PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 121, Number 3, July 1994

MOMENTS OF THE LIFETIME OF CONDITIONED BROWNIAN MOTION IN CONES

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(Communicated by Lawrence F. Gray)

ABSTRACT. Let τ be the time it takes standard d-dimensional Brownian motion, started at a point inside a cone Γ in $\mathbb{R}^d$ which has aperture angle θ, to leave the cone. Burkholder has determined the smallest p, denoted $p(\theta, d)$, such that $E\tau^p = \infty$. We show that if $y \in \partial\Gamma$ then the smallest p, such that $E(\tau^p | B_\tau = y) = \infty$, is $p = 2p(\theta, d) + (d-2)/2$.

We will be working with spherical coordinates in $\mathbb{R}^d$, $d \geq 2$. Let, for a point $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, $|x| = (\sum x_i^2)^{1/2}$, and let φ be the angle that the line segment connecting the origin 0 and x makes with the line segment connecting the origin and $1 = (0, 0, \dots, 0, 1)$. Let, for $0 < \theta < \pi$, $\Gamma = \Gamma(d, \theta)$ be the cone $\{\varphi < \theta\}$. We use τ_D to designate the exit time of a process from a domain D, and we shorten τ_Γ to τ. Probability and expectation for standard d-dimensional Brownian motion started at x will be denoted by P_x and E_x, and if $y \in \partial\Gamma$ (boundary of Γ), P_x^y and E_x^y designate probability and expectation for this motion conditioned to exit Γ at y or, more formally, of the h-process, with h the Poisson kernel of Γ for the boundary point y. We will discuss h-processes in more detail later.

Let $p(\theta, 2) = \pi/2\theta$, and, for $d > 2$, put $p(\theta, d) = 2\sup\{x : \theta < \lambda_{x,d}\}$, where $\lambda_{x,d}$ is the smallest positive zero of the hypergeometric function

$$h(w) = F(-x, x+d-2, (d-1)/2; (1-\cos w)/2),$$

with $F(a, b, c; t) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k t^k}{(c)_k k!}$, and $(r)_k = r(r+1)\cdots(r+k)$. In [1] it is shown that for $x \in \Gamma$ and $p > 0$, $E_x\tau^p < \infty$ if and only if $p < p(\theta, d)$. This was sharpened and generalized by [4]. Our main result is

Theorem 1. *Let $x \in \Gamma$, $y \in \partial\Gamma$, and $p > 0$. Then $E_x^y\tau^p < \infty$ if and only if $p < 2p(\theta, d) + (d-2)/2$.*

Proof. Our proof of this theorem essentially involves giving a new proof of Burkholder's result which, with little alteration, can be used for conditioned Brownian motion, although we note that this "new" proof rests on a calculation originally made by Burkholder. Let $\Gamma_n = \Gamma \cap \{|x| \leq 2^n\}$, $S_n = \Gamma \cap \{|x| = 2^n\}$,

Received by the editors May 18, 1992 and, in revised form, October 26, 1992.
1991 *Mathematics Subject Classification.* Primary 60J65, 60J05.
Key words and phrases. Conditioned Brownian motion, h-processes.
This research was supported by a National Science Foundation Grant.

and $H_n = S_n \cap \{\varphi \leq \theta/2\}$ be the middle half of S_n. We first prove Theorem 1 in the case $x = 1$ and $y = 0$, and then explain how to extend the proof to the general case. Let τ_n be the first time a process hits S_n. Then

$$E_1\tau^p = \sum_{n=0}^{\infty} E_1(\tau^p|\tau_n < \tau \leq \tau_{n+1})P_1(\tau_n < \tau \leq \tau_{n+1}) \tag{1}$$

and

$$E_1^0\tau^p = \sum_{n=0}^{\infty} E_1^0(\tau^p|\tau_n < \tau \leq \tau_{n+1})P_1^0(\tau_n < \tau \leq \tau_{n+1}). \tag{2}$$

