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ON INDUCED CHARACTERS

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ABSTRACT. Suppose that H is a normal subgroup of a finite group G, $\varphi \in \operatorname{Irr}(H)$, and $\operatorname{Irr}(\varphi^G)$ is the set of all irreducible constituents of the induced character φ^G. If $|\operatorname{Irr}(\varphi^G)| > |G:H|/4$ then G/H is solvable.

If τ is a character of a group, then by $\operatorname{Irr}(\tau)$ we denote the set of all irreducible constituents of τ. Set $s(\tau) = |\operatorname{Irr}(\tau)|$ and $w(\tau) = \sum\langle\tau, \chi\rangle$, where χ runs over the set $\operatorname{Irr}(G)$ of all irreducible characters of G. Obviously $s(\tau) \leq w(\tau)$.

In this note we prove the following

Theorem. *Suppose that H is a proper normal subgroup of a finite group G, p is the smallest prime dividing $|G:H|$, and φ is an irreducible character of H.*

(a) *If $s(\varphi^G) \geq |G:H|/p^2$ then G/H is solvable unless φ is G-invariant, $\varphi^G = p(\chi^1 + \cdots + \chi^s)$, where $\operatorname{Irr}(\varphi^G) = \{\chi^1, \ldots, \chi^s\}$, $s(\varphi^G) = |G:H|/p^2$.*

(b) *If $w(\varphi^G) \geq |G:H|/p$ then G/H is solvable unless the same exception holds as in* (a).

We consider only finite groups.

A group G is said to be p-nilpotent (p is always a prime) if it has a normal p-complement. A group G is said to be dispersive if its arbitrary subgroup A is p-nilpotent for the smallest prime p dividing $|A|$.

We fix the following notation. Let $\operatorname{Irr}(G) = \{\chi^1, \ldots, \chi^k\}$, where $k = k(G)$ is the class number of G. The number $\operatorname{mc}(G) = k(G)/|G|$ is called the measure of commutativity of G. Obviously $0 < \operatorname{mc}(G) \leq 1$ and $\operatorname{mc}(G) = 1$ iff G is abelian. Denote by $T(G)$ the sum of degrees of all irreducible characters of G and set $f(G) = T(G)/|G|$. Note that G is abelian iff $f(G) = 1$.

Lemma 1 [7]. *Let H be a subgroup of a group G. Then*:

(a) $\operatorname{mc}(H) \geq \operatorname{mc}(G)$.
(b) $f(H) \geq f(G)$.
(c) *If H is normal in G then* $\operatorname{mc}(G/H) \geq \operatorname{mc}(G)$.
(d) $\operatorname{mc}(G) \geq f(G)^2$; $\operatorname{mc}(G) = f(G)^2$ *iff G is abelian.*

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Proof. (a) By reciprocity

$$k(G) = |\operatorname{Irr}(G)| \leq \sum_{\psi\in\operatorname{Irr}(H)} |\operatorname{Irr}(\psi^G)| \leq |G:H|\,|\operatorname{Irr}(H)| = |G:H|k(H)$$

and

$$\operatorname{mc}(G) = k(G)/|G| \leq k(H)/|H| = \operatorname{mc}(H).$$

(b) By reciprocity

$$T(G) = \sum_{\chi\in\operatorname{Irr}(G)} \chi(1) \leq \sum_{\psi\in\operatorname{Irr}(H)} \psi^G(1) = |G:H| \sum_{\psi\in\operatorname{Irr}(H)} \psi(1) = |G:H|T(H)$$

and

$$f(G) = T(G)/|G| \leq T(H)/|H| = f(H).$$

(c) Let $k_G(M)$ denote the number of G-classes (= classes of conjugate elements of G) having nonempty intersections with the subset M of G. For $x \in G$ denote by $K(x)$ the G-class containing x. Obviously $k_G(K(x)H) = k_G(xH) \leq |H|$. Obviously there exists a subset $\mathfrak{M} \subseteq G$ such that

