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A BERNSTEIN-TYPE INEQUALITY FOR THE JACOBI POLYNOMIAL

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ABSTRACT. Let $P_n^{(\alpha,\beta)}(x)$ be the Jacobi polynomial of degree n. For $-\frac{1}{2} \le \alpha, \beta \le \frac{1}{2}$ and $0 \le \theta \le \pi$, it is proved that

$$(\sin\frac{\theta}{2})^{\alpha+\frac{1}{2}}(\cos\frac{\theta}{2})^{\beta+\frac{1}{2}}|P_n^{(\alpha,\beta)}(\cos\theta)| \le \frac{\Gamma(q+1)}{\Gamma(\frac{1}{2})}\binom{n+q}{n}N^{-q-\frac{1}{2}},$$

where $q = \max(\alpha, \beta)$ and $N = n + \frac{1}{2}(\alpha+\beta+1)$. When $\alpha = \beta = 0$, this reduces to a sharpened form of the well-known Bernstein inequality for the Legendre polynomial.

1. INTRODUCTION

It is well known that the Legendre polynomial $P_n(x)$ satisfies the inequality

$$(1.1) \qquad (\sin\theta)^{\frac{1}{2}}|P_n(\cos\theta)| < (\frac{2}{\pi})^{\frac{1}{2}}n^{-\frac{1}{2}}, \quad 0 \le \theta \le \pi;$$

see [9, (7.3.8), p. 165]. This inequality is due to S. N. Bernstein, who was the first to determine the least possible constant, $(\frac{2}{\pi})^{\frac{1}{2}}$. Recently, by using complex variable methods, Antonov and Holševnikov [1] have shown that the factor $n^{-\frac{1}{2}}$ in (1.1) can be replaced by $(n+\frac{1}{2})^{-\frac{1}{2}}$; that is, they have demonstrated the sharper result

$$(1.2) \qquad (\sin\theta)^{\frac{1}{2}}|P_n(\cos\theta)| < (\frac{2}{\pi})^{\frac{1}{2}}(n+\frac{1}{2})^{-\frac{1}{2}}, \quad 0 \le \theta \le \pi.$$

Later, Lorch [7] has provided an alternative proof of (1.2), by utilizing essentially a sharpened form of Bernstein's real variable method. Furthermore, in [8] he has shown that the ultraspherical (Gegenbauer) polynomial $P_n^{(\lambda)}(x)$ satisfies the inequality

$$(1.3) \qquad (\sin\theta)^{\lambda}|P_n^{(\lambda)}(\cos\theta)| < 2^{1-\lambda}\{\Gamma(\lambda)\}^{-1}(n+\lambda)^{\lambda-1}$$

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for $0 < \lambda < 1$ and $0 \le \theta \le \pi$, which of course improves the customary inequality

$$(1.4) \qquad (\sin\theta)^{\lambda}|P_n^{(\lambda)}(\cos\theta)| < 2^{1-\lambda}\{\Gamma(\lambda)\}^{-1}n^{\lambda-1}, \qquad 0 \le \theta \le \pi,$$

given in [9, (7.33.5), p. 171]. Inequality (1.3) also follows from a more general inequality given by Durand [3, (23)]; see a remark made in [8].

As regards the more general Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$, there does not seem to exist an inequality generalizing (1.4). Except for the simple, yet important, estimate

$$(1.5) \qquad |P_n^{(\alpha,\beta)}(x)| \le \binom{n+q}{n}, \qquad -1 \le x \le 1, \; q = \max(\alpha, \beta) \ge -\frac{1}{2},$$

all we have is the following more recent result of Baratella [2]:

$$(1.6) \qquad (\sin\frac{\theta}{2})^{\alpha+\frac{1}{2}}(\cos\frac{\theta}{2})^{\beta+\frac{1}{2}}|P_n^{(\alpha,\beta)}(\cos\theta)| \le 2.821\binom{n+\alpha}{n}N^{-\alpha-\frac{1}{2}},$$

