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A SHARP ESTIMATE ON THE BERGMAN KERNEL OF A PSEUDOCONVEX DOMAIN

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ABSTRACT. In this note we obtain a sharp estimate of the Bergman kernels near $\mathscr{C}^2$ boundary points of pseudoconvex domains by induction on the dimension and a theorem of Ohsawa-Takegoshi.

Let Ω be a bounded domain in $\mathbb{C}^n$, and let $A^2(\Omega)$ be the set of holomorphic functions on Ω in $L^2(\Omega)$. The Bergman kernel function of Ω on the diagonal can be defined by

$$K_\Omega(z, \bar{z}) = \sup\{|f(z)|^2;\ f \in A^2(\Omega),\ \|f\|_2 \leq 1\}.$$

For general properties of the Bergman kernel, we refer the readers to Bergman's book [B]. The following theorem was first proved by Pflug [P]:

Theorem (Pflug). *If Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$ with $\mathscr{C}^2$ boundary near $p \in \partial\Omega$, then for each $\varepsilon > 0$, there exist a constant $C > 0$ and a neighborhood U of p such that*

$$K_\Omega(z, \bar{z}) \geq C \frac{1}{d^{2-\varepsilon}(z)}$$

for all $z \in U \cap \Omega$, where $d(z) = \operatorname{dist}(z, \partial\Omega)$.

A generalization of this theorem was given by Catlin [C, Lemma 1]. The proofs given by Pflug and Catlin were based on the deep results of Skoda [S] on the L^2 estimate of $\bar{\partial}$ solutions.

In this note, we show that the ε in the previous theorem can be eliminated. This result is implicit in the work of Ohsawa-Takegoshi. The purpose of this note is to draw attention to it.

Theorem. *If Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$ with $\mathscr{C}^2$ boundary near $p \in \partial\Omega$, then there exist a constant $C > 0$ and a neighborhood U of p such that*

$$K_\Omega(z, \bar{z}) \geq C \frac{1}{d^2(z)} \quad \textit{for all } z \in U \cap \Omega.$$

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We shall prove the theorem by induction on the dimension and the deep theorem of Ohsawa-Takegoshi [O-T, p. 197].

Proof of the Theorem. Let U be a tubular neighborhood of $\partial\Omega$ near p such that for all $z \in U \cap \Omega$ there exists unique projection to the boundary $\pi(z)$ such that $d(z) = \operatorname{dist}(z, \pi(z))$. Let $n_{\pi(z)}$ be the outward normal direction at $\pi(z)$. We may assume that Ω is disjoint from the ball with center at $\pi(z) + \delta n_{\pi(z)}$ and radius δ for some $\delta > 0$ independent of $z \in U \cap \Omega$.

Now for each $z_0 \in U \cap \Omega$, after a translation and a unitary transformation, we may assume that $\pi(z_0)$ is the origin and $\operatorname{Re} z_1$-axis is the outward normal direction at $\pi(z_0)$. Hence $z_0 = (-d(z_0), 0, \dots, 0)$. If we set $H_1 = \{(z_1, z_2, \dots, z_n) \in \mathbb{C}^n; z_n = 0\}$, then there exists $f \in A^2(\Omega \cap H_1)$ with $\|f\|_2 \le 1$ such that $K_{\Omega\cap H_1}(z_0, \overline{z}_0) = |f(z_0)|^2$. By the above-mentioned theorem of Ohsawa-Takegoshi, there exists $F \in A^2(\Omega)$ such that $\|F\|_2^2 \le C_1$, $F|_{\Omega\cap H_1} = f$ for some constant $C_1 > 0$ depending only on the diameter of Ω. Therefore,

$$K_\Omega(z_0, \overline{z}_0) \ge \frac{1}{C_1} K_{\Omega\cap H_1}(z_0, \overline{z}_0). \tag{1}$$

Since the component of $\Omega \cap H_1$ containing z_0 is also a pseudoconvex domain, by induction on the dimension, we obtain

$$K_\Omega(z_0, \overline{z}_0) \ge \frac{1}{C_1'} K_{\Omega\cap H_{n-1}}(z_0, \overline{z}_0), \tag{2}$$

for some $C_1' > 0$ depending only on diameter of Ω, where $H_{n-1} = \{(z_1, z_2, \dots, z_n) \in \mathbb{C}^n; z_2 = z_3 = \cdots = z_n = 0\}$. However, by our choices of the neighborhood U and the coordinate system, we know that $\Omega \cap H_{n-1} \subset D_\delta = \{(z_1, z_2, \dots, z_n) \in \mathbb{C}^n; |z_1 - \delta| > \delta, z_2 = \cdots = z_n = 0\}$. Therefore,

$$K_{\Omega\cap H_{n-1}}(z_0, \overline{z}_0) \ge K_{D_\delta}(z_0, \overline{z}_0). \tag{3}$$

On the other hand, it is clear that there exists $C_2 > 0$, depending only on δ such that

$$K_{D_\delta}(z_0, \overline{z}_0) \ge C_2 \frac{1}{d^2(z_0)}. \tag{4}$$

Combining (2), (3), and (4), we conclude the proof of the theorem. □

References

[B] S. Bergman, *The kernel function and conformal mapping*, Amer. Math. Soc., Providence, RI, 1970.

[C] D. Catlin, *Necessary conditions for subellipticity of the $\overline{\partial}$-Neumann problem*, Ann. of Math. (2) **117** (1983), 147–171.

[O-T] T. Ohsawa and K. Takegoshi, *On the extension of L^2 holomorphic functions*, Math. Z. **195** (1987), 197–204.

[P] P. Pflug, *Quadratintegrable holomorphische Funktionen und die Serre-Vermutung*, Math. Ann. **216** (1975), 285–288.

[S] H. Skoda, *Application des techniques L^2 à la théorie des ideaux d'un algèbre de fonctions holomorphes avec poids*, Ann. Sci. École Norm. Sup. (4) **5** (1972), 545–580.

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