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BORSUK'S ANTIPODAL AND FIXED-POINT THEOREMS FOR SET-VALUED MAPS

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ABSTRACT. The purpose of this paper is to obtain the extensions of Borsuk's antipodal and fixed-point theorems for set-valued maps.

Borsuk [3] related the following relative results:

Borsuk's antipodal theorem. *A single-valued antipodal-preserving continuous map* $f: S^n \to E^n$ *has a zero value.*

Borsuk's fixed-point theorem. *Let* U *be a bounded symmetric convex open neighborhood of the origin in* E^n, *and let* $f: \overline{U} \to E^n$ *be antipodal-preserving on* ∂U, *i.e.,* $-f(a) = f(-a)$ *for each* $a \in \partial U$. *Then* f *has a fixed point.*

In this note we consider the set-valued maps. With a proper definition of antipodality, we prove that Borsuk's antipodal theorem can be extended for set-valued maps on the boundary of a symmetric bounded balanced neighborhood of the origin. Such boundaries are more general than the homeomorphic image of spheres. At the same time we also prove Borsuk's fixed-point theorem on a symmetric balanced set of a locally convex space. The main results are summarized in Theorems 4, 5, 6, and 7. Lemmas 1 and 2 are two approximation properties which play a crucial role in this paper.

Let us first recall some definitions. Let X and Y be two topological spaces; capital letters $F: X \to Y$ denote nonempty set-valued maps while noncapital letters $f: X \to Y$ will denote single-valued functions. F is called *upper semicontinuous* (u.s.c.) if for each open set G of Y the set $\{x \in X | F(x) \subset G\}$ is open in X. F is said to *have open lower sections* if, for each $y \in Y$, $F^{-1}(y) = \{x \in X | y \in F(x)\}$ is open in X. The set $\operatorname{Gr} F$ is the graph of F which is the set $\{(x, y) | y \in F(x), x \in X\}$. When Y is a topological vector space and $S \subset Y$, $\operatorname{co} S$ denotes the convex hull of S. A set $B \subset Y$ is said to be *balanced* if $rB \subset B$ for every real number r with $|r| \leq 1$.

Let E be the normed space of all those sequences $x = (x_1, x_2, \dots)$ of real numbers having at most finitely many $x_n \neq 0$, with the norm $\|x\| = \sum |x_i|$. The subset $\{x \in E | x_i = 0 \text{ for all } i > n\}$ is denoted by E^n; the unit n-sphere $S^n = \{x \in E^{n+1} | \|x\| = 1\}$. Let X and Y be subsets of E, $S \subset X$ be a

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subset which is symmetric with respect to the origin, and $F: X \to Y$. We say that F is *antipodal* on S if $F(x) \cap (-F(-x)) \neq \varnothing$ for all $x \in S$. This is a generalization of the original single-valued functions to set-valued maps.

The following lemma is a generalization of a theorem of Cellina (see [1, Lemma 13.1, p. 67]). For different versions of the statement, we refer to [2, Theorem 1, p. 19].

Lemma 1. *Let X be a compact set and $\mathscr{B}_1$ an open symmetric neighborhood base at 0 in a topological vector space E_1. Let Y be a set and $\mathscr{B}_2$ an open convex neighborhood base at 0 in a locally convex space E_2. Suppose the correspondence $F: X \to Y$ is u.s.c. with convex values. For $V \in \mathscr{B}_1$ define $F^V: X + V \to E_2$ via $F^V(x) = \operatorname{co} \bigcup_{z \in (x+V) \cap X} F(z)$. Then for each $W_1 \in \mathscr{B}_1$ and $W_2 \in \mathscr{B}_2$ there exists $V \in \mathscr{B}_1$ such that*

$$\operatorname{Gr} F^V \subset \operatorname{Gr} F + W_1 \times W_2$$

and F^V has open lower sections.

Furthermore, if $S \subset X$ is symmetric with the origin and F is antipodal on S, then F^V is antipodal on $S + V$.

