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FUNCTIONS WITH A UNIQUE MEAN VALUE AND AMENABILITY

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ABSTRACT. It is shown that there exist many amenable locally compact groups for which the sets of functions with unique left invariant mean values are not closed under addition. This resolves negatively a problem raised by T. Miao.

Let G be a locally compact group with a fixed left Haar measure λ and let $L^p(G)$ $(1 \leq p \leq \infty)$ be the associated Lebesgue spaces. (As usual, if G is compact, then λ is assumed to be normalized, i.e., $\lambda(G) = 1$.) A subspace S of $L^\infty(G)$ is said to be *admissible* if it contains the constants and ${}_xf$ for each $f \in S$ and $x \in G$ (where ${}_xf$ is the left translate of f by x, ${}_xf(y) = f(xy)$ $(y \in G)$). If $f \in L^\infty(G)$, let S_f denote the smallest admissible subspace containing f. We say that $f \in L^\infty(G)$ has a *unique left invariant mean value* if $\mathrm{LIM}(S_f)$ is nonempty and there exists a constant c such that $m(f) = c$ for each $m \in \mathrm{LIM}(S_f)$. (For an admissible subspace S of $L^\infty(G)$, $\mathrm{LIM}(S)$ stands for the set of left invariant means on S, i.e., all $m \in S^*$ with $m \geq 0$, $m(1) = 1$, and $m({}_xf) = m(f)$ $(x \in G,\ f \in S)$.) The set of functions with a unique left invariant mean value is denoted by $U(G)$. Note that $U(G)$ is always closed under scalar multiplication. Recall also that if G is amenable as a discrete group, then $U(G)$ coincides with the sum of the constants and the norm closed linear span of $\{f - {}_xf: f \in L^\infty(G),\ x \in G\}$ (see [6, Theorem 1.1]). In particular, $U(G)$ is closed under addition for such a G. Recently Miao [3, Theorem 3.4] proved that if $U(G)$ is closed under addition, then G is amenable, thus answering a question raised by Rosenblatt and Yang in [6, p. 747]. The following was posed as an open problem in [3, p. 1083]: Is $U(G)$ closed under addition if G is amenable? The purpose of the present note is to give many examples which show that the answer to Miao's problem is negative.

Recall that a compact group G is said to have the *mean zero weak containment property* if there exists a net $\{g_\alpha\}$ in

$$L_0^2(G) = \left\{f \in L^2(G):\ \int_G f\,d\lambda = 0\right\}$$

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such that $\|g_\alpha\|_2 = 1$ for all α and $\lim_\alpha \|_x g_\alpha - g_\alpha\|_2 = 0$ for all $x \in G$ (see [5]). The negative answer to Miao's problem is a direct consequence of the following result.

Theorem. *Let G be an infinite compact group and suppose that $U(G)$ is closed under addition. Then G has the mean zero weak containment property.*

Proof. Assume, by way of contradiction, that G does not have the mean zero weak containment property. We denote by $\mathbb{C}$ and H the constant functions on G and the linear span of $\{f - {}_x f : f \in L^\infty(G),\ x \in G\}$, respectively. Let us first show that $H + \mathbb{C}$ is included in $U(G)$. Obviously, $\mathbb{C}$ is contained in $U(G)$. Let h be in $\{f - {}_x f : f \in L^\infty(G),\ x \in G\}$. Since λ induces a left invariant mean on S_h, $\mathrm{LIM}(S_h)$ is nonempty. Observe that

