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ON THE TRANSLATES OF A SET WHICH MEET IT IN A SET OF POSITIVE MEASURE

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ABSTRACT. Given a singular Borel regular measure m_a on R^n and a Borel subset E of R^n, it is shown that the set of vectors x for which $m_a((E+x)\cap E) > 0$ is of Lebesgue measure 0. This fact is then used to show that subsets of finite, nonzero, Hausdorff s-measure are nonmeasurable sets with respect to any approximating measure $s\text{-}m_\delta$.

Given a set $E \subset R^n$, let $E+x$ denote the translation of E by the vector x; that is, $E+x = \{y : \text{there is } t \in E \text{ and } y = t+x\}$. If m_a is a measure on R^n, let $T_E(m_a) = \{x : m_a(E \cap (E+x)) > 0\}$. Let $m(A)$ denote the n-dimensional Lebesgue measure of A. Theorem 1 shows that $m(T_E(m_a)) = 0$ when m_a is a singular, σ-finite, Borel regular measure on R^n and $E \subset R^n$ is a Borel set. A specific application of and motivation for the result is Theorem 2, which shows that, for $s < n$ and $\delta > 0$, every s-measurable set with nonzero, finite measure (such sets are called s-sets) is a nonmeasurable set with respect to $s\text{-}m_\delta$. Here, the outer measures $s\text{-}m_\delta^*$ and $s\text{-}m^*$ are given by $s\text{-}m_\delta^*(E) = \inf\sum \operatorname{diam}(E_i)^s$ where the infimum is over all sequences $\{E_i\}$ with $E \subset \bigcup E_i$ and each $\operatorname{diam}(E_i) < \delta$; $s\text{-}m^*(E) = \lim_{\delta\to 0^+} s\text{-}m_\delta^*(E)$. Of course, all Borel sets are s-measurable, and, if $s\text{-}m(E) = 0$, then each $s\text{-}m_\delta(E) = 0$ and E is measurable with respect to $s\text{-}m_\delta$.

Given a set $E \subset R^n$, E^c will denote the complement of E with respect to R^n. Also, $C_E(x)$ will denote the characteristic function of E; that is, $C_E(x) = 1$ if $x \in E$, and $C_E(x) = 0$ if $x \in E^c$. The symbol $\int$ will denote the integral over all of R^n.

Theorem 1. *Let m_a be a singular, σ-finite, Borel regular measure on R^n. Let A be a Borel set with $m(A) = 0$ so that $m_a(A^c) = 0$. Then $m(T_A(m_a)) = 0$, and thus, for any Borel set E, $m(T_E(m_a)) = 0$.*

Proof. For $x \in R^n$, define $f(x) = m_a(A \cap (A+x))$ where A is the Borel set with $m(A) = 0$ and $m_a(A^c) = 0$. Since A is a Borel set, so is $A+x$; thus, $C_A(t)\cdot C_A(t+x) = g(x, t)$ is a measurable function with respect to the product

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measure $m \times m_a$. Furthermore, $f(x) = m_a(\{t : t \in A \text{ and } t + x \in A\}) = \int C_A(t)C_A(t+x)\,dm_a(t)$ is a nonnegative measurable function. (Cf. [2, pp. 82 ff] for this fact and the generalized Fubini's theorem for product spaces used below.) To prove the theorem, it suffices to show that $f(x) = 0$ except on a set of Lebesgue measure 0. Since $f(x) \geq 0$, this can be obtained by showing that $\int f(x)\,dm(x) = 0$. But

$$\begin{aligned}\int f(x)\,dm(x) &= \int \left(\int C_A(t)C_A(t+x)\,dm_a(t)\right) dm(x)\\ &= \int C_A(t)\left(\int C_A(t+x)\,dm(x)\right) dm_a(t)\\ &= \int C_A(t)\cdot 0\,dm_a(t) = 0.\end{aligned}$$

Note. The above argument is parallel to one frequently used to show that the set of all distances between points of a Lebesgue measurable set of positive measure contains an interval. A slight variation applies to any measure m_b which is absolutely continuous with respect to Lebesgue measure. It shows that $T_E(m_b)$ must contain an open set when $m_b(E) > 0$. This is because a compact set $F \subset E$ with $0 < m_b(F) < \infty$ gives rise to the continuous function

$$f(x) = \int C_F(x+t)C_F(t)\,dm_b(t) = \int C_F(x+t)C_F(t)g(t)\,dt$$

where g is the Radon-Nikodym derivative of m_b with respect to m. But

$$\begin{aligned}\int f(x)\,dx &= \iint C_F(x+t)C_F(t)g(t)\,dt\,dx\\ &= \int C_F(t)g(t)\left(\int C_F(x+t)\,dx\right) dt\\ &= m(F)\cdot m_b(F) > 0.\end{aligned}$$

This is because $m_b(F) > 0$ and m_b absolutely continuous with respect to m implies $m(F) > 0$. This argument also implies that $f(x)$ is greater than 0 on an open set G, and thus $T_E(m_b)$ contains G. By the Lebesgue decomposition theorem, this will also hold for any such measure which has a nonzero, absolutely continuous component measure.

