CENTRAL SEQUENCES IN SUBFACTORS II

DIETMAR BISCH

Abstract. We studied in On the existence of central sequences in subfactors [Trans. Amer. Math. Soc. 321 (1990), 117-128] certain ergodicity properties of inclusions of $\text{II}_1$ factors $N \subset M$. We give here various explicit examples of pairs of $\text{II}_1$ factors $N \subset M$ which have or do not have these properties. In particular, we show that if $N \subset M$ are hyperfinite $\text{II}_1$ factors with finite Jones's index, then both situations may occur.

1. INTRODUCTION

We studied in [Bil] certain asymptotic commutativity properties of inclusions of $\text{II}_1$ factors $N \subset M$. Namely, we gave a list of equivalent conditions on the pair $N \subset M$ that assure that the subfactor $N$ contains nontrivial central sequences for the ambient algebra $M$. Let us recall the definition:

Definition 1.1. Let $N \subset M$ be (separable) $\text{II}_1$ factors with faithful normal normalized trace $\tau$. We say that $N \subset M$ has property $\Gamma$ if given any $\epsilon > 0$ and any operators $x_1, \ldots, x_n \in M$, there is a unitary $u \in N$, $\tau(u) = 0$, such that $\|[x_i, u]\|_2 \leq \epsilon$, $1 \leq i \leq n$.

The study of central sequences of $M$ contained in the subfactor $N$ is closely related to deciding whether a given subfactor of the hyperfinite $\text{II}_1$ factor is amenable [Po3, Po7], i.e., classified by its higher relative commutant invariant, or not. The existence of nontrivial central sequences for $M$ contained in the subfactor $N$ is a necessary condition for the amenability of $N$ in the sense of Popa [Po7]. However, as we shall see below, there are many examples of non-amenable subfactors of the hyperfinite $\text{II}_1$ factor $R$ that do contain nontrivial central sequences for the ambient factor.

If $\omega$ is a free ultrafilter of $\mathbb{N}$, then $N \subset M$ has property $\Gamma$ if and only if $M' \cap N^\omega \neq C$. We proved in [Bi1] that in this case the algebra $M' \cap N^\omega$ contains a diffuse abelian subalgebra.

In this note we will give examples of inclusions of $\text{II}_1$ factors $N \subset M$ such that $N$ and $M$ have both property $\Gamma$ in the sense of Murray and von Neumann ([MvN]), however, the inclusion $N \subset M$ does not have property $\Gamma$ (Definition 1.1). In particular, we will show that if $N$ and $M$ are the hyperfinite $\text{II}_1$ factor...
with finite Jones’s index \([M : N]\) ([Jo1]), then \(N \subset M\) may or may not have property \(\Gamma\). We give explicit examples for both cases.

2. Construction of examples

We showed in [Bi1] that if \(N\) is a II\(_1\) factor which has property \(\Gamma\) and \(G\) is an amenable group acting freely on \(N\), then the inclusion \(N \subset N \rtimes G\) has property \(\Gamma\). Note that if \(N\) is the hyperfinite II\(_1\) factor, we can make any countable discrete group \(G\) act on \(N\) in this way (via the Bernoulli shift action). Thus taking \(G\) to be a finite group or an infinite amenable group, for instance \(G = S\infty\), the group of finite permutations of an infinite countable set, we get examples of inclusions of property \(\Gamma\) factors \(N, M\) of finite index \((G\) finite) or infinite index \((G\) infinite), resp., such that \(N \subset M\) also has property \(\Gamma\).

Another class of examples comes from strongly amenable inclusions of hyperfinite II\(_1\) factors \(N \subset M\) with finite Jones’s index \([M : N]\) [Po3, Po7], in particular, finite depth subfactors [Oc2, Po2] (see also [Ka1, Ha, HSch, Oc3, We1, We2]) of \(R\), the hyperfinite II\(_1\) factor. In this case \(N \subset M\) is completely classified by the associated sequence of higher relative commutants \(\{N_k \cap N \subset N_k' \cap M\}_{k \in \mathbb{N}}\) since \(N < M \cong \bigcup N_k \cap N^\omega \subset \bigcup N_k \cap M^\omega\). But the pair \(N \otimes R \subset M \otimes R\) enjoys the same properties (it has the same higher relative commutants classifying it), in other words, \(N \subset M\) is stable, i.e., \(N \subset M \cong N \otimes R \subset M \otimes R\). In [Bi1] we showed that this is equivalent to the fact that the algebra \(M' \cap N^\omega\) is noncommutative, in particular, \(N \subset M\) has property \(\Gamma\).