Proof. We choose $V_1 \in \mathscr{B}_1$ such that $V_1 + V_1 + V_1 \subset W_1$. For any $x \in X$ there exists $V_x \in \mathscr{B}_1$ such that $\bigcup_{z \in x + V_x} F(z) \subset F(x) + W_2$. We choose $U_x \in \mathscr{B}_1$ such that $U_x + U_x + U_x \subset V_x$.

Since X is compact, there exists a finite cover $x_1 + (U_{x_1} \cap V_1), \ldots, x_n + (U_{x_n} \cap V_1)$ of X. Let $V = (\bigcap_{i=1}^n U_{x_i}) \cap V_1$. Fix $x \in X + V$. Then $x \in x_i + (U_{x_i} \cap V_1) + V$ for some i. Hence $x + V \subset x_i + (U_{x_i} \cap V_1) + V + V \subset x_i + (V_{x_i} \cap W_1)$. Thus $\bigcup_{z \in (x+V) \cap X} F(z) \subset \bigcup_{z \in (x_i + V_{x_i})} F(z) \subset F(x_i) + W_2$. Since F has convex values and W_2 is convex, $\operatorname{co} \bigcup_{z \in (x+V) \cap X} F(z) \subset F(x_i) + W_2$, and then $\{x\} \times F^V(x) \subset (\{x_i\} + W_1) \times (F(x_i) + W_2)$ for such i. Thus $\operatorname{Gr} F^V \subset \operatorname{Gr} F + W_1 \times W_2$.

Let $\operatorname{Gr} H = \operatorname{Gr} F + V \times \{0\}$. Then $H: X + V \to E_2$ has open lower sections obviously, and hence $\operatorname{co} H: X + V \to E_2$ has open lower sections where $(\operatorname{co} H)(x) = \operatorname{co}(H(x))$ for each $x \in X + V$. For each $x \in X + V$, $\operatorname{co}(H(x)) = \operatorname{co} \bigcup_{z \in (x+V) \cap X} F(z) = F^V(x)$. This implies that F^V has open lower sections.

For any $x \in S + V$ there exist $x' \in S$, $v \in V$ such that $x = x' + v$. Since S and V are symmetric with the origin, $-x = -x' + (-v) \in (S + V)$. Since F is antipodal on S, $\varnothing \neq F(x') \cap (-F(-x')) \subset F^V(x) \cap (-F^V(-x))$. Hence F^V is antipodal on $S + V$. This completes the proof.

Lemma 2. *Let U be an open symmetric balanced neighborhood of the origin in E^{n+1}. Define $g: U \backslash \{0\} \to E^{n+1}$ by $g(x) = d(x, \partial U)x/\|x\| + x$. Then g is continuous and symmetric*; *i.e., $g(-x) = -g(x)$. Furthermore, for any $\varepsilon > 0$ there exists k such that*

$$d(g^k(U \backslash \{0\}), \partial U) = \sup\{d(g^k(x), \partial U) | x \in U \backslash \{0\}\} < \varepsilon.$$

Proof. By the property of U, $d(x, \partial U)$ is continuous and $d(x, \partial U) = d(-x, \partial U)$ in $U \backslash \{0\}$. Thus g is continuous, and $g(-x) = -g(x)$ for all $x \in U \backslash \{0\}$; i.e., g is symmetric.

Let $r_1 = \inf\{\|x\| \,|\, x \in \partial U\}$ and $r_2 = \sup\{\|x\| \,|\, x \in \partial U\}$. For every $x \in U \backslash \{0\}$ let $x_I = r_1 x/\|x\|, x_T = r_x x$, where $r_x = \inf\{r | rx/\|x\| \in \partial U\}$. Then

$r_1 \le \|g(x)\| = d(x, \partial U) + \|x\| \le r_x$, and hence $g(x) = \|g(x)\|x/\|x\| \in \overline{x_I x_T}$, where $\overline{x_I x_T}$ denotes the line segment from x_I to x_T. Since $g^j(x) = d(g^{j-1}(x), \partial U)x/\|x\| + \|g^{j-1}(x)\|x/\|x\|$, it follows that $\|g^{j-1}(x)\| \le \|g^j(x)\| = d(g^{j-1}(x), \partial U) + \|g^{j-1}(x)\| \le r_{g^{j-1}(x)} = r_x$ and $g^j(x) \in \overline{g^{j-1}(x)x_T} \subset \overline{x_I x_T}$ for each $j = 2, 3, \dots$.