Theorem 2. *Suppose E is an s-set in R^n with $s < n$. Then for any $\delta > 0$, E is a nonmeasurable set with respect to the measure $s\text{-}m_\delta$.*

Proof. For each set $X \subset R^n$, let $m_a^*(X) = s\text{-}m^*(E \cap X)$. Then m_a is a singular, Borel regular, σ-finite measure on R^n, and, by Theorem 1, $m(T_E(m_a)) = 0$. Thus the complement of $T_E(m_a)$ is dense and contains vectors x of arbitrarily small norm so that $s\text{-}m((E+x) \cap E) = 0$. Let $\{E_i\}$ satisfy for each i $\operatorname{diam}(E_i) < \delta$, $E \subset \bigcup E_i$, and $\sum \operatorname{diam}(E_i)^s < \frac{5}{4}s\text{-}m_\delta^*(E)$. We may also assume that the E_i are open sets. There is $M > 0$ so that if $E_M = \{x \in E : \|x\| \leq M\}$ then

$$\sum{}' \operatorname{diam}(E_i)^s < \tfrac{1}{4}s\text{-}m_\delta^*(E)$$

where $\sum'$ is over all E_i which contain points x with $\|x\| > M$. Then $s\text{-}m_\delta^*(E \backslash E_M) < \frac{1}{4}s\text{-}m_\delta^*(E)$. By a well-known property of s-measure (cf. [1])

there is an increasing sequence of compact sets $\{F_n\}$ with $F_n \subset E_M$ so that $s\text{-}m(\bigcup F_n) = s\text{-}m(E_M)$. By the increasing sets lemma (cf. [1, pp. 70ff]) there is N so that $F = F_N$ satisfies $s\text{-}m_\delta^*(F) > \frac{3}{4}s\text{-}m_\delta^*(E_M)$; thus, since $s\text{-}m_\delta^*(E) \leq s\text{-}m_\delta^*(E\backslash E_M) + s\text{-}m_\delta^*(E_M)$, it follows that

$$\begin{aligned} s\text{-}m_\delta^*(E_M) &\geq s\text{-}m_\delta^*(E) - s\text{-}m_\delta^*(E\backslash E_M) \\ &\geq s\text{-}m_\delta^*(E) - \tfrac{1}{4}s\text{-}m_\delta^*(E) \geq \tfrac{3}{4}s\text{-}m_\delta^*(E). \end{aligned}$$

Thus $s\text{-}m_\delta^*(F) \geq \frac{3}{4}s\text{-}m_\delta^*(E_M) \geq (\frac{3}{4})^2 s\text{-}m_\delta^*(E)$ and $s\text{-}m_\delta^*(F) > \frac{1}{2}s\text{-}m_\delta^*(E)$. Let x be a vector so that $F+x \subset \bigcup E_i$ and $s\text{-}m((F+x)\cap F) = 0 = s\text{-}m_\delta(F+x\cap F)$. To show that E is not measurable, let $A = E \cup (F+x)$. Since $A \subset \bigcup E_i$ and $s\text{-}m_\delta^*$ is translation invariant,

$$\begin{aligned} s\text{-}m_\delta^*(A) &\leq \sum \operatorname{diam}(E_i)^s \leq s\text{-}m_\delta^*(E) + \tfrac{1}{4}s\text{-}m_\delta^*(E) \\ &\leq s\text{-}m_\delta^*(E) + \tfrac{1}{2}s\text{-}m_\delta^*(F) \\ &\leq s\text{-}m_\delta^*(E) + \tfrac{1}{2}s\text{-}m_\delta^*(F+x) \\ &< s\text{-}m_\delta^*(A\cap E) + s\text{-}m_\delta^*(A\backslash E). \end{aligned}$$

Thus, by the standard criterion for measurability, E is not measurable.

Note that the above argument could be generalized to measures determined by a nondecreasing continuous function h defined on $(0, \infty)$ with $\lim_{t\to 0^+} h(t) = 0$ where $h\text{-}m_\delta^*$ is defined by replacing $\operatorname{diam}(E)^s$ with $h(\operatorname{diam}(E))$.

While Theorem 1 guarantees that the Lebesgue measure of $T_E(s\text{-}m)$ is zero whenever E is an s-set, several natural questions arise as to whether the size of $T_E(s\text{-}m)$ is not considerably smaller. In particular,

(1) For s-sets E in R^n with $n > s$ (especially when s is not a whole number), what is the supremum over all E of $\dim(T_E(s\text{-}m))$ where $\dim(X) = \inf\{s : s\text{-}m(X) = 0\}$?

(2) If E is an s-set in R^n where s is not a whole number, is it possible that $E + E = \{z : t = x + y, x \in E, y \in E\}$ is also an s-set?

References

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