smooth f on M. Since $h \geq k/2$, and hence $\lambda_1 \geq k^2/16$, it follows that $\rho_0 \geq k^2/\alpha$ for some numerical constant $\alpha > 0$. This relation clarifies and improves parts of [R2].

On the other hand, the content of Theorem 2 is that

$$\rho_0 \leq C(k\sqrt{K} + k^2),$$

where C only depends on the dimension.

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