The standard technique of constructing subfactors of the hyperfinite II\(_1\) factor is as follows: construct an inclusion of four finite-dimensional \(C^*\)-algebras

\[
B_0 \subset B_1 \\
A_0 \subset A_1,
\]

(satisfying the commuting square condition [PiPo, Po2, HSch, GHJ, We1], i.e., \(E_{B_0} E_{A_1} = E_{A_0}\), where \(E_{B_i}\) and \(E_{A_i}\) denote the trace-preserving conditional expectation onto \(B_i\), resp., \(A_i\).) Let \(B_0 \subset B_1 \subset B_2 := \langle B_1, e_{B_0}\rangle\) be the Jones basic construction [Jo1], and suppose the initial commuting square \((*)\) is symmetric [HSch] or nondegenerate [Po7], i.e., \(A_0 \subset A_1 \subset \langle A_1, e_{B_0}\rangle\) is isomorphic to the basic construction for \(A_0 \subset A_1\). Iterating the basic construction gives then a sequence of commuting squares

\[
B_n \subset B_{n+1} \\
A_n \subset A_{n+1},
\]

and one obtains an inclusion of hyperfinite II\(_1\) factors with finite Jones’ index in the usual way as \(N := \bigcup_n A_n^{\omega} \subset M := \bigcup_n B_n^{\omega}\) by performing the GNS construction with respect to the Markov trace on \(\bigcup_n B_n\). The inclusions \(N \subset M\) constructed in this way have property \(\Gamma\) (Definition 1.1) since the Jones projections \(\{e_{B_n}\}\) form a nontrivial central sequence for \(M\) contained in \(N\). Taking the noncommuting central sequences \(\{e_{B_n}\}\) and \(\{e_{B_{n+1}}\}\) one sees that these inclusions actually have the relative McDuff property [Bi1]. In particular, all the
examples of irreducible subfactors of the hyperfinite II\textsubscript{1} factor with index > 4 constructed by Haagerup and Schou [HSch] have property Γ in the sense of Definition 1.1 (or stronger, the relative McDuff property in the sense of [Bi1]). Note that Haagerup proved in recent work [Ha] that all infinite depth subfactors of \( R \) with index between 4 and \( 3 + \sqrt{3} \) have principal graph \( A_\infty \) and are therefore nonamenable in the sense of [Po7]. As pointed out by Popa, it would be interesting to study centrally trivial automorphisms and compute the analogue of Connes’s invariant \( \chi(M) \) as introduced in [Ka2] for these subfactors, since this invariant may lead to a distinction of nonamenable subfactors of the hyperfinite II\textsubscript{1} factor with same higher relative commutant invariant.

Now let us give an example of a subfactor of the hyperfinite II\textsubscript{1} factor \( R \) of infinite index that does not contain nontrivial central sequences for the ambient algebra [Po4].

**Lemma 2.1.** Let \( M \) be a finite von Neumann algebra, \( N \subset M \) a subalgebra, and suppose there is a unitary \( u \in M \) such that \( uNu^* \perp N \). Then \( N \) does not contain nontrivial central sequences for \( M \); in other words, \( N \subset M \) does not have property Γ.

