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THE UNITABILITY OF l-PRIME LATTICE-ORDERED RINGS WITH SQUARES POSITIVE

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ABSTRACT. It is shown that an l-prime lattice-ordered ring with squares positive and an f-superunit can be embedded in a unital l-prime lattice-ordered ring with squares positive.

In [1, p. 325] Steinberg asked if an l-prime l-ring with squares positive and an f-superunit can be embedded in a unital l-prime l-ring with squares positive. In this paper we show that the answer is yes.

If R is a lattice-ordered ring (l-ring) and $a \in R^+$, then a is called an f-element of R if $b \wedge c = 0$ implies $ab \wedge c = ba \wedge c = 0$. Let $T = T(R) = \{a \in R : |a| \text{ is an } f\text{-element of } R\}$. Then T is a convex f-subring of R, and R is a subdirect product of totally ordered T-T bimodules [2, Lemma 1]. R is an f-ring precisely when $T = R$. Throughout this paper T will denote the subring of f-elements of R. An element $e \geq 0$ of an l-ring R is called a superunit if $ex \geq x$ and $xe \geq x$ for each $x \in R^+$; e is an f-superunit if it is a superunit and an f-element. R is infinitesimal if $x^2 \leq x$ for each x in R^+. The l-ring R is called l-prime if the product of two nonzero l-ideals is nonzero and l-semiprime if it has no nonzero nilpotent l-ideals. An l-ring R is called squares positive if $a^2 \geq 0$ for each a in R.

Let A be any ring and $a \in A$. If a satisfies $ab = ba = nb$ for some fixed integer n and all $b \in A$, then a is said to be an n-fier of A and n is said to have an n-fier a in A. Let $K = \{n \in Z : n \text{ has } n\text{-fiers in } A\}$. Then K is an ideal in the ring Z of integers. The ideal K is called the modal ideal of A; its nonnegative generator k is called the mode of A. If R is an f-ring with mode $k > 0$, then R has a unique k-fier $x \geq 0$ [3, III, Lemma 2.1].

Let R be an l-ring, and let $S = S(R) = \{a \in R : |a| \geq d,\ \forall d \in T^+\}$ and $T^\perp = \{a \in R : |a| \wedge d = 0,\ \forall d \in T^+\}$. It is clear that $T^\perp$ is a convex l-subgroup of R.

The referee has pointed out to us that the following lemma, on which our arguments are based, is a special case of [4, Lemma 6.2], and P. Conrad attributes it to A. H. Clifford.

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Lemma 1. *Let R be an l-ring which contains nonzero f-elements and T a totally ordered subring of R. Then $R = S \cup (T \oplus T^{\perp})$, where the direct sum is regarded as the direct sum of l-subgroups and $S \cap (T \oplus T^{\perp}) = \varnothing$.*

Proof. If $a \in R$ and $a \notin S$, then $|a| \wedge b < b$ for some $b \in T^+$. Let $a_1 = |a| - |a| \wedge b$ and $b_1 = b - |a| \wedge b$. Then $a_1 \wedge b_1 = 0$, and hence $(a_1 \wedge d) \wedge b_1 = 0$ for each $d \in T^+$. Since $(a_1 \wedge d)$, $b_1 \in T$, which is a totally ordered ring, and $b_1 > 0$, we have $a_1 \wedge d = 0$ for each $d \in T^+$, and hence $a_1 \in T^{\perp}$. Since $0 \leq a^+$, $a^- \leq |a| = a_1 + (|a| \wedge b)$, there are a_2, a_3, b_2, $b_3 \in R$ such that $0 \leq a_2$, $a_3 \leq a_1$, $0 \leq b_2$, $b_3 \leq |a| \wedge b$, $a^+ = a_2 + b_2$, $a^- = a_3 + b_3$, and $a = a^+ - a^- = (b_2 - b_3) + (a_2 - a_3)$. But T and $T^{\perp}$ are the convex l-subgroups of R, so $a \in T \oplus T^{\perp}$.

If $a \in S \cap (T \oplus T^{\perp})$, then $a = a_1 + a_2$, where $a_1 \in T$ and $a_2 \in T^{\perp}$, and hence $2|a_1| \leq |a| \leq |a_1| + |a_2|$ since $a \in S$. Thus $a_1 = 0$ and $a = a_2$, and hence $T = \{0\}$. This contradicts $T \neq \{0\}$, so $S \cap (T \oplus T^{\perp}) = \varnothing$.