where $0 \le \theta \le \frac{\pi}{2}$, $-\frac{1}{2} \le \alpha, \beta \le \frac{1}{2}$, and

$$(1.7) \qquad N = n + \frac{\alpha+\beta+1}{2}.$$

In view of the reflection formula [9, p. 59]

$$(1.8) \qquad P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x),$$

Baratella's result in (1.6) can be expressed in the form

$$(1.9) \qquad (\sin\frac{\theta}{2})^{\alpha+\frac{1}{2}}(\cos\frac{\theta}{2})^{\beta+\frac{1}{2}}|P_n^{(\alpha,\beta)}(\cos\theta)| \le 2.821\binom{n+q}{n}N^{-q-\frac{1}{2}}$$

for $0 \le \theta \le \pi$ and $-\frac{1}{2} \le \alpha, \; \beta \le \frac{1}{2}$, where $q = \max(\alpha, \beta)$. In this note, the inequality in (1.6) will be sharpened. Indeed, we shall show that

$$(1.10) \qquad (\sin\frac{\theta}{2})^{\alpha+\frac{1}{2}}(\cos\frac{\theta}{2})^{\beta+\frac{1}{2}}|P_n^{(\alpha,\beta)}(\cos\theta)| \le \frac{\Gamma(q+1)}{\Gamma(\frac{1}{2})}\binom{n+q}{n}N^{-q-\frac{1}{2}}$$

for $0 \le \theta \le \pi$ and $-\frac{1}{2} \le \alpha, \; \beta \le \frac{1}{2}$. When α and β are restricted to the interval $[-\frac{1}{2}, \frac{1}{2}]$, it is known that $\Gamma(q+1) \le \Gamma(\frac{1}{2})$. Hence (1.10) improves (1.9) by a factor of 2.821. Baratella's proof is based on an integral equation satisfied by the Jacobi polynomial, whereas our approach is motivated by the complex variable method of Antonov and Holševnikov [1].

If $\alpha = \beta = 0$, then our result (1.10) immediately yields (1.2). In the case of ultraspherical polynomials, i.e., when $\alpha = \beta = \lambda - \frac{1}{2}$, we can also show that (1.10) reduces to (1.3), provided that $0 < \lambda < \frac{1}{2}$. If $\frac{1}{2} < \lambda < 1$, then our result reduces to one which is only slightly weaker than (1.3). For a more detailed discussion of this case, we refer to a remark in Section 4.

2. A Mehler-type integral

Let N be given as in (1.7) and put

$$(2.1) \qquad K(\alpha, \beta, \theta) = \frac{2^{(\alpha+\beta+1)/2}\Gamma(\alpha+1)}{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{1}{2})}(1-\cos\theta)^{-\alpha}(1+\cos\theta)^{-(\alpha+\beta)/2}.$$

For $0 < \theta < \pi$ and $\operatorname{Re}\alpha > -\frac{1}{2}$, Gasper [4] has given the following Mehler-type integral for the Jacobi polynomial

(2.2)
$$\frac{P_n^{(\alpha,\beta)}(\cos\theta)}{P_n^{(\alpha,\beta)}(1)} = K(\alpha,\beta,\theta)\int_0^\theta \frac{\cos N\phi}{(\cos\phi-\cos\theta)^{\frac{1}{2}-\alpha}} \times F\left(\frac{\alpha+\beta}{2}, \frac{\alpha-\beta}{2}; \alpha+\frac{1}{2}; \frac{\cos\theta-\cos\phi}{1+\cos\theta}\right)d\phi,$$

where $F(a,b;c;z)$ denotes the hypergeometric function and

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}. \tag{2.3}$$

Motivated by the method in [1], we consider the remainder

$$R_n(x,\theta) = \sum_{m=n}^{\infty} \frac{P_m^{(\alpha,\beta)}(\cos\theta)}{P_m^{(\alpha,\beta)}(1)} x^m. \tag{2.4}$$

