CYCLIC APPROXIMATION OF IRRATIONAL ROTATIONS

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(Communicated by Andrew Bruckner)

Abstract. We prove that an irrational number \( \alpha \) admits a rational approximation \( |\alpha - p/q| = o(f(q)) \) iff the irrational rotation \( Tx = \{x + \alpha\} \) admits cyclic approximation with speed \( o(f(n)) \). As an application to earlier results we obtain that a generic Anzai skew product over every irrational rotation is rank-1 and for a.e. \( \alpha \) most skew products admit cyclic approximation with speed \( o(1/n^2 \log n) \).

1. Introduction

Let \((X, \mu)\) be a Lebesgue space. If \( \xi_n \) is a sequence of (finite measurable) partitions then we write \( \xi_n \to \varepsilon \) if for any measurable set \( A \) there exists a union \( A_n \) of cells of \( \xi_n \) such that \( \mu(A \Delta A_n) \to 0 \).

Let \( 0 < f(n) \to 0 \). According to Katok and Stepin [5] we say that an invertible measure-preserving transformation \( T \) of \( X \) admits a cyclic approximation with speed \( o(f(n)) \) if there exists a sequence of partitions \( \xi_n \to \varepsilon \) and a transformation \( T_n \) cyclically permuting the cells of \( \xi_n \) such that

\[
\sum_{j=0}^{h_n-1} \mu(TC_j \Delta T_n C_j) = o(f(h_n)),
\]

where \( \xi_n = \{C_0, \ldots, C_{h_n-1}\}, C_j = T^j \xi C_0 \).

It is well known that cyclic approximation with speed \( o(1/n) \), called good cyclic approximation, implies rigidity, singular simple spectrum (see [5, 4]), and rank 1 of \( T \).

As was pointed out in [5, §1, 1 and 2]) for an a.e. irrational number \( \alpha \) the irrational rotation \( Tx = \{x + \alpha\} \) (where \( \{x\} \) denotes the fractional part of \( x \)) of the unit interval \([0, 1)\) admits a good cyclic approximation. In fact, it is easy to see that if \( |\alpha - p_n/q_n| = o(1/q_n^2) \) then \( T_n x = \{x + p_n/q_n\} \) and \( C_0 = [0, 1/q_n) \) will do.

It is our aim to prove that the speed of cyclic approximation of the rotation \( T \) is essentially as good as the speed of rational approximation of \( \alpha \) (see Theorem 1). In particular, every irrational rotation admits a good cyclic approximation (this was observed earlier by del Junco, see [2]), while for a.e. \( \alpha \) an admissible
speed is \(o(1/n^2 \log n)\) (see Corollary). Our argument is similar to that in del Junco \([1]\); now, however, we have to ensure an error term of order \(o(f(h_n))\). Instead of using cyclic partitions for rational rotations we exploit properties of continued fractions and use two disjoint \(T\)-stacks covering the interval.

### 2. Cyclic approximation

Let \(\alpha\) be an irrational number. We denote by \(p_n/q_n\) its continued fraction approximation. It is well known that \(|\alpha - p_n/q_n| < 1/q_n^2\). Moreover,

\[ q_{n+1} \| q_n \alpha \| + q_n \| q_{n+1} \alpha \| = 1, \]

where \(\|x\|\) denotes the distance from \(x\) to the nearest integer. This formula reflects splitting the interval into two Rokhlin towers \(\zeta_n\) and \(\zeta'_n\) for the irrational rotation \(T_x = \{x + \alpha\}\). More precisely, if \(n\) is even then \(\zeta_n\), the large tower, consists of the sets \(J_j = \{ja\}, \{(q_n + j)\alpha\}\), \(j = 0, \ldots, q_n + 1\), while \(\zeta'_n\), the small one, consists of \(J'_0 = \{q_{n+1} \alpha, 1\}\) and \(J'_j = \{(q_{n+1} + j)\alpha, \{j\alpha\}\}, j = 1, \ldots, q_n - 1\).