$$G = \sum_{x\in\mathfrak{M}} K(x)H$$

is a partition. Since $|\mathfrak{M}| = k(G/H)$ then

$$k(G) = \sum_{x\in\mathfrak{M}} k_G(K(x)H) \leq |\mathfrak{M}||H| = k(G/H)|H|$$

so that

$$\operatorname{mc}(G) = k(G)/|G| \leq k(G/H)|H|/|G| = \operatorname{mc}(G/H).$$

(d) (Mann) Consider two k-dimensional vectors $(k = k(G))$

$$\mathbf{a} = (\chi^1(1), \dots, \chi^k(1)), \qquad \mathbf{b} = (1, \dots, 1).$$

Then by the Cauchy-Schwartz inequality

$$(|G|f(G))^2 = T(G)^2 = (\mathbf{a}\cdot\mathbf{b})^2 \leq ||\mathbf{a}||||\mathbf{b}|| = |G|k(G) = |G|^2\operatorname{mc}(G)$$

and our inequality follows. If $f(G)^2 = \operatorname{mc}(G)$ then vectors $\mathbf{a}$ and $\mathbf{b}$ are linearly dependent. In this case $\chi^1(1) = \cdots = \chi^k(1) = 1$ and G is abelian. □

We note that Lemma 1(c) is a consequence of the inequality

$$k(G) \leq k(G/H)k(H)$$

which is due to Gallagher. For the other proof of Lemma 1(d) see [7].

Lemma 2. *Suppose that $G = RG'$ is a Frobenius group with the kernel G' and a complementary factor R, $|R| = q$ is a prime, G', the commutator subgroup of G, is an elementary abelian group of order p^b, and p is a prime.*

(a) *If $r = \min\{p, q\}$ then $\operatorname{mc}(G) < (r+1)/r^2$.*

(b) *If $p < q$ then $\operatorname{mc}(G) < 1/p^2$ unless $G \cong A_4$, the alternating group of degree 4.*

Proof. One has
$$\mathrm{mc}(G) = q^{-2}p^{-b}(p^b - 1 + q^2).$$

(a) Suppose that $\mathrm{mc}(G) \geq (r+1)/r^2$.

(1a) Let $p < q$. Then $b > 1$ and
$$p^b - 1 + q^2 \geq q^2 p^{b-2}(p+1) = q^2 p^{b-1} + q^2 p^{b-2}$$
which is impossible.

(2a) Let $q < p$. Then
$$p^b - 1 + q^2 \geq (q+1)p^b = qp^b + p^b,$$
$$q^2 - 1 \geq qp^b \geq q(q+1) = q^2 + q,$$
a contradiction, and (a) is proved.

(b) Suppose that $\mathrm{mc}(G) \geq 1/p^2$. Then
$$p^b - 1 + q^2 \geq q^2 p^{b-2}.$$
If $b = 2$ then $p = 2$, $q = 3$, and $G \cong A_4$. Let $b > 2$. Then
$$(p^b - 1)/(p^{b-2} - 1) \geq q^2 \geq (p+1)^2,$$
$$p^b - 1 \geq (p^{b-2} - 1)(p+1)^2 = p^{b-2}(p+1)^2 - (p+1)^2$$
$$= p^b + 2p^{b-1} + p^{b-2} - (p+1)^2,$$
$$(p+1)^2 \geq 2p^{b-1} + p^{b-2} + 1 \geq 2p^2 + p + 1,$$
a contradiction. □

Lemma 3. *Suppose that* p *is the smallest prime divisor of the order of a group* G.

(a) *If* $\mathrm{mc}(G) \geq (p+1)/p^2$ *then* G *is abelian.*
(b) *If* $\mathrm{mc}(G) \geq 1/p^2$ *then* G *is solvable, and* G *is dispersive if* $p > 2$.
(c) *If* $f(G) \geq 1/p$ *then* G *is dispersive unless* $|G'| \in \{2^2, 2^3\}$.