Let R be an l-ring as in Lemma 1 and $a \in R$. If $a \in T \oplus T^{\perp}$, then a can be uniquely represented as the sum of elements of T and $T^{\perp}$, and we may write $a = a_T + a_{T^{\perp}}$, where $a_T \in T$ and $a_{T^{\perp}} \in T^{\perp}$ are respectively called the components of a in T and in $T^{\perp}$. It is clear that $a \vee 0 = (a_T \vee 0) + (a_{T^{\perp}} \vee 0)$.

Throughout this paper the following fact is used frequently. Let R be an l-ring and e an f-superunit of R. Then, for each $x \in R$, $x \geq 0$ if and only if $ex \geq 0$ or $xe \geq 0$. More generally, if $0 < a \in T$ is a regular element (or a non-zero-divisor), then, for each $x \in R$, $x \geq 0$ if and only if $ax \geq 0$ or $xa \geq 0$.

Lemma 2. *Let R be an l-ring with an f-superunit and T a totally ordered ring. Let e be an f-superunit of R.*

(a) *If $0 < b \in S$, then $ne + be \geq 0$ for each $n \in Z$.*
(b) *If $b \in S$ and $me + be \geq 0$ for some $m \in Z$, then $b > 0$.*

Proof. (a) Since $0 < b \in S$, we have $-ne \leq |n|e \leq |n|e^2 \leq be$ for each $n \in Z$.

(b) Let $me + be \geq 0$ for some $m \in Z$. If $m \leq 0$, then $be \geq -me \geq 0$, so $be = |be| = |b|e$; that is, $(|b| - b)e = 0$. But e is an f-superunit, so $b = |b| > 0$. If $m > 0$, then $me \geq -be$, and hence $me \geq (-be) \vee 0 = (-b \vee 0)e \geq -b \vee 0$, so $-b \vee 0 \in T$. Since $b \vee 0 = b + (-b \vee 0)$ and $b \in S$, we have $b \vee 0 \in S$ by Lemma 1. Thus $b + (-b \vee 0) = b \vee 0 > -b \vee 0$; that is, $b > 0$.

Lemma 3. *Let R be an l-ring with an f-superunit and T a totally ordered ring. If the mode of R is n and the mode of T is k, then $n = k$ and the k-fier of R equals the k-fier of T.*

Proof. Let e be an f-superunit of R. If x is a k-fiber of T, then $kd = dx = xd$ for each $d \in T$, especially $ke = ex = xe$. Thus $e(ka) = exa$ and $(ka)e = axe$ for each $a \in R$. Since e is an f-superunit of R, $ka = xa = ax$ for each $a \in R$, so $n | k$. Let y be an n-fier of R. Then $na = ay = ya$ for each $a \in R$. By Lemma 1 we have $R = S \cup (T \oplus T^{\perp})$. If $y \in S$, then since e is an f-superunit of R and $ne = ey = ye$, we have $ne = |ne| = |ey| = e|y| \geq |y| \geq (n+1)e$. This is a contradiction, so $y \in T \oplus T^{\perp}$. Let $y = y_T + y_{T^{\perp}}$. Then $ne = ey = ey_T + ey_{T^{\perp}}$, and hence $ne - ey_T = ey_{T^{\perp}} \in T \cap T^{\perp} = \{0\}$, so $y_{T^{\perp}} = 0$ and $y = y_T \in T$. Thus $k | n$, and we have $k = n$.

Lemma 4. *Let R be an l-ring which has an f-superunit and T a totally ordered ring. If e is an f-superunit of R and $n \in Z$, $b \in T \oplus T^{\perp}$ satisfy $ne + be \geq 0$, then $b_{T^{\perp}} \geq 0$ and $ne + b_T e \geq 0$.*

Proof. If $ne + be \geq 0$, then $(ne + b_T e) + b_{T^{\perp}} e \geq 0$. Thus $ne + b_T e \geq 0$ and $b_{T^{\perp}} e \geq 0$, so $b_{T^{\perp}} \geq 0$.

