ACCP IN POLYNOMIAL RINGS: A COUNTEREXAMPLE

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Abstract. We describe an example showing that ACCP need not extend from a ring to a polynomial ring over it.

A ring $R$ (always commutative with unity) is said to satisfy the ascending chain condition on principal ideals (ACCP) if, for any infinite ascending chain of principal ideals $a_1R \subseteq a_2R \subseteq \cdots$, there is a positive integer $n$ for which $a_nR = a_{n+1}R = \cdots$ (cf. [Gi]). This property has been studied, even in the case of a noncommutative ring, in a number of papers, for example, [AAZ, AN, Gr, N, R]. It is well known and easy to see that if $R$ is an integral domain satisfying ACCP, then for any family $X$ of indeterminates, the polynomial ring $R[X]$ also satisfies ACCP. (In [Gr], where it seems to be asserted that this holds for any ring, there appears to be a tacit hypothesis of domain.)

Example. A ring $R$ that satisfies ACCP but for which the polynomial ring $R[x]$, in a single indeterminate $x$, does not satisfy ACCP. Let $k$ be a field and $A_1, A_2, \ldots$ be indeterminates over $k$, and set

$$S = k[A_1, A_2, \ldots] / (\{A_n(A_{n-1} - A_n) : n \geq 2\})k[A_1, A_2, \ldots].$$

Denote by $a_n$ the image of $A_n$ in $S$ and by $R$ the localization of $S$ at the ideal $(a_1, a_2, \ldots)S$. We note two facts about these rings: (1) the elements of $S$ that become units in $R$ are nonzerodivisors, so $R$ contains (an isomorphic copy of) $S$; and (2) in $S$, no power of $a_{n-1}$ annihilates the difference $a_n - a_{n-1}$. For (1), note that, as the factor ring of a polynomial ring over $k$ by a homogeneous ideal (in total degree in the $A_n$'s), $S$ is a graded ring. Thus, we can refer to the order of an element of $S$, i.e., the degree of the smallest-degree nonzero term in that element; and for elements $f, g$ in $S$, $\text{ord}(fg) \geq \text{ord}(f) + \text{ord}(g)$. Since an element outside $(a_1, a_2, \ldots)S$ has unit degree-0 term, its product with any
nonzero element \( g \) of \( S \) has the same order as \( g \); in particular, the product is not zero.  

For (2), we regard \( S \) as the limit of the rings \( S_n \) where \( S_1 = k[a_1] \) and  
\[
S_n = S_{n-1}[a_n] = S_{n-1}[A_n]/A_n(a_{n-1} - A_n)S_{n-1}[A_n]
\]
for \( n \geq 2 \). Since \( A_n(a_{n-1} - A_n) \) is the negative of a monic polynomial of degree 2, \( S_n \) is a free module over \( S_{n-1} \) on the generators \( 1, a_n \). Thus, \( S_{n-1} \) is a subring of \( S_n \), and a power of \( a_{n-1} \) annihilates \( a_{n-1} - a_n \) only if \( a_{n-1} \) is nilpotent. Assume \( a_{n-1}^m = 0 \) where, without loss of generality, \( m > n - 2 \). Then, using the defining relations of \( S \), we have \( a_{n-1}^m = a_1^{m-n+2}a_2a_3\cdots a_{n-1} \). Since each \( a_k \) is an element of a free basis over \( S_{k-1} \), we conclude that \( a_1 \) is nilpotent. But that is a contradiction, since \( S_1 \) is (isomorphic to) the polynomial ring in the indeterminate \( a_1 \) over \( k \).

To see that \( R \) satisfies ACCP, we again use the grading on \( S \): Suppose we have \( f_1R \subset f_2R \), where \( f_1, f_2 \) are chosen from \( S \). Then since elements of \( S \) of order 0 are units in \( R \), we must have elements \( g, h \) in \( S \) for which \( \text{ord}(g) > 0, \text{ord}(h) = 0 \), and \( f_2(g/h) = f_1 \) in \( R \), so that in \( S \), using (1) above, we have \( f_1h = f_2g \). It follows that \( \text{ord}(f_1) > \text{ord}(f_2) \). Since orders in \( S \) are bounded below by 0, it follows that \( R \) satisfies ACCP.

Now in \( R[x] \), we have  
\[
(a_nx + 1)((a_{n-1} - a_n)x + 1) = a_{n-1}x + 1
\]
for each \( n \geq 2 \), so  
\[
(a_1x + 1)R[x] \subseteq (a_2x + 1)R[x] \subseteq \cdots.
\]

To see that these containments are proper, suppose by way of contradiction that, for some \( n \geq 2 \), \( (a_{n-1}x + 1)b(x) = a_nx + 1 \) where \( b(x) \) is the polynomial \( b(x) = b_0 + b_1x + \cdots \). Then we must have \( b_0 = 1 \) and by induction \( b_m = (-1)^{m-1}a_{n-1}^{m-1}(a_n - a_{n-1}) \) for each \( m > 0 \). By (2), all the \( b_m \) are nonzero, so \( b(x) \) is not a polynomial, which is the desired contradiction.

Remark. It is shown in [HL, (3.8) and (3.9)] that if \( R \) is a quasilocal ring having the property that the annihilator of each finitely generated ideal in \( R \) has only finitely many minimal primes or if \( R \) is of dimension zero, then ACCP extends to a polynomial ring over \( R \). The \( R \) in the example above is quasilocal of dimension one and has a countably infinite number of minimal primes. Moreover, by using [HL, Proposition 2.1] it follows that \( R \) satisfies the ascending chain condition on \( n \)-generated ideals for every positive integer \( n \) (i.e., \( R \) has “pan-acc”).

REFERENCES


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