Proof. (a) Suppose that G is a counterexample of minimal order. Then (Lemma 1(a), (c)) G is a minimal nonabelian group. By the Miller-Moreno Theorem [6] one of the following assertions holds:

(i) $|G| = p^n$, $|G'| = p$, $|G : Z(G)| = p^2$.
(ii) $G = QG'$, a semidirect product of Q, $G' \in \mathrm{Syl}(G)$, G' is elementary abelian.

If (i) holds one obtains
$$\mathrm{mc}(G) = p^{-n}k(G) = p^{-n}(p^{n-2} + p^{n-1} - p^{n-3})$$
$$= p^{-2}(p+1) - p^{-3} < p^{-2}(p+1),$$
a contradiction.

If (ii) holds one obtains $\mathrm{mc}(G) \leq \mathrm{mc}(G/Z(G))$, and a contradiction follows from Lemma 2(a).

(b) At first we prove that G is solvable. Suppose that G is a counterexample of minimal order. Then all proper subgroups and epimorphic images of G are solvable, but G is nonsolvable (Lemma 1(a), (c)). So G is a nonabelian simple group. Suppose that $\chi^1(1) \leq \cdots \leq \chi^k(1)$. Then $\chi^i(1) \geq p$ for $i > 1$ and $\chi^k(1) \geq p+1$ [5, Theorem 6.9]. Hence

$$\begin{aligned}|G| &= 1 + \sum_{i=2}^{k-1} \chi^i(1)^2 + \chi^k(1)^2 \geq 1 + (k-2)p^2 + (p+1)^2 \\ &= (k-1)p^2 + 2p + 2 \geq (|G|/p^2 - 1)p^2 + 2p + 2 \\ &= |G| + 2p + 2 - p^2.\end{aligned}$$

Hence if $p = 2$ then we have $|G| \geq |G|+2$, a contradiction. Suppose that $p > 2$ and prove that G is dispersive. In view of Lemma 1(a) it is sufficient to prove that G is p-nilpotent. Suppose that G is a counterexample of minimal order. Then G is a minimal nonnilpotent group with a normal Sylow p-subgroup [4, Satz 4.5.4]. Then (Lemmas 1(c) and 2(b)) one has

$$\mathrm{mc}(G) \leq \mathrm{mc}(G/Z(G)) < 1/p^2,$$

a contradiction.

(c) is proved in [7]. □

Proof of the Theorem. (a) At first suppose that $H = 1$. Without loss of generality we may assume that $\varphi = 1_H$. Then $\varphi^G = \rho_G$, the regular character of G, $\mathrm{Irr}(\varphi^G) = \mathrm{Irr}(G) = \{\chi^1, \dots, \chi^k\}$, where $k = k(G)$, the class number of G. In our case $s(\varphi^G) = k(G)$. Therefore by the condition $k(G) \geq |G|/p^2$, $\mathrm{mc}(G) \geq 1/p^2$, and G is solvable (Lemma 3(b)).

Suppose that $H > 1$. Let

$$\mathrm{Irr}(\varphi^G) = \{\chi^1, \dots, \chi^s\} \quad \text{and} \quad \varphi^G = e_1\chi^1 + \cdots + e_s\chi^s.$$

If

$$\chi^i_H = e_i(\varphi_1 + \cdots + \varphi_t)$$

is the Clifford decomposition, $\varphi_1 = \varphi$, $t = |G : I_G(\varphi)|$, where $I_G(\varphi)$ is the inertia group of φ in G, then $\chi^i(1) = e_i t\varphi(1)$ for all i and

$$\begin{aligned}|G : H|\varphi(1) = \varphi^G(1) &= t\varphi(1)(e_1^2 + \cdots + e_s^2), \\ |I_G(\varphi) : H| &= e_1^2 + \cdots + e_s^2.\end{aligned}$$

Since $s = s(\varphi^G) \geq |G : H|/p^2$, we have

$$(*) \qquad |G : H| = t(e_1^2 + \cdots + e_s^2) \geq ts \geq t|G : H|/p^2;$$

then $t \leq p^2$. Since t is a divisor of $|G : H|$ then $t \in \{1, p^2, q\}$, where q is a prime (we recall that p is the smallest prime divisor of $|G : H|$).