In view of the asymptotic behavior of $P_n^{(\alpha,\beta)}(\cos\theta)$, the series in (2.4) clearly converges uniformly in $\theta \in (0,\pi)$. Inserting (2.2) in (2.4) gives

$$R_n(x,\theta) = K(\alpha,\beta,\theta)\int_0^\theta F\left(\frac{\alpha+\beta}{2}, \frac{\alpha-\beta}{2}; \alpha+\frac{1}{2}; \frac{\cos\theta-\cos\phi}{1+\cos\theta}\right) \times \frac{1}{(\cos\phi-\cos\theta)^{\frac{1}{2}-\alpha}} \sum_{m=n}^{\infty}(\cos M\phi)x^m\,d\phi, \tag{2.5}$$

where $M = m + \frac{1}{2}(\alpha+\beta+1)$. Since the series under the integral sign can be summed up as

$$\frac{1}{2}x^n\left(\frac{e^{iN\phi}}{1-xe^{i\phi}} + \frac{e^{-iN\phi}}{1-xe^{-i\phi}}\right),$$

we may rewrite (2.5) in the form

$$\frac{1}{x^n}R_n(x,\theta) = \frac{1}{2}K(\alpha,\beta,\theta)\cdot I(\alpha,\beta,\theta), \tag{2.6}$$

where

$$I(\alpha,\beta,\theta) = \int_{-\theta}^{\theta} F\left(\frac{\alpha+\beta}{2}, \frac{\alpha-\beta}{2}; \alpha+\frac{1}{2}; \frac{\cos\theta-\cos\phi}{1+\cos\theta}\right) \times \frac{1}{(\cos\phi-\cos\theta)^{\frac{1}{2}-\alpha}}\cdot\frac{e^{iN\phi}}{1-xe^{i\phi}}d\phi. \tag{2.7}$$

To the last integral, we now apply the quadratic transformation [6, p. 251]

(2.8)
$$F\left(a,b;a+b+\frac{1}{2};z\right) = \left(\frac{1+\sqrt{1-z}}{2}\right)^{\frac{1}{2}-a-b} \times F\left(a-b+\frac{1}{2}, b-a+\frac{1}{2}; a+b+\frac{1}{2}; \frac{1-\sqrt{1-z}}{2}\right),$$

which is valid for $|\arg(1-z)| < \pi$ and $a+b+\frac{1}{2} \neq 0, -1, -2, \ldots$. In our case, we have

$$a = \frac{\alpha+\beta}{2}, \quad b = \frac{\alpha-\beta}{2}, \quad \text{and} \quad z = \frac{\cos\theta-\cos\phi}{1+\cos\theta}. \tag{2.9}$$

Using the trigonometric identity $1+\cos\phi = 2\cos^2(\phi/2)$, it is easily seen that $\cos\phi - \cos\theta = 2[\cos^2(\phi/2) - \cos^2(\theta/2)]$ and

$$\frac{1\pm\sqrt{1-z}}{2} = \frac{\cos(\theta/2)\pm\cos(\phi/2)}{2\cos(\theta/2)}. \tag{2.10}$$

Consequently, it follows from (2.7) that

(2.11)

$$I(\alpha,\beta,\theta) = \frac{1}{2^{1-2\alpha}[\cos(\theta/2)]^{\frac12-\alpha}} \times \int_{-\theta}^{\theta} F\left(\beta+\frac12, -\beta+\frac12; \alpha+\frac12; \frac{\cos(\theta/2)-\cos(\phi/2)}{2\cos(\theta/2)}\right) \times \frac{1}{[\cos(\phi/2)-\cos(\theta/2)]^{\frac12-\alpha}} \cdot \frac{e^{iN\phi}}{1-xe^{i\phi}}d\phi.$$

(Note that $\operatorname{Re}\alpha > -\frac12$ and hence $\alpha+\frac12 \neq 0, -1, -2, \ldots$.) Since $1-\cos\theta = 2\sin^2(\theta/2)$, equation (2.1) can be written as

$$K(\alpha,\beta,\theta) = \frac{\Gamma(\alpha+1)}{\Gamma(\frac12)\Gamma(\alpha+\frac12)} 2^{\frac12-\alpha}\left(\sin\frac{\theta}{2}\right)^{-2\alpha}\left(\cos\frac{\theta}{2}\right)^{-\alpha-\beta}. \tag{2.12}$$

