A SHARP ESTIMATE ON THE BERGMAN KERNEL
OF A PSEUDOCONVEX DOMAIN

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Abstract. In this note we obtain a sharp estimate of the Bergman kernels near
$\partial^2$ boundary points of pseudoconvex domains by induction on the dimension
and a theorem of Ohsawa-Takegoshi.

Let $\Omega$ be a bounded domain in $\mathbb{C}^n$, and let $A^2(\Omega)$ be the set of holomorphic
functions on $\Omega$ in $L^2(\Omega)$. The Bergman kernel function of $\Omega$ on the diagonal
can be defined by

$$K_\Omega(z, \bar{z}) = \sup \{ |f(z)|^2 ; f \in A^2(\Omega), \|f\|_2 \leq 1 \}.$$

For general properties of the Bergman kernel, we refer the readers to
Bergman’s book [B]. The following theorem was first proved by Pflug [P]:

Theorem (Pflug). If $\Omega$ is a bounded pseudoconvex domain in $\mathbb{C}^n$ with $\partial^2$
boundary near $p \in \partial\Omega$, then for each $\varepsilon > 0$, there exist a constant $C > 0$
and a neighborhood $U$ of $p$ such that

$$K_\Omega(z, \bar{z}) \geq C \frac{1}{d^{2-\varepsilon}(z)}$$

for all $z \in U \cap \Omega$, where $d(z) = \text{dist}(z, \partial\Omega)$.

A generalization of this theorem was given by Catlin [C, Lemma 1]. The
proofs given by Pflug and Catlin were based on the deep results of Skoda [S] on
the $L^2$ estimate of $\overline{\partial}$ solutions.

In this note, we show that the $\varepsilon$ in the previous theorem can be eliminated.
This result is implicit in the work of Ohsawa-Takegoshi. The purpose of this
note is to draw attention to it.

Theorem. If $\Omega$ is a bounded pseudoconvex domain in $\mathbb{C}^n$ with $\partial^2$
boundary near $p \in \partial\Omega$, then there exist a constant $C > 0$ and a neighborhood $U$ of $p$
such that

$$K_\Omega(z, \bar{z}) \geq C \frac{1}{d^2(z)}$$

for all $z \in U \cap \Omega$. 

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We shall prove the theorem by induction on the dimension and the deep theorem of Ohsawa-Takegoshi [O-T, p. 197].

**Proof of the Theorem.** Let $U$ be a tubular neighborhood of $\partial \Omega$ near $p$ such that for all $z \in U \cap \Omega$ there exists unique projection to the boundary $\pi(z)$ such that $d(z) = \text{dist}(z, \pi(z))$. Let $n_{\pi(z)}$ be the outward normal direction at $\pi(z)$. We may assume that $\Omega$ is disjoint from the ball with center at $\pi(z) + \delta n_{\pi(z)}$ and radius $\delta$ for some $\delta > 0$ independent of $z \in U \cap \Omega$.

Now for each $z_0 \in U \cap \Omega$, after a translation and a unitary transformation, we may assume that $\pi(z_0)$ is the origin and $\text{Re} z_1$-axis is the outward normal direction at $\pi(z_0)$. Hence $z_0 = (-d(z_0), 0, \ldots, 0)$. If we set $H_1 = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n; z_n = 0\}$, then there exists $f \in A^2(\Omega \cap H_1)$ with $\|f\|_2 \leq 1$ such that $K_{\Omega \cap H_1}(z_0, \overline{z}_0) = |f(z_0)|^2$. By the above-mentioned theorem of Ohsawa-Takegoshi, there exists $F \in A^2(\Omega)$ such that $\|F\|_2^2 \leq C_1$, $F|_{\Omega \cap H_1} = f$ for some constant $C_1 > 0$ depending only on the diameter of $\Omega$. Therefore,

$$(1) \quad K_{\Omega}(z_0, \overline{z}_0) \geq \frac{1}{C_1} K_{\Omega \cap H_1}(z_0, \overline{z}_0).$$

Since the component of $\Omega \cap H_1$ containing $z_0$ is also a pseudoconvex domain, by induction on the dimension, we obtain

$$(2) \quad K_{\Omega}(z_0, \overline{z}_0) \geq \frac{1}{C'_1} K_{\Omega \cap H_{n-1}}(z_0, \overline{z}_0),$$

for some $C'_1 > 0$ depending only on diameter of $\Omega$, where $H_{n-1} = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n; z_2 = z_3 = \ldots = z_n = 0\}$. However, by our choices of the neighborhood $U$ and the coordinate system, we know that $\Omega \cap H_{n-1} \subset D_\delta = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n; |z_1 - \delta| > \delta, z_2 = \ldots = z_n = 0\}$. Therefore,

$$(3) \quad K_{\Omega \cap H_{n-1}}(z_0, \overline{z}_0) \geq K_{D_\delta}(z_0, \overline{z}_0).$$

On the other hand, it is clear that there exists $C_2 > 0$, depending only on $\delta$ such that

$$(4) \quad K_{D_\delta}(z_0, \overline{z}_0) \geq C_2 \frac{1}{d^2(z_0)}.$$

Combining (2), (3), and (4), we conclude the proof of the theorem. $\square$

**References**


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