PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 121, Number 3, July 1994

CYCLIC APPROXIMATION OF IRRATIONAL ROTATIONS

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(Communicated by Andrew Bruckner)

ABSTRACT. We prove that an irrational number α admits a rational approximation $|\alpha - p/q| = o(f(q))$ iff the irrational rotation $Tx = \{x + \alpha\}$ admits cyclic approximation with speed $o(f(n))$. As an application to earlier results we obtain that a generic Anzai skew product over every irrational rotation is rank-1 and for a.e. α most skew products admit cyclic approximation with speed $o(1/n^2 \log n)$.

1. INTRODUCTION

Let (X, μ) be a Lebesgue space. If ξ_n is a sequence of (finite measurable) partitions then we write $\xi_n \to \varepsilon$ if for any measurable set A there exists a union A_n of cells of ξ_n such that $\mu(A \triangle A_n) \to 0$.

Let $0 < f(n) \to 0$. According to Katok and Stepin [5] we say that an invertible measure-preserving transformation T of X admits a *cyclic approximation with speed* $o(f(n))$ if there exists a sequence of partitions $\xi_n \to \varepsilon$ and a transformation T_n cyclically permuting the cells of ξ_n such that

$$\sum_{j=0}^{h_n-1} \mu(TC_j \triangle T_n C_j) = o(f(h_n)),$$

where $\xi_n = \{C_0, \dots, C_{h_n-1}\}$, $C_j = T_n^j C_0$.

It is well known that cyclic approximation with speed $o(1/n)$, called *good cyclic approximation*, implies rigidity, singular simple spectrum (see [5, 4]), and rank 1 of T.

As was pointed out in [5, §1, 1) and 2)] for an a.e. irrational number α the irrational rotation $Tx = \{x + \alpha\}$ (where $\{x\}$ denotes the fractional part of x) of the unit interval $[0, 1)$ admits a good cyclic approximation. In fact, it is easy to see that if $|\alpha - p_n/q_n| = o(1/q_n^2)$ then $T_n x = \{x + p_n/q_n\}$ and $C_0 = [0, 1/q_n)$ will do.

It is our aim to prove that the speed of cyclic approximation of the rotation T is essentially as good as the speed of rational approximation of α (see Theorem 1). In particular, every irrational rotation admits a good cyclic approximation (this was observed earlier by del Junco, see [2]), while for a.e. α an admissible

Received by the editors August 7, 1992.
1991 *Mathematics Subject Classification*. Primary 28D05.

speed is $o(1/n^2 \log n)$ (see Corollary). Our argument is similar to that in del Junco [1]; now, however, we have to ensure an error term of order $o(f(h_n))$. Instead of using cyclic partitions for rational rotations we exploit properties of continued fractions and use two disjoint T-stacks covering the interval.

2. Cyclic approximation

Let α be an irrational number. We denote by p_n/q_n its continued fraction approximation. It is well known that $|\alpha - p_n/q_n| < 1/q_n^2$. Moreover,

$$q_{n+1}\|q_n\alpha\| + q_n\|q_{n+1}\alpha\| = 1,$$

where $\|x\|$ denotes the distance from x to the nearest integer. This formula reflects splitting the interval into two Rokhlin towers ζ_n and ζ_n' for the irrational rotation $Tx = \{x+\alpha\}$. More precisely, if n is even then ζ_n, the large tower, consists of the sets $J_j = [\{j\alpha\}, \{(q_n + j)\alpha\})$, $j = 0, \dots, q_{n+1} - 1$, while ζ_n', the small one, consists of $J_0' = [\{q_{n+1}\alpha\}, 1)$ and $J_j' = [\{(q_{n+1} + j)\alpha\}, \{j\alpha\})$, $j = 1, \dots, q_n - 1$.

Theorem 1. *Let $0 < f(n) \to 0$. The irrational rotation $Tx = \{x + \alpha\}$ admits cyclic approximation with speed $o(f(n))$ iff there exists a sequence of rational numbers $p/q \to \alpha$ such that*

$$|\alpha - p/q| = o(f(q)).$$

Proof. To prove the "if" part it clearly suffices to construct a sequence of Rokhlin towers $\tilde{\eta}_n \to \varepsilon$ of height h_n, such that the complement of the tower $\tilde{\eta}_n$ has measure $o(f(h_n))$. We are going to select a subsequence m_n and then construct $\tilde{\eta}_n$ out of the pair ζ_{m_n}, ζ_{m_n}' by forming a new base from levels of the two towers.

Without loss of generality we may assume that the p/q are continued fraction convergents of α and $qf(q) \to 0$. Denote $M(q) = [1/qf(q)]$. We start by choosing an arbitrary sequence ε_n decreasing to zero. By compactness, we can find integers K_n such that for every $x \in [0, 1)$ there is a j with $0 \le j < K_n$ and $T^j x \in [0, \varepsilon_n/2)$. Now choose N_n such that $K_n/N_n < 1/n$. It is clear that for k_n large enough we have

$$N_n M(q_{k_n})\|q_{k_n}\alpha\| < \varepsilon_n/2$$

and $K_n < q_{k_n}$. Finally choose m_n satisfying

$$M(q_{k_n})q_{k_n}^2/q_{m_n} < 1/n.$$

For the sake of convenience we may assume that the numbers k_n, m_n are even and the sequences k_n, m_n are increasing.