A combination of (2.6), (2.11) and (2.12) gives

$$\frac{1}{x^n}R_n(x,\theta) = 2^{\alpha-\frac32}\frac{\Gamma(\alpha+1)}{\Gamma(\frac12)\Gamma(\alpha+\frac12)}\left(\sin\frac{\theta}{2}\right)^{-2\alpha}\left(\cos\frac{\theta}{2}\right)^{-\beta-\frac12} I^*(\alpha,\beta,\theta), \tag{2.13}$$

where

$$I^*(\alpha,\beta,\theta) = \int_{-\theta}^{\theta} F\left(\beta+\frac12, -\beta+\frac12; \alpha+\frac12; \frac{\cos(\theta/2)-\cos(\phi/2)}{2\cos(\theta/2)}\right) \times \frac{1}{[\cos(\phi/2)-\cos(\theta/2)]^{\frac12-\alpha}} \cdot \frac{e^{iN\phi}}{1-xe^{i\phi}}d\phi. \tag{2.14}$$

So far the only conditions which we require are

$$0<\theta<\pi \quad \text{and} \quad \operatorname{Re}\alpha > -\tfrac12. \tag{2.15}$$

Now we deform the path of integration in (2.14) into two vertical lines $\operatorname{Re}\phi = \theta$ and $\operatorname{Re}\phi = -\theta$. This can be achieved by showing that the contribution from the horizontal line segment, $\operatorname{Im}\phi = T$ and $-\theta \le \operatorname{Re}\phi \le \theta$, approaches zero as $T \to +\infty$. Thus we have

$$I^*(\alpha,\beta,\theta) = iI_-(\alpha,\beta,\theta) - iI_+(\alpha,\beta,\theta), \tag{2.16}$$

where

(2.17)

$$I_\pm(\alpha,\beta,\theta) = \int_0^\infty F\left(\beta+\frac12, -\beta+\frac12; \alpha+\frac12; \frac{\cos\frac12\theta - \cos\frac12(\pm\theta+i\tau)}{2\cos\frac12\theta}\right) \times \frac{e^{iN(\pm\theta+i\tau)}}{[\cos\frac12(\pm\theta+i\tau)-\cos\frac12\theta]^{\frac12-\alpha}}\frac{d\tau}{1-xe^{i(\pm\theta+i\tau)}}.$$

The validity of (2.17) requires that

$$\alpha + \beta > 0 \quad \text{and} \quad \beta > -\tfrac{1}{2}; \tag{2.18a}$$

see the conditions for equation (3.1) below. Since the hypergeometric function $F(a, b; c; z)$ is symmetric in a and b, (2.17) is also valid under the conditions

$$\alpha > \beta > -\tfrac{1}{2}. \tag{2.18b}$$

One could have proceeded with the deformation of contour directly from the integral in (2.7), but this would yield a smaller region of validity for the parameters α and β.

Our next step is to estimate the integrals in (2.17).

3. Proof of (1.10)

We first recall the integral representation [6, p. 239]

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\, dt, \tag{3.1}$$

where $\operatorname{Re} c > \operatorname{Re} b > 0$ and $|\arg(1-z)| < \pi$. If $\operatorname{Re} z < 0$ and $\operatorname{Re} a > 0$, then it is easily seen from (3.1) that $F(a, b; c; z)$ is bounded by 1 in absolute value. Since the real part of

$$\frac{\cos\frac{1}{2}\theta - \cos\frac{1}{2}(\pm\theta + i\tau)}{2\cos\frac{1}{2}\theta}$$

is negative for $\tau > 0$, it follows that

$$\left|F\left(\beta + \frac{1}{2}, -\beta + \frac{1}{2}; \alpha + \frac{1}{2}; \frac{\cos\frac{1}{2}\theta - \cos\frac{1}{2}(\pm\theta + i\tau)}{2\cos\frac{1}{2}\theta}\right)\right| \le 1 \tag{3.2}$$

either under the conditions

$$\alpha + \beta > 0 \quad \text{and} \quad \tfrac{1}{2} > \beta > -\tfrac{1}{2}, \tag{3.3a}$$

or under the conditions

$$\alpha > \beta \quad \text{and} \quad \tfrac{1}{2} > \beta > -\tfrac{1}{2}. \tag{3.3b}$$