(i) Suppose that $t = p^2$. Then $e_1 = \cdots = e_s = 1$ by $(*)$. By [5, Theorem 6.11] we have

$$\mathrm{Irr}(\varphi^{I_G(\varphi)}) = \{\psi_1, \dots, \psi_s\},$$

$$(**) \qquad \varphi^{I_G(\varphi)} = e_1\psi_1 + \cdots + e_s\psi_s = \psi_1 + \cdots + \psi_s.$$

Then by reciprocity $(\psi_1)_H = \varphi$ and (by Gallagher's Theorem [5, Corollary 6.17]) $|\mathrm{Irr}(I_G(\varphi)/H)| = s$, $\mathrm{Irr}(I_G(\varphi)/H) = \{\beta_1, \dots, \beta_s\}$, $\beta_i(1) = e_i$, and then $\psi_i = \psi_1\beta_i$ (after possible reordering) for $i = 1, \dots, s$. Now

$$\mathrm{mc}(I_G(\varphi)/H) = s|I_G(\varphi)/H|^{-1} \geq (|G : H|/p^2)(|G : H|/t)^{-1} = t/p^2 = 1,$$

and $I_G(\varphi)/H$ is abelian. Then G/H is solvable as a product of $I_G(\varphi)/H$ and the Sylow p-subgroup of G/H (see, for example, [4, Satz 6.4.11]).

(ii) Suppose that t is a prime. Then $I_G(\varphi)/H$ is maximal in G/H. By [5, Theorem 6.11] the equalities $(**)$ are true.

Suppose that all $e_i > 1$. Then all $e_i \geq p$ since $e_1, \dots, e_s$ as degrees of irreducible projective representations of $I_G(\varphi)/H$ are divisors of $|I_G(\varphi)/H|$. Hence

$$\begin{aligned}|I_G(\varphi)/H| &= e_1^2 + \cdots + e_s^2 \geq p^2 s \geq p^2(|G:H|/p^2)\\ &= |G:H| = t|I_G(\varphi)/H| > |I_G(\varphi)/H|,\end{aligned}$$

a contradiction. Supposing $e_1 \leq \cdots \leq e_s$, we have $e_1 = 1$. The $(\psi_1)_H = \varphi$ and, as in (i), using [5, Corollary 6.17], we obtain $\mathrm{mc}(I_G(\varphi)/H) \geq t/p^2$. Then $I_G(\varphi)/H$ is solvable by Lemma 3(b).

If $t = p$ then $I_G(\varphi)/H$ is normal in G/H, and $G/I_G(\varphi)$ is cyclic of order p. Therefore G/H is solvable in this case.

Suppose that $t > p$. Then

$$\mathrm{mc}(I_G(\varphi)/H) \geq (p+1)/p^2$$

and $I_G(\varphi)/H$ is abelian by Lemma 3(a), so that G/H is solvable by Herstein's Theorem [3].

(iii) Suppose that $t = 1$. Then φ is G-invariant. If $e_i = 1$ for some $i \in \{1, \dots, s\}$ then as above G/H is solvable. Suppose that all $e_i > 1$. Then all $e_i \geq p$ and

$$|G:H| = e_1^2 + \cdots + e_s^2 \geq sp^2 \geq (|G:H|/p^2)p^2 = |G:H|.$$

Hence $s = |G:H|/p^2$, $e_1 = \cdots = e_s = p$, and assertion (a) is proved.

(b) Suppose that $H = 1$. Without loss of generality we may assume that $\varphi = 1_H$. Then $\varphi^G = \rho_G$, the regular character of G, and $|G|/p \leq w(\varphi^G) = T(G) = |G|f(G)$ and G is solvable by Lemma 3(c).