**Theorem 1.** Let \(0 < f(n) \to 0\). The irrational rotation \(T_x = \{x + \alpha\}\) admits cyclic approximation with speed \(o(f(n))\) iff there exists a sequence of rational numbers \(p/q \to \alpha\) such that

\[ |\alpha - p/q| = o(f(q)). \]

**Proof.** To prove the “if” part it clearly suffices to construct a sequence of Rokhlin towers \(\tilde{\eta}_n\) of height \(h_n\), such that the complement of the tower \(\tilde{\eta}_n\) has measure \(o(f(h_n))\). We are going to select a subsequence \(m_n\) and then construct \(\tilde{\eta}_{m_n}\) out of the pair \(\zeta_{m_n}, \zeta'_{m_n}\) by forming a new base from levels of the two towers.

Without loss of generality we may assume that the \(p/q\) are continued fraction convergents of \(\alpha\) and \(qf(q) \to 0\). Denote \(M(q) = [1/qf(q)]\). We start by choosing an arbitrary sequence \(\varepsilon_n\) decreasing to zero. By compactness, we can find integers \(K_n\) such that for every \(x \in [0, 1)\) there is a \(j\) with \(0 \leq j < K_n\) and \(T^jx \in [0, \varepsilon_n/2)\). Now choose \(N_n\) such that \(K_n/N_n < 1/n\). It is clear that for \(k_n\) large enough we have

\[ N_n M(q_{k_n}) \| q_{k_n} \alpha \| < \varepsilon_n/2 \]

and \(K_n < q_{k_n}\). Finally choose \(m_n\) satisfying

\[ M(q_{k_n}) q_{k_n}^2 / q_{m_n} < 1/n. \]

For the sake of convenience we may assume that the numbers \(k_n, m_n\) are even and the sequences \(k_n, m_n\) are increasing.

Now we form two new towers \(\eta_n, \eta'_n\) of height \(q_{k_n}\). We construct the base \(A\) of \(\eta_n\) by selecting certain levels of \(\zeta_{m_n}\). The first group of \(N_n M(q_{k_n})\) many levels are

\[ J_0, J_{q_{k_n}}, \ldots, J_{(N_n M(q_{k_n}) - 1) q_{k_n}}. \]

Each of them is clearly contained in \([0, \varepsilon_n]\). Now by the choice of \(K_n\) we can find \(s_1\) with \(N_n M(q_{k_n}) q_{k_n} \leq s_1 < N_n M(q_{k_n}) q_{k_n} + K_n\) such that \(J_{s_1} \subset [0, \varepsilon_n/2)\). The next group of selected levels are

\[ J_{s_1}, J_{s_1 + q_{k_n}}, \ldots, J_{s_1 + (N_n M(q_{k_n}) - 1) q_{k_n}}. \]
We continue up the tower in the same manner until we define the last group,

$$J_{s_p}, J_{s_p+q_{k_n}}, \ldots, J_{s_p+(M-1)q_{k_n}},$$

where $1 \leq M \leq N_n M(q_{k_n})$, with at least $q_{k_n} - 1$ but less than $2q_{k_n} + K_n - 2 < 3q_{k_n} - 2$ top levels of $\zeta_{m_n}$ left over. The set

$$A = J_0 \cup J_{q_{k_n}} \cup \cdots \cup J_{(N_n M(q_{k_n})-1)q_{k_n}}$$

$$\cup \cdots \cup J_{s_1} \cup J_{s_1+q_{k_n}} \cup \cdots \cup J_{s_1+(N_n M(q_{k_n})-1)q_{k_n}}$$

$$\cup \cdots \cup J_{s_p} \cup J_{s_p+q_{k_n}} \cup \cdots \cup J_{s_p+(M-1)q_{k_n}}$$

is the base of $\eta_n$. This new tower is contained entirely in $\zeta_{m_n}$ and only a small part of measure $\delta_n$ of the tower is not covered by $\eta_n$. Since less than $K_n$ levels have been skipped over every $N_n M(q_{k_n})q_{k_n}$th step and less than $2q_{k_n}$ levels have been left on top, we obtain

$$\delta_n < \frac{K_n}{N_n M(q_{k_n})q_{k_n}} + \frac{2q_{k_n}}{q_{m_n}+1}.$$

Moreover, we have $A \subset [0, \varepsilon_n)$ by the choice of $k_n$.