Now we form two new towers η_n, η_n' of height q_{k_n}. We construct the base A of η_n by selecting certain levels of ζ_{m_n}. The first group of $N_n M(q_{k_n})$ many levels are

$$J_0, J_{q_{k_n}}, \dots, J_{(N_n M(q_{k_n})-1)q_{k_n}}.$$

Each of them is clearly contained in $[0, \varepsilon_n)$. Now by the choice of K_n we can find s_1 with $N_n M(q_{k_n})q_{k_n} \le s_1 < N_n M(q_{k_n})q_{k_n} + K_n$ such that $J_{s_1} \subset [0, \varepsilon_n/2)$. The next group of selected levels are

$$J_{s_1}, J_{s_1+q_{k_n}}, \dots, J_{s_1+(N_n M(q_{k_n})-1)q_{k_n}}.$$

We continue up the tower in the same manner until we define the last group,

$$J_{s_p}, J_{s_p+q_{k_n}}, \dots, J_{s_p+(M-1)q_{k_n}},$$

where $1 \le M \le N_n M(q_{k_n})$, with at least $q_{k_n} - 1$ but less than $2q_{k_n} + K_n - 2 < 3q_{k_n} - 2$ top levels of ζ_{m_n} left over. The set

$$\begin{aligned} A = J_0 &\cup J_{q_{k_n}} \cup \dots \cup J_{(N_n M(q_{k_n})-1)q_{k_n}} \\ &\cup \dots \cup J_{s_1} \cup J_{s_1+q_{k_n}} \cup \dots \cup J_{s_1+(N_n M(q_{k_n})-1)q_{k_n}} \\ &\cup \dots \cup J_{s_p} \cup J_{s_p+q_{k_n}} \cup \dots \cup J_{s_p+(M-1)q_{k_n}} \end{aligned}$$

is the base of η_n. This new tower is contained entirely in ζ_{m_n} and only a small part of measure δ_n of the tower is not covered by η_n. Since less than K_n levels have been skipped over every $N_n M(q_{k_m})q_{k_n}$th step and less than $2q_{k_n}$ levels have been left on top, we obtain

$$\delta_n < \frac{K_n}{N_n M(q_{k_n})q_{k_n}} + \frac{2q_{k_n}}{q_{m_n+1}}.$$

Moreover, we have $A \subset [0, \varepsilon_n)$ by the choice of k_n.

We repeat the construction to form η'_n. By selecting levels as before we obtain the base

$$\begin{aligned} A' = J'_0 &\cup J'_{q_{k_n}} \cup \dots \cup J'_{(N_n M(q_{k_n})-1)q_{k_n}} \\ &\cup \dots \cup J'_{s'_1} \cup J'_{s'_1+q_{k_n}} \cup \dots \cup J'_{s'_1+(N_n M(q_{k_n})-1)q_{k_n}} \\ &\cup \dots \cup J'_{s'_r} \cup J'_{s'_r+q_{k_n}} \cup \dots \cup J'_{s'_r+(M'-1)q_{k_n}}. \end{aligned}$$

Clearly $A' \subset (-\varepsilon_n, \varepsilon_n)$; η'_n is contained in ζ'_{m_n} and covers it up to

$$\delta'_n < \frac{K_n}{N_n M(q_{k_n})q_{k_n}} + \frac{2q_{k_n}}{q_{m_n}}.$$

Finally we form a single tower $\tilde{\eta}_n$ by joining η_n with η'_n. Its base is $\widetilde{A} = A \cup A'$, and the height is q_{k_n}. It covers the space up to

$$\tilde{\delta}_n = \delta_n + \delta'_n < \frac{6}{nM(q_{k_n})q_{k_n}} = o(f(q_{k_n})).$$

Moreover, $\widetilde{A} \subset (-\varepsilon_n, \varepsilon_n)$, so each level of $\tilde{\eta}_n$ has diameter at most $2\varepsilon_n$ and consequently $\tilde{\eta}_n \to \varepsilon$, which ends the proof of sufficiency.