Applying (3.2) to (2.17), we obtain

$$|I_\pm(\alpha, \beta, \theta)| \le \int_0^\infty \frac{e^{-N\tau}}{|\cos\frac{1}{2}(\pm\theta + i\tau) - \cos\frac{1}{2}\theta|^{\frac{1}{2}-\alpha}} \cdot \frac{d\tau}{|1 - xe^{i(\pm\theta + i\tau)}|}. \tag{3.4}$$

Simple calculation shows

$$\left|\cos\left(\frac{\pm\theta + i\tau}{2}\right) - \cos\frac{\theta}{2}\right|^2 = 4\sinh^2\frac{\tau}{4} \cdot \left(\sinh^2\frac{\tau}{4} + \sin^2\frac{\theta}{2}\right).$$

Hence

$$\left|\cos\left(\frac{\pm\theta + i\tau}{2}\right) - \cos\frac{\theta}{2}\right| \ge 2\sinh\frac{\tau}{4}\sin\frac{\theta}{2} \ge \frac{\tau}{2}\sin\frac{\theta}{2}, \qquad 0 < \theta < \pi. \tag{3.5}$$

Since

$$\lim_{x\to 0} |1 - xe^{i(\pm\theta + i\tau)}| = 1, \qquad 0 < \theta < \pi,$$

coupling (3.4) and (3.5) yields

$$\lim_{x\to 0} |I_{\pm}(\alpha, \beta, \theta)| \le \left(\frac{1}{2}\sin\frac{\theta}{2}\right)^{\alpha-\frac{1}{2}} \frac{\Gamma(\alpha+\frac{1}{2})}{N^{\alpha+\frac{1}{2}}}, \tag{3.6}$$

provided that either

$$\alpha+\beta > 0, \quad \beta > -\tfrac{1}{2}, \quad \text{and} \quad -\tfrac{1}{2} < \alpha < \tfrac{1}{2}, \tag{3.7a}$$

or

$$\alpha > \beta, \quad \beta > -\tfrac{1}{2}, \quad \text{and} \quad -\tfrac{1}{2} < \alpha < \tfrac{1}{2}. \tag{3.7b}$$

A combination of (2.13), (2.16), and (3.6) gives

$$\lim_{x\to 0} \frac{1}{x^n} |R_n(x, \theta)| \le \left(\sin\frac{\theta}{2}\right)^{-\alpha-\frac{1}{2}} \left(\cos\frac{\theta}{2}\right)^{-\beta-\frac{1}{2}} \frac{\Gamma(\alpha+1)}{\Gamma(\frac{1}{2})N^{\alpha+\frac{1}{2}}}. \tag{3.8}$$

From (2.4), (2.3), and (3.8) it follows that

$$\left(\sin\frac{\theta}{2}\right)^{\alpha+\frac{1}{2}} \left(\cos\frac{\theta}{2}\right)^{\beta+\frac{1}{2}} |P_n^{(\alpha,\beta)}(\cos\theta)| \le \frac{\Gamma(\alpha+1)}{\Gamma(\frac{1}{2})} \binom{n+\alpha}{n} N^{-\alpha-\frac{1}{2}}. \tag{3.9}$$

Let $\theta = \pi - \phi$. By (3.9) and the reflection formula (1.8),

$$\left(\cos\frac{\phi}{2}\right)^{\alpha+\frac{1}{2}} \left(\sin\frac{\phi}{2}\right)^{\beta+\frac{1}{2}} |P_n^{(\beta,\alpha)}(\cos\phi)| \le \frac{\Gamma(\alpha+1)}{\Gamma(\frac{1}{2})} \binom{n+\alpha}{n} N^{-\alpha-\frac{1}{2}}.$$