We will show

$$E_1(\tau^p|\tau_n < \tau \leq \tau_{n+1}) \sim E_1^0(\tau^p|\tau_n < \tau \leq \tau_{n+1}) \sim 2^{2np}, \tag{3}$$

where $a_n \sim b_n$ means that a_n/b_n is bounded above and below by positive constants which, while they may depend on θ, d, and p, do not depend on $n \geq 0$. We also show that there is an $\alpha = \alpha(\theta) > 0$ such that both of the following hold:

$$P_1(\tau_n < \tau \leq \tau_{n+1}) \sim 2^{-n\alpha}. \tag{4}$$

$$P_1^0(\tau_n < \tau \leq \tau_{n+1}) \sim 2^{-n[2\alpha+d-2]}. \tag{5}$$

Relationships (1)–(5) imply $E_1\tau^p = \infty$ if and only if $\sum_{n=1}^{\infty} 2^{2np}2^{-n\alpha} = \infty$ and $E_1^0\tau^p = \infty$ if and only if $\sum_{n=1}^{\infty} 2^{2np}2^{-n[2\alpha+d-2]} = \infty$. Thus $E_1\tau^p = \infty$ if and only if $p \geq \alpha/2$, which with Burkholder's result gives $p(d, \theta) = \alpha/2$, while $E_1^0\tau^p = \infty$ if and only if $p \geq [2\alpha + d - 2]/2 = 2p(d, \theta) + (d-2)/2$, verifying Theorem 1 in the special case $x = 1$, $y = 0$.

To complete the proof of Theorem 1 in this special case we need to prove (3)–(5). Before we do, we collect some of the tools we will use. We let $P_x^{h,D} = P_x^h$ and $E_x^{h,D} = E_x^h$ denote probability and expectation for the h-process in a domain D with associated harmonic function h. Here, the only h-processes we will be concerned with are Brownian motion conditioned to exit a domain at a specified point or set. For a formal description of h-processes and proofs of the properties of h-processes stated below, see [5]. Let G be a subdomain of D, $x \in G$, h harmonic in D. Then the exit distribution from G under P_x^h is given by

$$P_x^h(B_{\tau_G} \in A) = \int_A \frac{h(z)}{h(x)}\, dP_x(B_{\tau_G} = z), \qquad A \subset \partial G,\ A \text{ Borel}. \tag{6}$$

Furthermore, conditioned on B_{τ_G}, the process B_t, $0 \leq t \leq \tau_G$, has the same distribution under both P_x and P_x^h. Especially, the distribution of the exit time of B_t from the open ball $B(x, \delta) \subset D$, of center x and radius δ, is the same under both P_x and P_x^h, since this distribution conditioned on the exit position from the ball is the same and by symmetry does not depend on the exit position, under P_x.

In the following inequalities, c, C, C_p, etc., stand for generic positive constants, which may depend on θ and d but do not depend on n. Let the harmonic functions u and v be defined in Γ_1 by $u(x) = P_x(B_{\tau_{\Gamma_1}} \in S_1)$ and $v(x) = P_x(B_{\tau_{\Gamma_1}} \in H_1)$.

Lemma 1. *If* $x \in \Gamma_1$ *and* $|x| \le 1$, *then* $u(x) < Cv(x)$.

Proof. A direct probabilistic proof is not too difficult, but since Lemma 1 follows immediately from the boundary Harnack principle for Lipschitz domains (see [7]), we take this route. This principle implies that, given $x \in \partial\Gamma \cup \{|x| \le 1\}$, there is a $\delta(x) > 0$, such that $u(y) < Cv(y)$ if $y \in \Gamma \cap B(x, \delta(x))$. Since we can pick a finite number of x such that the union of the $B(x, \delta(x))$ for these x contains $\{\partial\Gamma\} \cap \{|x| \le 1\}$ and since clearly $u(y) < Cv(y)$ for y in a compact subset of Γ_1, Lemma 1 follows. □

Now let $K(x)$ be the Poisson kernel for Γ with respect to the point 0; that is, K is the unique function in Γ which is harmonic and positive, has limit zero as either ∞ or a nonzero boundary point is approached, and satisfies (is normalized so that) $K(1) = 1$. Scaling shows there is a positive number β and a positive function g on $[0, \theta)$ such that $K(x) = g(\varphi)/|x|^\beta$. The exponent $\beta = \beta(\theta) > 0$ was found in [1]. We also note that $M(x) = K(x/|x|^2)/|x|^{d-2} = |x|^{\beta+2-d} g(\varphi)$ is harmonic in Γ (see [6, p. 36]). Let $\alpha = \beta + 2 - d$, so $M(x) = |x|^\alpha g(\varphi)$.