**Proof (see [Po4]).** Let \((x_n)_{n \in \mathbb{N}} \subset N\) be a central sequence for \( M \). Replacing \( x_n \) by \( x_n - \tau(x_n) \) we may assume that \( \tau(x_n) = 0 \). Then \( \|ux_nu^* - x_n\|_2 \to 0 \) as \( n \to \infty \). We compute then \( \|ux_nu^* - x_n\|_2^2 = 2\|x_n\|_2^2 \), which implies that \((x_n)_{n \in \mathbb{N}}\) is trivial. \( \Box \)

Now it is not difficult to construct a subfactor of \( R \) satisfying the conditions of Lemma 2.1. All we need to do is to take an amenable I.C.C. (infinite conjugacy classes) group \( G \) and an I.C.C. subgroup \( H \) such that \( gHg^{-1} \cap H = \{e\} \) for some \( g \in G \). Then \( \lambda(g)L(H)\lambda(g)^{-1} \perp L(H) \) (\( \lambda \) denotes as usual the left regular representation of \( G \)). Hence the group von Neumann algebra \( L(H) \subset L(G) \) contains only trivial central sequences for \( L(G) \); i.e., \( L(H) \subset L(G) \) does not have property Γ. We give explicit examples of such groups (see [Po4]). Consider the group \( H \) of affine transformations over \( \mathbb{Q} \), which is amenable and I.C.C. Let \( S_\infty H \) be the group of all permutations of \( H \) and \( SH \) the subgroup of finite permutations. Identify \( H \) with the subgroup of translations in \( S_\infty H \), and let \( G \) be the group generated by \( SH \) and \( H \) in \( S_\infty H \). Then \( G/SH \cong H \). Since \( SH \) is amenable, this implies that \( G \) is amenable, and it is easy to see that \( gHg^{-1} \cap H = \{e\} \) for all \( g \in G \setminus H \). This implies that \( G \) is I.C.C.

A different class of examples arises from the subfactors \( N_G \subset M \) as introduced in [Po1] and studied in [Bi2]. We recall the construction. Let \( R \) be the hyperfinite II\textsubscript{1} factor and \( G \) be a finitely generated countable discrete group acting on \( R \) via \( \sigma \), and fix a generating set \( \{g_1, \ldots, g_n\} \subset G \). Take automorphisms \( \theta_i \in \text{Aut } R \), \( 0 \leq i \leq 2n \), such that \( \theta_0 = \text{id} \), \( \theta_i = \sigma_{g_i} \mod \text{Int } R \), \( 1 \leq i \leq n \), and \( \theta_i = \sigma_{g_i^{-1}} \mod \text{Int } R \), \( n + 1 \leq i \leq 2n \). Set

\[
N_G = \left\{ \sum_{i=0}^{2n} \theta_i(x)e_{ii} \mid x \in R \right\} \subset M = M_{2n+1 \times 2n+1}(R) = R \otimes M_{2n+1 \times 2n+1}(\mathbb{C})
\]

where \( \{e_{ij}\}_{0 \leq i, j \leq 2n} \) denote the usual matrix units in \( M_{2n+1 \times 2n+1}(\mathbb{C}) \). Note that \( N_G \subset M \) is an inclusion of hyperfinite II\textsubscript{1} factors of index \( [M : N_G] = (2n+1)^2 \) and nontrivial relative commutant \( N_G' \cap M \neq \mathbb{C} \). We prove
Theorem 2.2. Let \( G, \sigma, N_G, \) and \( M \) be as above. Then \( N_G \subset M \) has property \( \Gamma \) if and only if \( R \subset R \rtimes_\sigma G \) has property \( \Gamma \).

The proof uses the following three lemmas.

Lemma 2.3. \( N_G \subset M \) has property \( \Gamma \) if and only if given any \( \epsilon > 0 \) and \( x_1, \ldots, x_m \in R \), there is a unitary \( u \in R \), \( \tau(u) = 0 \), such that

\[ \|ux_r - x_r\theta_i^{-1}\theta_j(u)\|_2 \leq \epsilon, \]

for all \( 1 \leq r \leq m \), \( 0 \leq i, j \leq 2n \).

Proof. The proof is straightforward and left as an exercise to the reader. \( \square \)

Denote by \( \{u_g\}_{g \in G} \) the unitaries that implement the action \( \sigma \) of \( G \) on \( R \), i.e., \( u_gxu_g^* = \sigma_g(x), x \in R \). Since the \( * \)-algebra \( \{\sum_{\text{finite}} xgu_g | x_g \in R\} \) is \( \| \cdot \|_2 \) dense in \( R \rtimes_\sigma G \), the following lemma is trivial.