Fix $x \in U\backslash\{0\}$, where $\|x\| \ge r_1$. If $d(x, \partial U) = \delta$, then there is $p_x \in \partial U$ such that $\|x - p_x\| = \delta$. If $p_x = x_T$, then for each $x' \in \overline{x x_T}$, $d(x', \partial U) < \delta$. If $p_x \ne x_T$, then each $x' \in \overline{x x_T}$, the line l passing through x' and parallel to $\overline{x p_x}$, intersects $\overline{o p_x}$ at $p_{x'}$. Then $p_{x'} \notin U$ and $d(x', \partial U) \le \|x' - p_{x'}\| = \|x'\|\delta/\|x\| \le r_2\delta/r_1$. Hence, if $\|x\| \ge r_1$ and $d(x, \partial U) < r_1\varepsilon/r_2$, then $d(\overline{x x_T}, \partial U) < \varepsilon$. Then $d(g^j(x), \partial U) < \varepsilon$ for $j = 0, 1, 2, \dots$.

Fix $y \in U\backslash\{0\}$. Then $\|g(y)\| \ge r_1$, and

$$\begin{aligned} g^k(y) &= d(g^{k-1}(y), \partial U)y/\|y\| + g^{k-1}(y) \\ &= [d(g^{k-1}(y), \partial U) + d(g^{k-2}(y), \partial U) + \cdots + d(g(y), \partial U)]y/\|y\| + g(y) \\ &= [d(g^{k-1}(y), \partial U) + \cdots + d(g(y), \partial U) + \|g(y)\|]y/\|y\|. \end{aligned}$$

Since

$$\|g^k(y)\| = d(g^{k-1}(y), \partial U) + \cdots + d(g(y), \partial U) + \|g(y)\| \le r_2$$

and $\|g(y)\| \ge r_1$, $d(g^{k-1}(y), \partial U) + \cdots + d(g(y), \partial U) \le r_2 - r_1$, and hence there is some i where $1 \le i \le k-1$ such that $d(g^i(y), \partial U) \le (r_2 - r_1)/(k-1)$. If $k > (r_2 - r_1)r_2/r_1\varepsilon + 2$, then $d(g^i(y), \partial U) < (r_2 - r_1)r_1\varepsilon/[(r_2 - r_1)r_2] = r_1\varepsilon/r_2 < \varepsilon$, and hence

$$d(g^k(y), \partial U) = d(g^{k-i}(g^i(y)), \partial U) < \varepsilon.$$

This completes the proof.

Lemma 3. *Let E_1 and E_2 be two topological vector spaces. Let $S \subset E_1$ and S be compact and symmetric with respect to 0. Let $F: S \to E_2$ be antipodal-preserving, convex-valued, and with open lower sections. Then F has a single-valued continuous antipodal selection $f: S \to E_2$.*

Proof. By definition, $\{F^{-1}(y) \cap [-F^{-1}(-y)] | y \in E_2\}$ is an open cover of S; hence, there is a finite subcover $\Gamma = \{F^{-1}(y_i) \cap [-F^{-1}(-y_i)] | i = 1, 2, \dots, n\}$. Let $\{\varphi_i\}_{i=1}^n$ be a partition of unity of S with respect to Γ. It implies that each φ_i is nonnegative continuous on S, $\sum \varphi_i(x) = 1$, and $\varphi_i(x) > 0 \Rightarrow x \in F^{-1}(y_i) \cap [-F^{-1}(-y_i)]$. Define $p: S \to E_2$ by

$$p(x) = \frac{\sum_{i=1}^n \varphi_i(x)y_i - \sum_{i=1}^n \varphi_i(-x)y_i}{\sum_{i=1}^n \varphi_i(x) + \sum_{i=1}^n \varphi_i(-x)}.$$

Then p is continuous, and $-p(-x) = p(x)$ for all $x \in S$. If $\varphi_i(x) \ne 0$, then $x \in F^{-1}(y_i)$, and hence $y_i \in F(x)$. If $\varphi_i(-x) \ne 0$, then $-x \in -F^{-1}(-y_i)$, and hence $-y_i \in F(x)$. Since F is convex-valued, $p(x) \in F(x)$ for all $x \in S$.