Replacing ϕ by θ and reversing the roles of α and β, we have

$$\left(\sin\frac{\theta}{2}\right)^{\alpha+\frac{1}{2}} \left(\cos\frac{\theta}{2}\right)^{\beta+\frac{1}{2}} |P_n^{(\alpha,\beta)}(\cos\theta)| \le \frac{\Gamma(\beta+1)}{\Gamma(\frac{1}{2})} \binom{n+\beta}{n} N^{-\beta-\frac{1}{2}}, \tag{3.10}$$

either under the conditions

$$\beta+\alpha > 0, \quad \alpha > -\tfrac{1}{2}, \quad \text{and} \quad -\tfrac{1}{2} < \beta < \tfrac{1}{2}, \tag{3.11a}$$

or under the conditions

$$\beta > \alpha, \quad \alpha > -\tfrac{1}{2}, \quad \text{and} \quad -\tfrac{1}{2} < \beta < \tfrac{1}{2}. \tag{3.11b}$$

The desired inequality now follows from (3.9) and (3.10), using the set of conditions given in (3.7b) and (3.11b). The special case $\alpha = \beta$ can be treated by a limiting argument.

4. Remarks

1. If α and β are both restricted to the interval $(-\frac{1}{2}, \frac{1}{2})$, then the two sets of conditions in (3.7a) and (3.11a) are the same. Hence inequality (1.10) holds with $q = \min(\alpha, \beta)$, instead of $q = \max(\alpha, \beta)$. However, the validity of this stronger result is only in half of the unit square $(-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2})$, namely, $\alpha+\beta > 0$. It would be desirable to extend (1.10) to allow $\alpha+\beta < 0$, $\alpha, \beta \in (-\frac{1}{2}, \frac{1}{2})$.

2. If $\alpha = \beta$, then the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ reduces to the ultraspherical polynomial $P_n^{(\lambda)}(x)$, $\lambda = \alpha + \frac{1}{2}$. More precisely, we have

$$P_n^{(\lambda)}(x) = \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} \cdot \frac{\Gamma(n+2\alpha+1)}{\Gamma(n+\alpha+1)} P_n^{(\alpha,\alpha)}(x), \qquad \alpha = \lambda - \frac{1}{2}; \tag{4.1}$$

see [9, (4.7.1), p. 80]. Since $P_n^{(\alpha,\beta)}(x)$ is continuous in β, we may let β approach α in (3.9). In view of (4.1) and the duplication formula for the gamma function, this gives

$$(\sin\theta)^\lambda |P_n^{(\lambda)}(\cos\theta)| \le \frac{2^{1-\lambda}}{\Gamma(\lambda)} \cdot \frac{\Gamma(n+2\lambda)}{\Gamma(n+1)}(n+\lambda)^{-\lambda}. \tag{4.2}$$

Lorch's result (1.3) now follows from the inequality

$$\frac{\Gamma(n+2\lambda)}{\Gamma(n+1)} < \frac{1}{(n+\lambda)^{1-2\lambda}},$$

provided that $0 < 2\lambda < 1$; see [8, (8)] or [5, (1.3)]. If $1 < 2\lambda < 2$, i.e., $0 < \alpha < \frac{1}{2}$, then by the inequality [8, (10)]

$$\frac{\Gamma(n+2\lambda)}{\Gamma(n+1)} < \frac{1}{(n+2\lambda)^{1-2\lambda}}$$

we have from (4.2)

$$(\sin\theta)^\lambda |P_n^{(\lambda)}(\cos\theta)| \le \frac{2^{1-\lambda}}{\Gamma(\lambda)} \cdot \frac{(n+2\lambda)^{2\lambda-1}}{(n+\lambda)^\lambda},$$

which is only slightly weaker than (1.3).

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