Lemma 2.4. \( R \subset R \rtimes_\sigma G \) has property \( \Gamma \) if and only if given any \( \epsilon > 0 \), \( x_1, \ldots, x_m \in R \), and \( u_{h_1}, \ldots, u_{h_i} \in R \rtimes_\sigma G \), there is a unitary \( u \in R \), \( \tau(u) = 0 \), such that

\[ \|x_iu - ux_i\|_2 \leq \epsilon, \quad 1 \leq i \leq m, \quad \|\sigma_{h_i}(u) - u\|_2 \leq \epsilon, \quad 1 \leq i \leq l. \]

We need another lemma.

Lemma 2.5. Let \( G \) be generated by \( \{g_i, 1 \leq i \leq 2n\} \), where \( g_{n+i} = g_i^{-1}, 1 \leq i \leq n \), and let \( g_0 = e \), the identity of \( G \). Fix some \( h \in G \), and let \( h = g_{i_1} \cdots g_{i_r} \) be some expression of \( h \) in terms of the \( g_i \)'s. Then \( \|\sigma_{g_i}(x) - x\|_2 \leq \epsilon, 0 \leq i \leq 2n \), implies that \( \|\sigma_h(x) - x\|_2 \leq r\epsilon \).

Proof. Use induction on the number of generators that appear in some expression of the element \( h \in G \). \( \square \)

Proof of Theorem 2.2. First note that there are unitaries \( u_i \in R \) with \( \sigma_{g_i}(x) = u_i\theta_i(x)u_i^* \), \( x \in R \), \( 0 \leq i \leq 2n \).

Suppose now that \( N_G \subset M \) has property \( \Gamma \), and let \( \epsilon > 0 \), \( x_1, \ldots, x_m \in R \) be given. By Lemma 2.3 there is a unitary \( u \in R \), \( \tau(u) = 0 \), such that

\[ \|ux_r - x_r\theta_i^{-1}\theta_j(u)\|_2 \leq \epsilon, \quad 0 \leq i, j \leq 2n, \quad 1 \leq r \leq m, \]

\[ \|ux_s - x_s\theta_i^{-1}\theta_j(u)\|_2 \leq \epsilon, \quad 0 \leq i, j \leq 2n, \quad 0 \leq s \leq 2n. \]

In particular, choosing \( i = j \) in (1) we have \( \|ux_r - x_ru\|_2 \leq \epsilon, 1 \leq r \leq m \). Furthermore, (2) implies with \( i = 0 \) that \( \|ux_s - x_s\theta_j(u)\|_2 \leq \epsilon \); i.e., if \( s = j \), we get

\[ \|u - u_j\theta_j(u)u_j^*\|_2 = \|u - \sigma_{g_i}(u)\|_2 \leq \epsilon, \quad 0 \leq j \leq 2n. \]

Choosing \( \epsilon \) small enough from the start we get that \( R \subset R \rtimes_\sigma G \) has property \( \Gamma \) from Lemma 2.5 and then Lemma 2.4.

Conversely, assume \( R \subset R \rtimes_\sigma G \) has property \( \Gamma \). Let \( x_1, \ldots, x_m \in R \), \( \epsilon > 0 \), and set \( c = \sup_{1 \leq r \leq m} \|x_r\| \). By Lemma 2.4 we can find a unitary \( u \in R \),
\[ \tau(u) = 0, \] such that
\[ \|x_r u - u x_r\|_2 \leq \frac{\epsilon}{3}, \quad \|\sigma_{g_i^{-1} g_j}(u) - u\|_2 \leq \frac{\epsilon}{3c}, \]
\[ \|\theta_j^{-1} \theta_i(u_i^\ast) \theta_j^{-1}(u_j) u - u \theta_j^{-1} \theta_i(u_i^\ast) \theta_j^{-1}(u_j)\|_2 \leq \frac{\epsilon}{3c} \]
for all \( i, j, r \).