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{k=1}^{n} {}_{x^k}(f - {}_x f) \right\|_\infty = 0$$

for each $f \in L^\infty(G)$ and $x \in G$. Using this fact, we can see that $m(h) = 0$ for each $m \in \mathrm{LIM}(S_h)$. Thus h has a unique left invariant mean value. As $U(G)$ is closed under addition, $U(G)$ contains $H + \mathbb{C}$, as desired. On the other hand, since G does not have the mean zero weak containment property, it follows from the proofs of [8, Lemma and Proposition 2] that $L^\infty(G) = H + \mathbb{C}$. In conclusion, we obtain $L^\infty(G) = U(G)$. But this is a contradiction. In fact, let E be an open dense subset of G satisfying $\lambda(E) < 1$. (It is possible to find such an E because G is an infinite compact group.) Then there exists a left invariant mean m on S_{χ_E} such that $m(\chi_E) = 1$, where χ_E denotes the characteristic function of E (cf. [3, Lemma 3.1]). This means that both 1 and $\lambda(E)$ (< 1) can be left invariant mean values of χ_E on S_{χ_E}. We therefore have $\chi_E \notin U(G)$, which gives the desired contradiction. The proof is now complete.

It is well known that if an infinite compact group G is amenable as a discrete group, then G has the mean zero weak containment property (cf. [4, Theorem 1.3 and Lemma 3.1]). Some examples of compact groups which do not have the mean zero weak containment property can be found in [1, 2, 5, 7]. For example, $\mathrm{SO}(n)$ (the special orthogonal group) does not have the mean zero weak containment property for $n \geq 3$. Therefore our Theorem implies that the set of functions on $\mathrm{SO}(n)$ $(n \geq 3)$ with a unique left invariant mean value is not closed under addition. This resolves Miao's problem [3, p. 1083] negatively.

As a consequence of our Theorem we have the following. This gives a number of examples which show that the answer to Miao's problem is negative.

Proposition. *Let G_1 be an infinite compact group that does not have the mean zero weak containment property and let G_2 be an amenable locally compact group. Then $G_1 \times G_2$ is an amenable locally compact group for which $U(G_1 \times G_2)$ is not closed under addition.*

Proof. Let us find two functions in $U(G_1 \times G_2)$ whose sum does not have a unique left invariant mean value. Since, by the Theorem, $U(G_1)$ is not closed under addition, there exist h and k in $U(G_1)$ such that $h + k$ is not contained in $U(G_1)$. Now define functions f and g in $L^\infty(G_1 \times G_2)$ by

$$f(x, y) = h(x) \quad \text{and} \quad g(x, y) = k(x) \qquad ((x, y) \in G_1 \times G_2).$$

Then f and g have unique left invariant mean values but $f+g$ does not. This can be easily verified from [6, Proposition 1.3] and a straightforward argument. Thus we obtain that $U(G_1 \times G_2)$ is not closed under addition.

Remarks. (1) It is possible that a compact group G has the mean zero weak containment property, and $U(G)$ is not closed under addition. For example, take $G = \mathrm{SO}(n) \times \mathbb{T}$ $(n \geq 3)$, where $\mathbb{T}$ denotes the circle group. Then the Proposition implies that $U(G)$ is not closed under addition. Since $L^\infty(G)$ has more than one left invariant mean, it follows from [4, Theorem 1.3 and Lemma 3.1] that G has the mean zero weak containment property. Thus the converse of our Theorem is false.

(2) Recall Miao's Theorem [3, Theorem 3.4] quoted above: If $U(G)$ is closed under addition, then G is amenable. Our Theorem and Proposition may be regarded as partial improvements of this result. It would be of interest to determine the class of locally compact groups G for which $U(G)$ is closed under addition.

(3) Let H be as in the proof of our Theorem. It is worthwhile to observe the following: If G is a locally compact group and if $U(G)$ is closed under addition, then

$$H + \mathbb{C} \subseteq U(G) \subseteq \overline{H} + \mathbb{C} \ (= \overline{H + \mathbb{C}}),$$

where the over bar denotes the norm closure in $L^\infty(G)$. The argument used in the proof of the Theorem can apply to show the first inclusion relation. (Notice that the closure of $U(G)$ under addition implies amenability of G [3, Theorem 3.4].) The second inclusion relation follows from the proof of [3, Theorem 3.6].

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