Lemma 5. *Let R be an l-ring with squares positive and $a \in T^{\perp}$ or S. Then $d|a| \leq a^2$ and $|a|d \leq a^2$ for each $d \in T^{\perp}$. If R has an f-superunit, then $n|a| \leq a^2$ for each $n \in Z^+$.*

Proof. Since R is squares positive, $0 \leq (a \pm b)^2$ yields $|ad + da| \leq a^2 + d^2$. But R is a T-T f-bimodule, and $|ad| + |da| = |ad + da|$ holds in any totally ordered T-T bimodule which is a homomorphic image of R since $0 \leq d \in T$; so it also holds in R [2, Lemma 4]. If $a \in T^{\perp}$, then $|a|d = |ad| \leq |ad| + |da| = |ad + da| \leq a^2 + d^2$ and $|a| \wedge d = 0$ imply $|a|d = |a|d \wedge (a^2 + d^2) \leq (|a|d \wedge a^2) + (|a|d \wedge d^2) = |a|d \wedge a^2 \leq a^2$. Similarly, $d|a| \leq a^2$. If $a \in S$, then $d \leq |a|$ for each $d \in T^+$, and hence $|a|d + d|a| = |ad| + |da| = |ad + da| \leq a^2 + d^2 \leq a^2 + d|a|$. Thus $|a|d \leq a^2$. Similarly, $d|a| \leq a^2$. If e is an f-superunit of R, then, as in the previous paragraph, $n|a| \leq ne|a| \leq a^2$ for each $n \in Z^+$.

Theorem 1. *An l-prime l-ring R with squares positive and an f-superunit can be embedded in a unit l-prime l-ring with squares positive.*

Proof. We first note that R is a domain [2, Theorem 1], and hence T is a totally ordered ring. Thus $R = S \cup (T \oplus T^{\perp})$ by Lemma 1. Let e be an f-superunit of R.

Let $\bar{R}$ be the ring obtained by freely adjoining the integers to R, and let k be the mode of T and x the unique k-fier of T. Then k is the mode of R and x the unique k-fier of R by Lemma 3. Let $I(k, x)$ be the set of all integral multiples of $(k, -x) \in \bar{R}$. Then $I(k, x)$ is an ideal of $\bar{R}$. Denote the ring $\bar{R}/I(k, x)$ by R' which has identity $(\overline{1, 0})$. Since R is a domain, it is well known that R' is a domain. Consider the subset $(R')^+$ of R' defined by $(\overline{n, a}) \in (R')^+$ if and only if $nc + ac \geq 0$ for all $c \in R^+$, which is equivalent to $ne + ae \geq 0$. Since $nc + ac \geq 0$ if and only if $(n + mk)c + (a - mx)c \geq 0$ for each $m \in Z$, $(R')^+$ is well defined. It is obvious that $(R')^+$ has the following properties:

(i) $(\overline{0, 0}) \in (R')^+$,
(ii) $(R')^+ \cap [-(R')^+] = \{0\}$,
(iii) $(R')^+ + (R')^+ \subseteq (R')^+$,
(iv) $(R')^+ (R')^+ \subseteq (R')^+$.

Hence, the ring R' is a partially ordered ring, and $a' \leq b'$ if and only if $(b' - a') \in (R')^+$ for all a', $b' \in R'$. It is clear that, for each $x' \in R'$, $x' \geq 0$ if and only if $x'(\overline{0, e}) \geq 0$.

We show below that R' is a lattice-ordered ring under the partial order defined above. In fact, if $a \in S$, then

$$(\overline{n, a}) \vee 0 = \begin{cases} (\overline{n, a \vee 0}) & \text{if } a \vee 0 \in S, \\ (\overline{0, a \vee 0}) & \text{if } a \vee 0 \in T \oplus T^{\perp}. \end{cases}$$

If $a \in T \oplus T^{\perp}$, then since T is a totally ordered ring and $ne + a_T e \in T$,

$ne + a_T e \geq 0$ or $ne + a_T e \leq 0$. Thus

$$(\overline{n, a}) \vee 0 = \begin{cases} (\overline{n, a_T + a_{T^\perp} \vee 0}) & \text{if } ne + a_T e \geq 0, \\ (\overline{0, a_{T^\perp} \vee 0}) & \text{if } ne + a_T e \leq 0. \end{cases}$$

We show the above results according to the following four cases.

