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ACCP IN POLYNOMIAL RINGS: A COUNTEREXAMPLE

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ABSTRACT. We describe an example showing that ACCP need not extend from a ring to a polynomial ring over it.

A ring R (always commutative with unity) is said to satisfy the *ascending chain condition on principal ideals* (ACCP) if, for any infinite ascending chain of principal ideals $a_1R \subseteq a_2R \subseteq \cdots$, there is a positive integer n for which $a_nR = a_{n+1}R = \cdots$ (cf. [Gi]). This property has been studied, even in the case of a noncommutative ring, in a number of papers, for example, [AAZ, AN, Gr, N, R]. It is well known and easy to see that if R is an integral domain satisfying ACCP, then for any family X of indeterminates, the polynomial ring $R[X]$ also satisfies ACCP. (In [Gr], where it seems to be asserted that this holds for any ring, there appears to be a tacit hypothesis of domain.)

Example. A ring R that satisfies ACCP but for which the polynomial ring $R[x]$, in a single indeterminate x, does not satisfy ACCP. Let k be a field and $A_1, A_2, \ldots$ be indeterminates over k, and set

$$S = k[A_1, A_2, \ldots]/(\{A_n(A_{n-1} - A_n) : n \geq 2\})k[A_1, A_2, \ldots] .$$

Denote by a_n the image of A_n in S and by R the localization of S at the ideal $(a_1, a_2, \ldots)S$. We note two facts about these rings: (1) the elements of S that become units in R are nonzerodivisors, so R contains (an isomorphic copy of) S; and (2) in S, no power of a_{n-1} annihilates the difference $a_{n-1} - a_n$. For (1), note that, as the factor ring of a polynomial ring over k by a homogeneous ideal (in total degree in the A_n's), S is a graded ring. Thus, we can refer to the order of an element of S, i.e., the degree of the smallest-degree nonzero term in that element; and for elements f, g in S, $\operatorname{ord}(fg) \geq \operatorname{ord}(f) + \operatorname{ord}(g)$. Since an element outside $(a_1, a_2, \ldots)S$ has unit degree-0 term, its product with any

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nonzero element g of S has the same order as g; in particular, the product is not zero.

For (2), we regard S as the limit of the rings S_n where $S_1 = k[a_1]$ and

$$S_n = S_{n-1}[a_n] = S_{n-1}[A_n]/A_n(a_{n-1} - A_n)S_{n-1}[A_n]$$

for $n \geq 2$. Since $A_n(a_{n-1} - A_n)$ is the negative of a monic polynomial of degree 2, S_n is a free module over S_{n-1} on the generators $1, a_n$. Thus, S_{n-1} is a subring of S_n, and a power of a_{n-1} annihilates $a_{n-1} - a_n$ only if a_{n-1} is nilpotent. Assume $a_{n-1}^m = 0$ where, without loss of generality, $m > n - 2$. Then, using the defining relations of S, we have $a_{n-1}^m = a_1^{m-n+2}a_2a_3 \cdots a_{n-1}$. Since each a_k is an element of a free basis over S_{k-1}, we conclude that a_1 is nilpotent. But that is a contradiction, since S_1 is (isomorphic to) the polynomial ring in the indeterminate a_1 over k.

To see that R satisfies ACCP, we again use the grading on S: Suppose we have $f_1R \subset f_2R$, where f_1, f_2 are chosen from S. Then since elements of S of order 0 are units in R, we must have elements g, h in S for which $\operatorname{ord}(g) > 0$, $\operatorname{ord}(h) = 0$, and $f_2(g/h) = f_1$ in R, so that in S, using (1) above, we have $f_1h = f_2g$. It follows that $\operatorname{ord}(f_1) > \operatorname{ord}(f_2)$. Since orders in S are bounded below by 0, it follows that R satisfies ACCP.

Now in $R[x]$, we have

$$(a_nx + 1)((a_{n-1} - a_n)x + 1) = a_{n-1}x + 1$$

for each $n \geq 2$, so

$$(a_1x + 1)R[x] \subseteq (a_2x + 1)R[x] \subseteq \cdots .$$

To see that these containments are proper, suppose by way of contradiction that, for some $n \geq 2$, $(a_{n-1}x+1)b(x) = a_nx+1$ where $b(x)$ is the polynomial $b(x) = b_0 + b_1x + \cdots$. Then we must have $b_0 = 1$ and by induction $b_m = (-1)^{m-1}a_{n-1}^{m-1}(a_n - a_{n-1})$ for each $m > 0$. By (2), all the b_m are nonzero, so $b(x)$ is not a polynomial, which is the desired contradiction.

Remark. It is shown in [HL, (3.8) and (3.9)] that if R is a quasilocal ring having the property that the annihilator of each finitely generated ideal in R has only finitely many minimal primes or if R is of dimension zero, then ACCP extends to a polynomial ring over R. The R in the example above is quasilocal of dimension one and has a countably infinite number of minimal primes. Moreover, by using [HL, Proposition 2.1] it follows that R satisfies the ascending chain condition on n-generated ideals for every positive integer n (i.e., R has "pan-acc").

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