Let $H > 1$. Let as before

$$\mathrm{Irr}(\varphi^G) = \{\chi^1, \dots, \chi^s\}, \qquad \varphi^G = e_1\chi^1 + \cdots + e_s\chi^s.$$

Then

$$w(\varphi^G) = e_1 + \cdots + e_s \geq |G:H|/p.$$

As before one has

$$|G:H| = t(e_1^2 + \cdots + e_s^2), \qquad t = |G:I_G(\varphi)|.$$

Therefore

$$\begin{aligned}|I_G(\varphi):H| &= e_1^2 + \cdots + e_s^2 \geq e_1 + \cdots + e_s\\ &= w(\varphi^G) \geq |G:H|/p = t|I_G(\varphi):H|/p \Rightarrow t \leq p.\end{aligned}$$

So $I_G(\varphi)$ is normal in G and $G/I_G(\varphi)$ is cyclic.

Suppose that $e_i = 1$ for some $i \in \{1, \dots, s\}$. Then as in (a) one has

$$\begin{aligned}|I_G(\varphi)/H|f(I_G(\varphi)/H) &= T(I_G(\varphi)/H) = e_1 + \cdots + e_s = w(\varphi^G) \geq |G:H|/p\\ &= t|I_G(\varphi)/H)|/p \Rightarrow f(I_G(\varphi)/H) \geq 1/p\end{aligned}$$

and $I_G(\varphi)/H$ is solvable (Lemma 3(c)). Since $G/I_G(\varphi)$ is cyclic then G/H is solvable.

Suppose that all $e_i > 1$. Then $e_i \geq p$ for all i. Therefore

$$|I_G(\varphi) : H| = e_1^2 + \cdots + e_s^2 \geq p(e_1 + \cdots + e_s) = pw(\varphi^G) \geq p(|G : H|/p)$$
$$= |G : H| \Rightarrow I_G(\varphi) = G, \qquad e_1 = \cdots = e_s = p. \quad \square$$

Remark. If, in the Theorem, φ is reducible then G/H is solvable unless for any $\lambda \in \operatorname{Irr}(\varphi)$ one has $\lambda^G = p(\chi^1 + \cdots + \chi^s)$, $s = |G : H|/p^2$, and $\operatorname{Irr}(\lambda^G) = \{\chi^1, \dots, \chi^s\}$.

Corollary. *Suppose that H is a proper normal subgroup of a group G, and p is the smallest prime dividing $|G : H|$.*

(a) *If $w(\varphi^G) \geq |G : H|/p$ for all nonlinear $\varphi \in \operatorname{Irr}(H)$, then G/H is solvable or H' has a normal p-complement.*
(b) *If $s(\varphi^G) \geq |G : H|/p^2$ for all nonlinear $\varphi \in \operatorname{Irr}(H)$, then the same conclusion as in* (a) *holds.*

Proof. Suppose that G/H is nonsolvable. We may assume that H is nonabelian (so that $\operatorname{Irr}(H)$ contains a nonlinear character). Then for any nonlinear $\varphi \in \operatorname{Irr}(H)$ we have (by the Theorem)

$$\varphi^G = p(\chi^1 + \cdots + \chi^s), \qquad \operatorname{Irr}(\varphi^G) = \{\chi^1, \dots, \chi^s\}.$$

By reciprocity p divides degrees of all irreducible constituents of φ^G. Let

$$\operatorname{Irr}(G, p') = \{\chi \in \operatorname{Irr}(G) | \chi(1) > 1 \text{ and } p \text{ does not divide } \chi(1)\}$$

and let $G(p')$ be the intersection of kernels of all characters belonging to $\operatorname{Irr}(G, p')$. The subgroup $G(p')$ is p-nilpotent [2]. Take $\chi \in \operatorname{Irr}(G, p')$. Then by the above all irreducible constituents of χ_H are linear so that $H' \leq \ker \chi$. Therefore $H' \leq G(p')$ and H' is p-nilpotent. $\square$