(a) $a \in S$ and $a \vee 0 \in S$. By Lemma 2(a) we have $(\overline{n, a \vee 0}) \geq (\overline{n, a})$, 0. If $b' = (\overline{m, b}) \in R'$ and $b' \geq (\overline{n, a})$, 0, then $me + be \geq 0$ and $(m-n)e + (b-a)e \geq 0$. If $b \in S$, then $b > 0$ by Lemma 2(b), and hence $(m-n)e + be \geq 0$ by Lemma 2(a). Thus $(m-n)e + be \geq ae \vee 0 = (a \vee 0)e$; that is, $b' \geq (\overline{n, a \vee 0})$. If $b \in T \oplus T^\perp$, then $b_{T^\perp} \geq 0$ by Lemma 4. Since $(b-a) \in S$, $b - a > 0$ by Lemma 2(b), and hence $b_{T^\perp} > a$, so $b_{T^\perp} > a \vee 0$. Since $b_{T^\perp} > a \vee 0$, $b_{T^\perp}$ is in S, a contradiction.

(b) $a \in S$ and $a \vee 0 \in T \oplus T^\perp$. Since $a = (a \vee 0) - (-a \vee 0)$, we have $(-a \vee 0) \in S$. It follows from (a) that $(\overline{-n, -a}) \vee 0 = (\overline{-n, -a \vee 0})$, so $(\overline{n, a}) \vee 0 = (\overline{0, a \vee 0})$.

(c) $a \in T \oplus T^\perp$ and $ne + a_T e \geq 0$. It is evident that $(\overline{n, a_T + (a_{T^\perp} \vee 0)}) \geq (\overline{n, a})$, 0. Let $b' = (\overline{m, b}) \in R'$ and $b' \geq (\overline{n, a})$, 0. Then $me + be \geq 0$ and $(m-n)e + (b-a)e \geq 0$. If $b \in S$, then $b > 0$, and hence $0 < b - a_T \in S$. Since $(m-n)e + (b - a_T)e \geq 0$ by Lemma 2, we have $(m-n)e + (b - a_T)e \geq a_{T^\perp} e \vee 0 = (a_{T^\perp} \vee 0)e$; that is, $b' \geq (\overline{n, a_T + (a_{T^\perp} \vee 0)})$. If $b \in T \oplus T^\perp$, then $b_{T^\perp} \geq a_{T^\perp} \vee 0$ and $(m-n)e + (b_T - a_T)e \geq 0$ by Lemma 4. Thus $(m-n)e + (b - a_T - a_{T^\perp} \vee 0)e \geq 0$. Again we have $b' \geq (\overline{n, a_T + (a_{T^\perp} \vee 0)})$.

(d) $a \in T \oplus T^\perp$ and $ne + a_T e \leq 0$. It is clear that $(\overline{0, a_{T^\perp} \vee 0}) \geq (\overline{n, a})$, 0. If $b' = (\overline{m, b}) \in R'$ and $b' \geq (\overline{n, a})$, 0, then $(me + be) \geq 0$ and $(m-n)e + (b-a)e \geq 0$. If $b \in S$, then since $b > 0$ and $b > a$, $0 < (b - a_{T^\perp} \vee 0) \in S$, and hence $me + (b - a_{T^\perp} \vee 0)e \geq 0$ by Lemma 2(a); that is, $b' \geq (\overline{0, a_{T^\perp} \vee 0})$. If $b \in T \oplus T^\perp$, then $b_{T^\perp} \geq a_{T^\perp} \vee 0$ and $me + b_T e \geq 0$ by Lemma 4. Hence $me + (b - a_{T^\perp} \vee 0)e \geq 0$. Thus $b' \geq (\overline{0, a_{T^\perp} \vee 0})$.

So far we have shown that R' is an l-ring.

Finally, we show that R' is squares positive.