To prove the "only if" part assume there exist partitions

$$\xi_n = \{C_0, \dots, C_{h_n-1}\} \to \varepsilon$$

and cyclic approximations T_n of T such that

$$\sum_{j=0}^{h_n-1} \mu(TC_j \triangle T_n C_j) < \tilde{f}(h_n),$$

where $\tilde{f}(n) = o(f(n))$. Let $\chi(x) = e^{2\pi i x}$ on $[0, 1)$. As $\xi_n \to \varepsilon$, we can find, for any n large enough, an approximation $\chi_n = \sum_{j=0}^{h_n-1} \lambda_j 1_{C_j}$ of χ, where $|\lambda_j| = 1$ and $\|\chi_n - \chi\| < 1/4$ in L^1. Observe that

$$\|\chi_n \circ T_n^{-1} - \chi_n \circ T^{-1}\| = \left\|\sum \lambda_j (1_{T_n C_j} - 1_{TC_j})\right\| \le \sum \mu(T_n C_j \triangle TC_j) < \tilde{f}(h_n).$$

For every $k \geq 1$ the same argument applies to the function $\chi_n \circ T_n^{-k}$ which is also of the form $\sum \lambda_j^{(k)} 1_{C_j}$, so we get by induction

$$\begin{aligned}\|\chi_n \circ T_n^{-k} - \chi_n \circ T^{-k}\| &\leq \|(\chi_n \circ T_n^{-k+1}) \circ T_n^{-1} - (\chi_n \circ T_n^{-k+1}) \circ T^{-1}\| \\ &\quad + \|\chi_n \circ T_n^{-k+1} - \chi_n \circ T^{-k+1}\| \\ &< \tilde{f}(h_n) + (k-1)\tilde{f}(h_n) = k\tilde{f}(h_n).\end{aligned}$$

Now by the triangle inequality we obtain

$$\begin{aligned}|e^{2\pi i k\alpha} - 1| &= \|\chi \circ T^{-k} - \chi\| \\ &\leq \|\chi \circ T^{-k} - \chi_n \circ T^{-k}\| + \|\chi_n \circ T^{-k} - \chi_n \circ T_n^{-k}\| \\ &\quad + \|\chi_n \circ T_n^{-k} - \chi_n\| + \|\chi_n - \chi\| \\ &< k\tilde{f}(h_n) + \|\chi_n \circ T_n^{-k} - \chi_n\| + 1/2.\end{aligned}$$

In particular, for $k = lh_n$, $l = 1, 2, \ldots$, we have $|e^{2\pi i l h_n \alpha} - 1| < lh_n\tilde{f}(h_n) + 1/2$, which implies $\|lh_n\alpha\| < lh_n\tilde{f}(h_n)/4 + 1/8$. Notice that $\|lx\| = l\|x\|$ whenever $\|x\| \leq 1/2l$. Consequently, by choosing $1/4\|h_n\alpha\| \leq l \leq 1/2\|h_n\|$ we get on dividing by l that $\|h_n\alpha\| < h_n\tilde{f}(h_n)/2$. Therefore, there exist integers p_n such that $|\alpha - p_n/h_n| < \tilde{f}(h_n)/2 = o(f(h_n))$, which ends the proof of the theorem.

It is well known that for every irrational number the rational approximation of Theorem 1 is satisfied with $f(n) = 1/n$ (and even c_n/n^2 for any $c_n \to \infty$), so every irrational rotation admits good cyclic approximation (see [2]). It is also clear that if α has unbounded partial quotients then we may take $f(n) = 1/n^2$. An even better approximation is possible for almost all numbers α (which also form a residual subset of the unit interval).

Let $g(x) > 0$ for $x > 0$ be such that the function $xg(x)$ is nonincreasing and $\int_0^\infty g(x)\,dx = \infty$. Then the inequality $|\alpha - p/q| < g(q)/q$ has infinitely many solutions for a.e. α (see [6, Theorem 32]). Consequently, the conditions of Theorem 1 are satisfied if $g(n)/n = o(f(n))$. In particular, if $g(x) = (x \log x \log\log x)^{-1}$ for $x > 3$ then $g(n)/n = o(1/n^2 \log n)$. We have obtained the following

Corollary. *For a.e.* α *the rotation* T *admits cyclic approximation with speed* $o(1/n^2 \log n)$.

3. Application

Let G denote a compact metrizable monothetic group with normalized Haar measure. If T is an invertible measure-preserving transformation of X and $\phi: X \to G$ a measurable function (called *cocycle*), we can define a skew product extension T_ϕ of T acting on the product space by $T_\phi(x, g) = (Tx, g + \phi(x))$. Denote by Φ the space of all cocycles endowed with topology of convergence in measure (cocycles that are equal a.e. are identified). It has been shown in [3] that if T admits an approximation with speed $o(f(n))$ where $\sup f(n)/f(2n) < \infty$ then, for generic ϕ, the same is true of T_ϕ. Here "generic" means from a residual subset of Φ. In particular, T_ϕ is generically rank-1 if $f(n) = 1/n$ (similar results on simple spectrum have been obtained earlier by Robinson [7, 8]).

In the case where $X = G = \mathbf{T}$, the circle group, and T an irrational rotation of $\mathbf{T}$, T_ϕ is referred to as *Anzai skew product*; in multiplicative notation

$$T_\phi(z, w) = (e^{2\pi i\alpha} z, \phi(z)w).$$

The following result, which is now a consequence of Theorem 1 and Corollary, is an improved version of Corollary 2 in [3].

Theorem 2. *For every irrational rotation generic Anzai skew products have good cyclic approximation. In particular, they are rank-*1. *Moreover, for a.e. rotation a generic Anzai skew product admits cyclic approximation with speed* $o(1/n^2 \log n)$.

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