Lemma 2. *For each* $p > 0$ *there is a constant* C_p *such that if* h *is harmonic in* Γ_1 *and* $x \in \Gamma_1$,

$$E_x^h \tau^p < C_p. \tag{7}$$

Proof. That $\sup_{x,h} E_x^h \tau < \infty$ is a result of Cranston [2], and the argument that extends this to (7) is standard (see the end of the first section in [3]). □

Now we prove (3)–(5), starting with (4). Note that $\lambda = \max\{g(\varphi) : \varphi < \theta\} < \infty$ and $\eta = \min\{g(\varphi) : \varphi \le \theta/2\} > 0$. The fact that $1 = M(1) = EM(B_{\tau_n}) = EM(B_{\tau_n}) I(\tau_n < \tau)$, where I denotes indicator function, together with Lemma 1 and scaling, gives $cP_1(\tau_n < \tau)(2^n)^\alpha < 1 < CP_1(\tau_n < \tau)(2^n)^\alpha$. Clearly $P_x(\tau_{n+1} > \tau) > c$, $x \in S_n$, and this, together with the preceding inequalities, gives (4).

Next we prove (5). We have, by (6) with $h = K$, recalling that $g(1) = K(1) = 1$,

$$P_1^0(B_{\tau_{\Gamma_n}} \in S_n) \le \lambda(2^{-n})^\beta P_1(B_{\tau_{\Gamma_n}} \in S_n) \le C\lambda 2^{-n\beta} 2^{-n\alpha}, \tag{8}$$

where the last inequality follows from (4). Furthermore, again by (6) in the second inequality and Lemma 1 in the third,

$$\begin{aligned} P_1^0(B_{\tau_{\Gamma_n}} \in S_n) &\ge P_1^0(B_{\tau_{\Gamma_n}} \in H_n) \ge \eta P_1(B_{\tau_{\Gamma_n}} \in H_n)(2^{-n})^\beta \\ &\ge cP_1(B_{\tau_{\Gamma_n}} \in S_n) 2^{-n\beta} \ge c2^{-n\beta}2^{-n\alpha}. \end{aligned}$$

Together with (8), this proves (5).

Next we prove (3). Let $G_n = \{\tau_n < \tau \le \tau_{n+1}\}$. On G_n, $\tau = \tau_n + (\tau - \tau_n)$. That $E_1(\tau_n^p | \tau_n < \tau) < C_p 2^{2np}$ follows from Lemma 2, with $h = u$ and scaling, and since $P_x(\tau_{n+1} > \tau) > c$, $x \in S_n$, we have $P_x(G_n) > cP_x(\tau_n < \tau)$. Thus

$$E_1(\tau_n^p | G_n) < C_p 2^{2np}. \tag{9}$$

The inequality

$$E_1^0(\tau_n^p | \tau_n < \tau) < C_p 2^{2np} \tag{10}$$

follows from Lemma 2 with $h = u$ and scaling, recalling the first sentence after inequality (6). Now

$$P_x^0(\tau_{n+1} > \tau) > c, \qquad x \in H_n, \tag{11}$$

since $P_x^0(\tau_{n+1} > \tau)$ is a positive continuous function on H_n. That c may be chosen independently of n in (11) follows from scaling. Since

$$P_1^0(B_{\tau_n} \in H_n) > cP_1^0(B_{\tau_n} \in S_n), \tag{12}$$

by Lemma 1 and formula (6), we have from (11) that $P_1^0(G_n) > cP_1^0(\tau_n < \tau)$, and this together with (10), gives

$$E_1^0(\tau_n^p | G_n) < C_p 2^{2np}. \tag{13}$$

The inequalities

$$E_1^0((\tau - \tau_n)^p | G_n) < C_p 2^{2np} \tag{14}$$

and

$$E_1((\tau - \tau_n)^p | G_n) < C_p 2^{2np} \tag{15}$$

follow by very similar reasoning. We just prove (14). Now

$$\begin{aligned} E_1^0(\tau - \tau_n)^p I(G_n) &= E_1^0 E_1^0[(\tau - \tau_n)^p I(G_n) | B_{\tau_n}] \\ &= E_1^0 E_{B_{\tau_n}}^0 (\tau^p I(\tau < \tau_{n+1})) I(\tau_n < \tau) \\ &= E_1^0 E_{B_{\tau_n}}^{\xi, \Gamma_{n+1}} \tau^p P_{B_{\tau_n}}^0 (\tau < \tau_{n+1}) I(\tau_n < \tau) \\ &\le E_1^0 C_p 2^{2(n+1)p} P_{B_{\tau_n}}^0 (\tau < \tau_{n+1}) I(\tau_n < \tau) \\ &= C_p 2^{2(n+1)p} P_1^0(G_n), \end{aligned} \tag{16}$$

where the function ξ is the Poisson kernel for the point 0 for the domain Γ_{n+1}, and the inequality follows from Lemma 2 and scaling. We use (6) and the sentence after (6) to justify the third equality.