Let $a' = (\overline{n, a}) \in R'$, $a \in R$. Then $(a')^2 = (\overline{n^2, 2na + a^2})$. If $a \in S$, by Lemma 5 we have $2na + a^2 \geq 0$, and hence $(a')^2 \geq 0$. If $a \in T \oplus T^\perp$ and $a = a_T + a_{T^\perp}$, then $2na + a^2 = 2na_T + 2na_{T^\perp} + a_T^2 + a_T a_{T^\perp} + a_{T^\perp} a_T + (a_{T^\perp})^2$, and hence $(a')^2 = (\overline{n, a_T})^2 + (\overline{0, 2na_{T^\perp} + a_T a_{T^\perp} + a_{T^\perp} a_T + (a_{T^\perp})^2})$. Since T is a totally ordered ring, $(\overline{n, a_T})^2 \geq 0$, and, from Lemma 5 and $a_{T^\perp} \in T^\perp$, $2na_{T^\perp} + a_T a_{T^\perp} + a_{T^\perp} a_T + (a_{T^\perp})^2 \geq 0$, so $(a')^2 \geq 0$; that is, R' is squares positive.

Let ψ be the map given by $a \to (\overline{0, a})$ for every $a \in R$. It is evident that ψ is a ring monomorphism from R into R' and $\psi(a \vee 0) = (\overline{0, a \vee 0}) = (\overline{0, a}) \vee 0 = \psi(a) \vee 0$ for each $a \in R$; that is, ψ is an l-ring monomorphism.

The proof of Theorem 1 is now complete.

Corollary. *An l-semiprime l-ring with squares positive and an f-superunit can be embedded in a unital l-semiprime l-ring with squares positive.*

Proof. Since an l-semiprime l-ring with squares positive and an f-superunit is a subdirect product of its l-prime homomorphic images which are squares positive and contain f-superunits, the proof is completed by Theorem 1.

Lemma 6. *Let R be an l-prime l-ring with squares positive and an f-superunit, and let R' be as in Theorem* 1. *If e is an f-superunit of R, then $(\overline{0, e})$ is an f-superunit of R'.*

Proof. Since e is an f-superunit of R, $(\overline{1, 0}) \leq (\overline{0, e})$, and hence $(\overline{0, e})$ is a superunit of R'. We show $(\overline{0, e})$ is an f-element of R'. Suppose that $(\overline{n, x}) \wedge (\overline{m, y}) = 0$. First we show that $(ne + xe) \wedge (me + ye) = 0$ in R. Note that x and y are not both in S; otherwise, we have $(\overline{1, 0}) \leq (\overline{n, x})$, $(\overline{m, y})$ by Lemma 2. If $x \in S$ and $y \in T \oplus T^{\perp}$, then $(\overline{m, y_T}) = 0$ and $(\overline{n, x}) \wedge (\overline{0, y_{T^{\perp}}}) = 0$, so $me + y_T e = 0$ and $x \wedge y_{T^{\perp}} = 0$ since $0 \leq (\overline{0, x}) \leq 2(\overline{n, x})$. Thus $(ne + xe) \wedge (me + ye) = (ne + xe) \wedge y_{T^{\perp}} e = 0$. If $x, y \in T \oplus T^{\perp}$, then $(\overline{n, x_T}) = 0$ or $(\overline{m, y_T}) = 0$ and $(\overline{0, x_{T^{\perp}}}) \wedge (\overline{0, y_{T^{\perp}}}) = 0$. Thus $(ne+xe) \wedge (me+ye) \leq [(ne+x_T e) \wedge (me+y_T e)] + (x_{T^{\perp}} e \wedge y_{T^{\perp}} e) = 0$, and we get $(ne + xe) \wedge (me + ye) = 0$. Similarly, we also have $(ne + ex) \wedge (me + ey) = 0$. These yield $(\overline{n, x})(\overline{0, e}) \wedge (\overline{m, y})(\overline{0, e}) = (\overline{0, ne + xe}) \wedge (\overline{0, me + ye}) = 0$ and $(\overline{0, e})(\overline{n, x}) \wedge (\overline{0, e})(\overline{m, y}) = (\overline{0, ne + ex}) \wedge (\overline{0, me + ey}) = 0$. So $(\overline{0, e})$ is an f-element of R' since $(\overline{1, 0}) \leq (\overline{0, e})$.

Let R be an l-prime l-ring with squares positive and an f-superunit, and let R' be as in Theorem 1. It is easy to verify $T(R') = \{(\overline{n, a}) : a \in T\}$ by Lemma 6. Thus $R' = S(R') \cup [T(R') + T(R')^{\perp}]$ by Lemma 1, where $S(R') = \{(\overline{n, a}) : a \in S\}$ and $T(R')^{\perp} = \{(\overline{0, a}) : a \in T^{\perp}\}$.