Thus
\[ \|ux_r - x_r \theta_i^{-1} \theta_j(u)\|_2 \]
\[ \leq \|ux_r - x_r u\|_2 + c\|u - \sigma_{g_i^{-1} g_j}(u)\|_2 + c\|\sigma_{g_i^{-1} g_j}(u) - \theta_i^{-1} \theta_j(u)\|_2 \]
\[ \leq \frac{2\epsilon}{3} + c\|u_i^\ast \theta_i^{-1}(u_j) \theta_j^{-1}(u) \theta_i^{-1}(u_j)^\ast u_i - \theta_i^{-1} \theta_j(u)\|_2 \]
\[ = \frac{2\epsilon}{3} + c\|\theta_i(u_i^\ast)u_j \theta_j(u) - \theta_i(u) \theta_i(u_i^\ast)u_j\|_2 \]
\[ = \frac{2\epsilon}{3} + c\|\theta_i(u_i^\ast) \theta_j^{-1}(u_j) u - u \theta_j^{-1} \theta_i(u_i^\ast) \theta_j^{-1}(u_j)\|_2 \]
\[ = \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \quad \square \]

We can now use Theorem 2.2 to construct examples of subfactors of the hyperfinite \( \Pi_1 \) factor of finite index which do not contain nontrivial central sequences for the ambient algebra. Let \( G \) be a group with property \( T \) (which is then necessarily finitely generated), for instance \( \text{Sl}(n, \mathbb{Z}), n \geq 3 \), that acts on \( R \) via the Bernoulli shift action \( \sigma \), a properly outer and ergodic action. Then we have

**Lemma 2.6.** \( R \subset R \rtimes G \) does not have property \( \Gamma \).

**Proof.** It was shown in [Ch1] that the crossed product \( R \rtimes G \) does not have property \( \Gamma \); hence, the inclusion \( R \subset R \rtimes G \) cannot have property \( \Gamma \). \( \square \)

Theorem 2.2 implies then that the inclusion of hyperfinite \( \Pi_1 \) factors \( N_G \subset M \) does not have property \( \Gamma \). This construction gives a large class of examples of inclusions of hyperfinite \( \Pi_1 \) factors not having property \( \Gamma \), since there are uncountably many mutually nonconjugate properly outer cocycle actions of an infinite property \( T \) group on \( R \) (see [Po1, Po5, Po6]). Note that \( N_G \subset M \) is amenable (in the sense of [Po7]) if and only if the group \( G \) is amenable. We obtain therefore in this way nonamenable subfactors of the hyperfinite \( \Pi_1 \) factor which do not have property \( \Gamma \) as in Definition 1.1.

**Remark 2.7.** Popa defines in [Po5] the notion (different from [Po7]) of amenability of an inclusion of \( \Pi_1 \) factors \( N \subset M \). In his terminology, if \( N \subset M = N \rtimes G \) then \( N \subset M \) is amenable if and only if \( G \) is amenable. In this case amenability of \( N \subset M \) implies that \( N \subset M \) has property \( \Gamma \). However, in general these two notions are different. Popa shows that if \( [M : N] < \infty \), then \( N \subset M \) is amenable. We have seen above that in finite index we may have both property \( \Gamma \) and nonproperty \( \Gamma \).
There are still a number of open problems related to the notion of property \( \Gamma \) of an inclusion. For instance, since property \( \Gamma \) of a single \( \text{II}_1 \) factor has a nice characterization in terms of \( \mathcal{M} \)-\( \mathcal{M} \)-bimodules (see [Po5]), it would be interesting to find such a characterization also for the property \( \Gamma \) of an inclusion \( N \subset M \).

Finally, let us remark that central sequences in crossed products have been studied extensively by Bédos, Choda, Jones, Jones and Hermann, Ocneanu, and Philipps [Ch1, Ch2, Be, Jo2, JoH, Oc1, Ph].

References


MATHEMATICAL SCIENCES RESEARCH INSTITUTE, 1000 CENTENNIAL DRIVE, BERKELEY, CALIFORNIA 94720

Current address: Department of Mathematics, University of California Berkeley, Berkeley, California 94720

E-mail address: bisch@math.berkeley.edu