Now if λ is the distance of H_1 from $\partial\Gamma$, then $2^n\lambda$ is the distance from H_n to $\partial\Gamma$, and if $v_n = \inf\{t : |B_t - x| = 2^n\lambda\}$, we have, as in (16),

$$\begin{aligned} E_1^0 \tau^p I(G_n) &\ge E_1^0(\tau - \tau_n)^p I(G_n) \\ &= E_1^0 E_{B_{\tau_n}}^{\xi, \Gamma_{n+1}} \tau^p P_{B_{\tau_n}}^0 (\tau < \tau_{n+1}) I(\tau_n < \tau) \\ &\ge E_1^0 E_{B_{\tau_n}}^{\xi, \Gamma_{n+1}} v_n^p P_{B_{\tau_n}}^0 (\tau < \tau_{n+1}) I(\tau_n < \tau,\ B_{\tau_n} \in H_n) \\ &= E_1^0 C_p 2^{2np} P_{B_{\tau_n}}^0 (\tau < \tau_{n+1}) I(\tau_n < \tau,\ B_{\tau_n} \in H_n) \\ &= C_p 2^{2np} P_1^0(G_n,\ B_{\tau_n} \in H_n) > C_p 2^{2np} P_1^0(G_n), \end{aligned}$$

where we recall the second sentence after (6), and use scaling to obtain the next to the last inequality, and use (11) and (12) to prove the last inequality. Rephrased, this becomes

$$E_1^0(\tau^p | G_n) > C_p 2^{2np}, \tag{17}$$

and similarly we can prove

$$E_1(\tau^p | G_n) > C_p 2^{2np}. \tag{18}$$

Together, (9), (13)–(15), (17), and (18) establish (3), and thus Theorem 1, in the special case that $x = 1$ and $y = 0$, is proved.

Finally, we prove the general case. For $y \in \partial\Gamma$, that $E_x^y \tau^p$ is either finite for all $x \in \Gamma$ or infinite for all $x \in \Gamma$, follows from the well-known argument that shows the analogous result for $E_x \tau^p$, which we will not repeat. Since, if a is positive, the distribution of τ under P_{ax}^{ay} is the distribution of $a^2\tau$ under P_x^y, evidently $E_x^y \tau^p$ is either finite for all $x \in \Gamma$ and nonzero $y \in \partial\Gamma$ or infinite for all these x, y. To finish the proof, it suffices to show that there is just one $y \neq 0$, $y \in \partial\Gamma$, such that for all p, $E_1^0 \tau^p$ and $E_1^y \tau^p$ are finite for exactly the same values of p. Pick y such that $|y| \leq \frac{1}{2}$. Let K' be the Poisson kernel for Γ for the point y, normalized so that $K'(1) = 1$. Now it follows easily from Theorem 5.20 of Jerison and Kenig (1982) that $cK(x) < K'(x) < CK(x)$, $x \in \Gamma$, $|x| \geq 1$, and thus the proof of the E_1^0 case of Theorem 1 works essentially without change, to show that $E_1^y \tau^p$ is finite for the same p for which $E_1^0 \tau^p$ is finite. This finishes our proof of Theorem 1. $\square$

References

1. D. L. Burkholder, *Exit times of Brownian motion, harmonic majorization, and Hardy spaces*, Adv. Math. **26** (1977), 182–205.
2. M. Cranston, *Lifetime of conditioned Brownian motion in Lipschitz domains*, Z. Wahrsch. Verw. Gebiete **70** (1985), 335–340.
3. B. Davis, *Conditioned Brownian motion in planar domains*, Duke Math. J. **57** (1988), 397–421.
4. R. D. DeBlassie, *Exit times from cones in* $\mathbb{R}^n$ *of Brownian motion*, Probab. Theory Related Fields **74** (1987), 1–29.
5. J. L. Doob, *Classical potential theory and its probabilistic counterpart*, Springer, New York, 1984.
6. L. L. Helms, *Introduction to potential theory*, Wiley, New York, 1969.
7. D. S. Jerison and C. E. Kenig, *Boundary value problems on Lipschitz domain*, Partial Differential Equations (Walter Littman, ed.), MAA Stud. Math., vol. 23, Math. Assoc. Amer., Washington, DC, 1982, pp. 1–68.

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