We repeat the construction to form $\eta'_n$. By selecting levels as before we obtain the base

$$A' = J'_0 \cup J'_{q_{k_n}} \cup \cdots \cup J'_{(N_n M(q_{k_n})-1)q_{k_n}}$$

$$\cup \cdots \cup J'_{s_1} \cup J'_{s_1+q_{k_n}} \cup \cdots \cup J'_{s_1+(N_n M(q_{k_n})-1)q_{k_n}}$$

$$\cup \cdots \cup J'_{s_p} \cup J'_{s_p+q_{k_n}} \cup \cdots \cup J'_{s_p+(M-1)q_{k_n}}.$$

Clearly $A' \subset (-\varepsilon_n, \varepsilon_n)$; $\eta'_n$ is contained in $\zeta'_{m_n}$ and covers it up to

$$\delta'_n < \frac{K_n}{N_n M(q_{k_n})q_{k_n}} + \frac{2q_{k_n}}{q_{m_n}}.$$

Finally we form a single tower $\tilde{\eta}_n$ by joining $\eta_n$ with $\eta'_n$. Its base is $\tilde{A} = A \cup A'$, and the height is $q_{k_n}$. It covers the space up to

$$\tilde{\delta}_n = \delta_n + \delta'_n < \frac{6}{n M(q_{k_n})q_{k_n}} = o(f(q_{k_n})).$$

Moreover, $\tilde{A} \subset (-\varepsilon_n, \varepsilon_n)$, so each level of $\tilde{\eta}_n$ has diameter at most $2\varepsilon_n$ and consequently $\tilde{\eta}_n \rightarrow \varepsilon$, which ends the proof of sufficiency.

To prove the “only if” part assume there exist partitions

$$\xi_n = \{C_0, \ldots, C_{h_n-1}\} \rightarrow \varepsilon$$

and cyclic approximations $T_n$ of $T$ such that

$$\sum_{j=0}^{h_n-1} \mu(TC_j \Delta T_n C_j) < \tilde{f}(h_n),$$

where $\tilde{f}(n) = o(f(n))$. Let $\chi(x) = e^{2\pi i x}$ on $[0, 1)$. As $\xi_n \rightarrow \varepsilon$, we can find, for any $n$ large enough, an approximation $\chi_n = \sum_{j=0}^{h_n-1} \lambda_j 1_{C_j}$ of $\chi$, where $|\lambda_j| = 1$ and $\|\chi_n - \chi\| < 1/4$ in $L^1$. Observe that

$$\|\chi_n \circ T_n^{-1} - \chi_n \circ T^{-1}\| = \left\| \sum \lambda_j (1_{T_n C_j} - 1_{T C_j}) \right\| \leq \sum \mu(T_n C_j \Delta T C_j) < \tilde{f}(h_n).$$
For every $k \geq 1$ the same argument applies to the function $\chi_n \circ T_n^{-k}$ which is also of the form $\sum \chi_{j}^{(k)} 1_{C_j}$, so we get by induction

$$
\| \chi_n \circ T_n^{-k} - \chi_n \circ T^{-k} \| \leq \| (\chi_n \circ T_n^{-k+1}) \circ T^{-1} - (\chi_n \circ T_n^{-k+1}) \circ T^{-1} \|
+ \| \chi_n \circ T_n^{-k+1} - \chi_n \circ T^{-k+1} \|
+ \| \chi_n \circ T_n^{-k} - \chi_n \circ T^{-k} \|
+ \| \chi_n - \chi \|
\leq f(h_n) + (k-1)f(h_n) = kf(h_n).
$$