Borsuk antipodal and Borsuk-Ulam theorems [3, §4, Theorem 5.2, p. 44] can be generalized to the following two theorems.

Theorem 4. *Let U be an open bounded symmetric neighborhood of the origin in E^{n+1}, and let $F: \partial U \to E^n$ be u.s.c., closed, convex-valued, and antipodal-preserving. Then F has a zero.*

Proof. Let $\mathbf{0}$ denote the zero map from ∂U into $\{0\}$. Suppose that F does not have zero. Then by [4, Theorem 1] there are open convex neighborhoods W of 0 in E^n and W_1 of 0 in E^{n+1} such that $[\operatorname{Gr} F + W_1 \times W] \cap \operatorname{Gr} \mathbf{0} = \varnothing$. By Lemma 1 there is an open neighborhood V of 0 in E^{n+1} such that $\operatorname{Gr} F^V \subset \operatorname{Gr} F + W_1 \times W$ and the domain of F^V is $\partial U + V$. Let V_1 be a closed bounded symmetric balanced neighborhood of 0 in E^{n+1} such that $V_1 + V_1 \subset V$ and $F^V|_{\partial U + V_1}$ denote the restriction of F^V on $\partial U + V_1$. Choose λ, $0 < \lambda < 1$, such that $\lambda S_n \subset U$. Define g on $U\backslash\{0\}$ as Lemma 2. Then there exists a positive integer k such that $g^k(x) \in \partial U + V_1$ for all $x \in \lambda S_n$. Let $K = g^k(\lambda S_n)$. By Lemma 1, $F^V|_{\partial U + V_1}$ is antipodal and has open lower sections. By Lemma 3, $F^V|_K$ has a single-valued continuous antipodal selection h. Define $f\colon S_n \to E^n$ by $f(x) = h(g^k(\lambda x))$. Then f is a single-valued continuous antipodal map with zero free. This is a contradiction to Borsuk's antipodal theorem. This completes the proof.

Theorem 5. *Let U be an open bounded symmetric neighborhood of the origin in E^{n+1}. Then every u.s.c. closed convex-valued map $F\colon \partial U \to E^n$ has an intersection point for at least one pair of antipodal points.*

Proof. Define $G\colon \partial U \to E^n$ by $G(x) = F(x) - F(-x)$. Then G is a u.s.c. closed convex-valued antipodal-preserving map. By Theorem 4, $0 \in F(x) - F(-x)$, and hence $F(x) \cap F(-x) \neq \varnothing$ for some $x \in \partial U$. This completes the proof.

Theorem 6. *Let U be an open bounded symmetric balanced neighborhood of the origin in E^{n+1}. Let $F\colon \overline{U} \to E^{n+1}$ be u.s.c., closed, convex-valued, and antipodal on ∂U; i.e., $F(a) \cap [-F(-a)] \neq \varnothing$ for each $a \in \partial U$. Then F has a zero value and a fixed point on $\overline{U}$.*

Proof. Suppose that F has no zero value. Let $\mathbf{0}$ denote the zero map from $\overline{U}$ into $\{0\}$. Then $\operatorname{Gr} F \cap \operatorname{Gr} \mathbf{0} = \varnothing$. By [4, Theorem 1] there is a closed convex neighborhood W of 0 in E^{n+1} such that $[\operatorname{Gr} F + W \times W] \cap [\operatorname{Gr} \mathbf{0} + W \times W] = \varnothing$. By Lemma 1 there exists a symmetric neighborhood V of 0 such that $\operatorname{Gr}(F^V) \subset \operatorname{Gr} F + W \times W$. Hence $0 \notin F^V(x)$ for all $x \in \overline{U} + V$. By Lemma 3, $F^V|_{\overline{U}}$ has a single-valued selection f such that f is antipodal on ∂U.