Theorem 2. *Let R be an l-prime l-ring with squares positive and an f-superunit, and let R' be as in Theorem* 1. *If R can be embedded as an l-subring in a unital l-ring B and some f-superunit of R is an f-superunit of B, then there are an l-subring R_1 of B containing R and an l-isomorphism ψ from l-ring R' to l-ring R_1, which satisfies $\psi(a) = a$ for each $a \in R$.*

Proof. Let 1_B be the identity element of B and R_1 the subring of B generated by 1_B and R. Let ψ be the map from R' into R_1 given by $(\overline{n, a}) \to n1_B + a$, and let e be an f-superunit of R which is also an f-superunit of B. If $(\overline{n, a}) = (\overline{m, b})$, then $(n-m)e + (a-b)e = 0$, and hence $[(n-m)1_B + (a-b)]e = 0$ (in B). Since e is an f-superunit of B, we have $(n-m)1_B + (a-b) = 0$, and hence $\psi((\overline{n, a})) = \psi((\overline{m, b}))$; that is, ψ is well defined. It is evident that ψ is an isomorphism of rings and $\psi((\overline{0, a})) = a$. If $a \in R$ and $(n1_B + a) \vee 0 = b$ (in B), then $be = (n1_B + a)e \vee 0 = (ne + ae) \vee 0$ since e is an f-element of B. Let $(\overline{n, a}) \vee 0 = (\overline{n_1, a_1})$ (in R'). Since $(\overline{0, e})$ is an f-element of R' by Lemma 6, we get $(\overline{n, a})(\overline{0, e}) \vee 0 = (\overline{n_1, a_1})(\overline{0, e})$ and $(\overline{0, ne + ae}) \vee 0 = (\overline{0, n_1 e + a_1 e})$, and hence $(ne + ae) \vee 0 = n_1 e + a_1 e$. Since R is an l-subring of B, $be = n_1 e + a_1 e = (n_1 1_B + a_1)e$, and hence $b = n_1 1_B + a_1 \in R_1$ since e is an f-superunit of B; that is, R_1 is an l-subring of B. Finally, it is easy to verify that ψ is also an l-isomorphism from R' to R_1.

Let R be an l-prime l-ring with squares positive and an f-superunit. Theorem 2 shows that the R' in Theorem 1 is the unique smallest unital l-ring which contains R and has an f-superunit that is an f-superunit of R. In general, no such uniqueness holds [1, p. 331].

Now we determine the conditions which ensure that an l-prime l-ring with squares positive which has nonzero f-elements can be embedded as a convex l-subring in a unital l-prime l-ring with squares positive. If R is an l-prime

l-ring with squares positive which has nonzero f-elements, then R is a domain [5, Theorem 1], and hence T is a totally ordered ring. Thus $R = S \cup (T \oplus T^{\perp})$ by Lemma 1. In [6, Remark, p. 367] Steinberg defined an l-ring R to be supertessimal if $n|x| \leq |x^2|$ for each $n \in Z^+$ and $x \in R$.

Theorem 3. *Let R be an l-prime l-ring with squares positive which has nonzero f-elements, and suppose R has no identity element. R can be embedded as a convex l-subring in a unital l-prime l-ring with squares positive if and only if R is supertessimal*; *or $S = \varnothing$, T is infinitesimal, and $n|a| \leq a^2$ for each $n \in Z^+$ and $a \in T^{\perp}$.*

Proof. Let B be a unital l-prime l-ring with squares positive and R a convex l-subring of B. Since R is a convex l-subring of B, $T^{\perp} \subseteq T(B)^{\perp}$, and, also, since R does not have an identity element, $R \subseteq T(B) + T(B)^{\perp}$ by Lemma 1. If $S \neq \varnothing$, then, for any $a \in S$, we have $|a| \in T(B)^{\perp}$. This yields $S \subseteq T(B)^{\perp}$, and hence $T \subseteq T(B)^{\perp}$, so $R \subseteq T(B)^{\perp}$ by Lemma 1. From Lemma 5 we get R to be supertessimal. If $S = \varnothing$, but T is not infinitesimal, then there is $d \in T$ which satisfies $0 < d < d^2$. Let $d = d_1 + d_2$, where $0 \leq d_1 \in T(B)$ and $0 \leq d_2 \in T(B)^{\perp}$. Since R is a convex l-subring of B, d_1, $d_2 \in R$, and hence d_1, $d_2 \in T$. But T is a totally ordered ring and $d_1 \wedge d_2 = 0$, so $d_1 = 0$ or $d_2 = 0$. If $d_2 = 0$, then $d \in T(B)$, and hence $d > 1_B$ (identity element of B) since d is comparable with 1_B and $d < d^2$. This contradicts the fact that R has no identity element, so $d_1 = 0$ and $d \in T(B)^{\perp}$. This yields $T \subseteq T(B)^{\perp}$, and hence $R \subseteq T(B)^{\perp}$. Thus, again, we get R to be supertessimal by Lemma 5.