Now by the triangle inequality we obtain

$$|e^{2\pi i k \alpha} - 1| = \| \chi \circ T^{-k} - \chi \|
\leq \| \chi \circ T^{-k} - \chi_n \circ T^{-k} \| + \| \chi_n \circ T^{-k} - \chi_n \circ T^{-k} \|
+ \| \chi_n \circ T^{-k} - \chi_n \| + \| \chi_n - \chi \|
\leq k f(h_n) + \| \chi_n \circ T^{-k} - \chi_n \| + 1/2.
$$

In particular, for $k = lh_n$, $l = 1, 2, \ldots$, we have $|e^{2\pi i l h_n \alpha} - 1| < lh_n f(h_n) + 1/2$, which implies $\| h_n \alpha \| < lh_n f(h_n)/4 + 1/8$. Notice that $\| x \| = l \| x \|$ whenever $\| x \| \leq 1/2l$. Consequently, by choosing $1/4 \| h_n \alpha \| \leq l \leq 1/2 \| h_n \|$ we get on dividing by $l$ that $\| h_n \alpha \| < lh_n f(h_n)/2$. Therefore, there exist integers $p_n$ such that $|\alpha - p_n/h_n| < f(h_n)/2 = o(f(h_n))$, which ends the proof of the theorem.

It is well known that for every irrational number the rational approximation of Theorem 1 is satisfied with $f(n) = 1/n$ (and even $c_n/n^2$ for any $c_n \to \infty$), so every irrational rotation admits good cyclic approximation (see [2]). It is also clear that if $\alpha$ has unbounded partial quotients then we may take $f(n) = 1/n^2$. An even better approximation is possible for almost all numbers $\alpha$ (which also form a residual subset of the unit interval).

Let $g(x) > 0$ for $x > 0$ be such that the function $xg(x)$ is nonincreasing and $\int_0^\infty g(x) \, dx = \infty$. Then the inequality $|\alpha - p/q| < g(q)/q$ has infinitely many solutions for a.e. $\alpha$ (see [6, Theorem 32]). Consequently, the conditions of Theorem 1 are satisfied if $g(n)/n = o(f(n))$. In particular, if $g(x) = (x \log x \log \log x)^{-1}$ for $x > 3$ then $g(n)/n = o(1/n^2 \log n)$. We have obtained the following

**Corollary.** For a.e. $\alpha$ the rotation $T$ admits cyclic approximation with speed $o(1/n^2 \log n)$.

### 3. Application

Let $G$ denote a compact metrizable monothetic group with normalized Haar measure. If $T$ is an invertible measure-preserving transformation of $X$ and $\phi : X \to G$ a measurable function (called cocycle), we can define a skew product extension $T_\phi$ of $T$ acting on the product space by $T_\phi(x, g) = (Tx, g + \phi(x))$. Denote by $\Phi$ the space of all cocycles endowed with topology of convergence in measure (cocycles that are equal a.e. are identified). It has been shown in [3] that if $T$ admits an approximation with speed $o(f(n))$ where $\sup f(n)/f(2n) < \infty$ then, for generic $\phi$, the same is true of $T_\phi$. Here “generic” means from a residual subset of $\Phi$. In particular, $T_\phi$ is generically rank-1 if $f(n) = 1/n$ (similar results on simple spectrum have been obtained earlier by Robinson [7, 8]).
In the case where $X = G = T$, the circle group, and $T$ an irrational rotation of $T$, $T_\phi$ is referred to as Anzai skew product; in multiplicative notation

$$T_\phi(z, w) = (e^{2\pi i \alpha} z, \phi(z)w).$$

The following result, which is now a consequence of Theorem 1 and Corollary, is an improved version of Corollary 2 in [3].

**Theorem 2.** For every irrational rotation generic Anzai skew products have good cyclic approximation. In particular, they are rank-1. Moreover, for a.e. rotation a generic Anzai skew product admits cyclic approximation with speed $o(1/n^2 \log n)$.

**REFERENCES**


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