Remarks. 1. We note the crucial role of Gallagher's Theorem [5, Corollary 6.17] in the proof of the Theorem. Note that the assertion converse to Gallagher's Theorem is also true. Namely, if N is a normal subgroup of G and $\chi \in \operatorname{Irr}(G)$ then $\chi\theta \in \operatorname{Irr}(G)$ for all $\theta \in \operatorname{Irr}(G/N)$ implies $\chi_N \in \operatorname{Irr}(N)$. We prove this assertion. Take $\lambda \in \operatorname{Irr}(\chi_N)$. It is sufficient to prove that $\lambda(1) = \chi(1)$. Take $\psi \in \operatorname{Irr}(G/N)$. Then

$$\langle \psi\chi, \lambda^G \rangle = \langle (\psi\chi)_N, \lambda \rangle = \psi(1)\langle \chi_N, \lambda \rangle$$

so that $\psi(1)\psi\chi$ is a constituent of λ^G. Now

$$(\chi_N)^G = (\chi_N \cdot 1_N)^G = \chi\rho_{G/N},$$

where $\rho_{G/N}$ is the regular character of G/N. Put $\operatorname{Irr}(G/N) = \{\theta_1, \dots, \theta_n\}$. Then

$$\chi\rho_{G/N} = \sum_{i=1}^{n} \theta_i(1)\theta_i\chi$$

by the above is a constituent of λ^G. Since

$$(\chi\rho_{G/N})(1) = |G : N|\chi(1) \geq |G : N|\lambda(1) = \lambda^G(1)$$

then $\chi\rho_{G/N} = \lambda^G$, $\lambda(1) = \chi(1)$, and $\lambda = \chi_N$. Therefore $\chi_N \in \operatorname{Irr}(N)$ and our assertion is proved.

2. If, in the Theorem,

$$s(\varphi^G) \geq (p+1)|G:H|/p^2$$

then G/H is abelian. In particular if $s(\varphi^G) = |G:H|$ then G/H is abelian.

3. Suppose that $H < G$, $\varphi \in \mathrm{Irr}(H)$, and $\chi \in \mathrm{Irr}(G)$. Then

$$w(\varphi^G) > |G:H|/2 \Rightarrow \min\{\langle \varphi^G, \tau\rangle | \tau \in \mathrm{Irr}(\varphi^G)\} = 1,$$
$$w(\chi_H) > |G:H|/2 \Rightarrow \min\{\langle \chi_H, \psi\rangle | \psi \in \mathrm{Irr}(\chi_H)\} = 1.$$

Analogous results hold for $s(\varphi^G)$ and $s(\chi_H)$.

4. Let $H < G$. If $s(\varphi^G) = |G:H|$ (or $w(\varphi^G) = |G:H|$) for all nonprincipal $\varphi \in \mathrm{Irr}(H)$ then H is normal in G. We prove the first part of this assertion. If $\varphi \in \mathrm{Irr}(H)$ and $s(\varphi^G) = |G:H|$ then degrees of all irreducible constituents of φ^G are equal to $\varphi(1)$. Take $\chi \in \mathrm{Irr}((1_H)^G)$ and suppose that $|\mathrm{Irr}(\chi_H)| > 1$. Take $\lambda \in \mathrm{Irr}(\chi_H) - \{1_H\}$. Since $s(\lambda^G) = |G:H|$ then by the above $\lambda(1) = \chi(1)$, a contradiction since $\lambda \in \mathrm{Irr}(\chi_H - 1_H)$. So all irreducible constituents of $(1_H)^G$ are linear and H is normal in G.

5. If $\mathrm{mc}(G) > 1/12$ then G is solvable [1]. So (Lemma 1(d)) if $f(G)^2 > 1/12$ then G is solvable.

Conjectures. Suppose that $H < G$.

1. If $s(\varphi^G) > |G:H|/4$ for all $\varphi \in \mathrm{Irr}(H)$ and $|G:H|$ is sufficiently large, then H is normal in G.
2. If $w(\varphi^G) > |G:H|/2$ for all $\varphi \in \mathrm{Irr}(H)$ and $|G:H|$ is sufficiently large, then H is normal in G.

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