Conversely, first suppose R is supertessimal. Let $Z \oplus R$ be as an l- group with multiplication defined by $(n, a)(m, b) = (nm, nb + ma + ab)$. Then $Z \oplus R$ is a unital l-ring with squares positive [6, p. 367]. If $(n, a)(m, b) = 0$, then $nm = 0$ and $nb + ma + ab = 0$. We may assume $n = 0$. If $a \neq 0$, then, since R is domain and $ma + ab = 0$, we have $mb + b^2 = 0$, and hence $(|m| + 1)|b| \leq b^2 = -mb = |mb| = |m||b|$ since R is supertessimal. This yields $b = m = 0$, and hence $(m, b) = 0$; that is, $Z \oplus R$ is a domain. Let ψ be the map given by $a \to (0, a)$. It is evident that ψ is an l-monomorphism and $\psi(R)$ is a convex l-subring of $Z \oplus R$.

Finally, let R satisfy the following conditions: $S = \varnothing$, T is infinitesimal, and $n|a| \leq a^2$ for each $a \in T^{\perp}$ and $n \in Z^+$. From Lemma 1 we have $R = T \oplus T^{\perp}$.

Let T' be the ring obtained by freely adjoining the Z to T with partial order defined by $(n, a) \geq 0$ if and only if $n > 0$ or $n = 0$ and $a \geq 0$. Then T' is a totally ordered domain ring with identity $(1, 0)$ [7, 1.4].

Let $R'' = T' \oplus T^{\perp}$ as an l-group. If $a'' = ((n, a_1), a_2)$ and $b'' = ((m, b_1), b_2) \in R''$, where a_1, $b_1 \in T$ and a_2, $b_2 \in T^{\perp}$, we define the multiplication in R'' as

$$a''b'' = ((n, a_1)(m, b_1) + (0, (a_2b_2)_T),\ nb_2 + a_1b_2 + ma_2 + a_2b_1 + (a_2b_2)_{T^{\perp}}).$$

It is easy to verify R'' to be a ring with identity $((1, 0), 0)$. Since T has mode 0 and $R = T \oplus T^{\perp}$, R has mode 0, so R'' is a domain. Also, the product of positive elements of R'' is clearly positive by the definition of the multiplication in R''; that is, R'' is an l-ring.

Let $a'' = ((n, a_1), a_2) \in R''$, where $a_1 \in T$ and $a_2 \in T^{\perp}$. Then $(a'')^2 = ((n, a_1)^2 + (0, (a_2^2)_T),\ 2na_2 + a_1a_2 + a_2a_1 + (a_2^2)_{T^{\perp}})$. From the hypothesis and

Lemma 5, we get $2na_2 + a_1a_2 + a_2a_1 + a_2^2 \geq 0$, and hence $2na_2 + a_1a_2 + a_2a_1 + (a_2^2)_{T^\perp} \geq 0$, $(a_2^2)_T \geq 0$. Thus $(a'')^2 \geq 0$.

So far we have shown that R'' is a unital l-prime l-ring with squares positive. Let ψ be the map from R into R'' given by $\psi(a) = ((0, a_T), a_{T^\perp})$ for each $a \in R$. It is clear that ψ is an l-monomorphism. If $((n, a_1), a_2)$ and $((0, b_1), b_2) \in R''$ satisfy $0 \leq ((n, a_1), a_2) \leq ((0, b_1), b_2)$, then $n = 0$ by the definition of the partial order of R'', and hence $\psi(R)$ is a convex l-subring of R''.

The proof of Theorem 3 is now completed.

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