Define $J\colon \overline{U} \to E^{n+2}$ by $J(x) = x + d(x, \partial U)\vec{e}_{n+2}$. Then J is one-to-one continuous, and $J(x) = x$ on ∂U. Let $V = \{x + u\vec{e}_{n+2} | x \in U, |u| < d(x, \partial U)\}$. Then V is open, bounded, symmetric, and balanced in E^{n+2} and $\partial V = J(\overline{U}) \cup \{-J(\overline{U})\}$.

Define $H\colon \partial V \to E^{n+1}$ by

$$H(x) = \begin{cases} f(J^{-1}(x)) & \text{if } x \in J(\overline{U}), \\ -f(J^{-1}(-x)) & \text{if } x \in -J(\overline{U}). \end{cases}$$

Then H is a continuous antipodal single-valued function with zero free. This is a contradiction to Theorem 4. Thus F has a zero value.

By the above conclusion, $F - I$ has a zero value; i.e., there exists $x \in \overline{U}$ such that $0 \in F(x) - \{x\}$. Hence $x \in F(x)$ for some $x \in \overline{U}$. This proves the theorem.

The following theorem is a generalization of a result of Borsuk [3, §4, Theorem 3.3, p. 57].

Theorem 7. *Let M be a closed bounded symmetric balance set at 0 in a locally convex space E. Let $F: M \to E$ be a u.s.c. closed convex-valued map such that the closure of $F(M)$ is compact and F is antipodal on the boundary of M. Then F has at least one fixed point.*

Proof. Let $\mathscr{B}$ denote a closed bounded symmetric convex neighborhood base at 0 in E. For each $V \in \mathscr{B}$ there is a finite subset S of $F(M)$ such that $(y+V)\cap S \neq \varnothing$ for each $y \in F(M)$. Let S_V be a finite subset of E such that $S \subset S_V$. Let H_{S_V} denote the finite-dimensional space spanned by S_V.

Define $G_V: M \cap H_{S_V} \to H_{S_V}$ by $G_V(x) = (F(x)+V)\cap H_{S_V}$. Then G_V is u.s.c. with nonempty compact convex values. Since $F(a)\cap -F(-a) \neq \varnothing$ for all $a \in \partial(M\cap H_{S_V})$, $\varnothing \neq \{[F(a)\cap(-F(-a))]+V\}\cap H_{S_V}$ for all $a \in \partial(M\cap H_{S_V})$; i.e., G_V is antipodal on its boundary. If 0 is a relative interior point of $M \cap H_{S_V}$, then by Theorem 6 there is x_V such that $x_V \in G_V(x_V)$; i.e., $x_V \in F(x_V)+V$. On the other hand, if $0 \in \partial(M\cap H_{S_V})$, then $M\cap H_{S_V} = \partial(M\cap H_{S_V})$. Since G_V is antipodal on $M \cap H_{S_V}$, we have $G_V(0) = -G_V(0)$. Since $G_V(0)$ is convex, we have $0 \in G_V(0)$.

From the above argument, we get that in any situation for each $V \in \mathscr{B}$ there exist $x_V \in M$ such that $x_V = y_V + v$, where $y_V \in F(x_V)$ and $v \in V$. Since the closure of $F(M)$ is compact, there is a subnet of $\{y_V\}$ that converges to x_0. Then $x_0 \in F(x_0)$. This completes the proof.

Now we give the following application of Theorem 4 which is a result of Krein, Krasnoselsky, and Milman [3, §5, Theorem 5.4, p. 80].

Theorem 8. *Let M, N be linear subspaces of a Banach space $(E, \|\ \|)$. If $\dim M > \dim N$, then there is a $x_0 \in M$ such that $d(x_0, N) = \|x_0\| > 0$.*

Proof. For each $x \in E$ let $F(x) = \{y \in N | d(y, x) = d(x, N)\}$. Then $F(x)$ is compact convex and $x \to F(x)$ is of closed graph. Furthermore, the map $F: E \to N$ has the property $F(-x) = -F(x)$ for all $x \in E$. Let S_M denote the unit sphere in M. Consequently, applying Theorem 4 to $F|_{S_M}: S_M \to N$, we obtain a point $x_0 \in S_M$ such that $0 \in F(x_0)$. Clearly, x_0 is the required point. This proves the theorem.

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