THE UNITABILITY OF \( l \)-PRIME LATTICE-ORDERED RINGS WITH SQUARES POSITIVE

MA JINGJING

(Communicated by Lance W. Small)

Abstract. It is shown that an \( l \)-prime lattice-ordered ring with squares positive and an \( f \)-superunit can be embedded in a unital \( l \)-prime lattice-ordered ring with squares positive.

In [1, p. 325] Steinberg asked if an \( l \)-prime \( l \)-ring with squares positive and an \( f \)-superunit can be embedded in a unital \( l \)-prime \( l \)-ring with squares positive. In this paper we show that the answer is yes.

If \( R \) is a lattice-ordered ring (\( l \)-ring) and \( a \in R^+ \), then \( a \) is called an \( f \)-element of \( R \) if \( b \wedge c = 0 \) implies \( ab \wedge c = ba \wedge c = 0 \). Let \( T = T(R) = \{a \in R : |a| \text{ is an } f\text{-element of } R \} \). Then \( T \) is a convex \( f \)-subring of \( R \), and \( R \) is a subdirect product of totally ordered \( T-T \) bimodules [2, Lemma 1]. \( R \) is an \( f \)-ring precisely when \( T = R \). Throughout this paper \( T \) will denote the subring of \( f \)-elements of \( R \). An element \( e \geq 0 \) of an \( l \)-ring \( R \) is called a superunit if \( ex \geq x \) and \( xe \geq x \) for each \( x \in R^+ \); \( e \) is an \( f \)-superunit if it is a superunit and an \( f \)-element. \( R \) is infinitesimal if \( x^2 \leq x \) for each \( x \) in \( R^+ \). The \( l \)-ring \( R \) is called \( l \)-prime if the product of two nonzero \( l \)-ideals is nonzero and \( l \)-semiprime if it has no nonzero nilpotent \( l \)-ideals. An \( l \)-ring \( R \) is called squares positive if \( a^2 \geq 0 \) for each \( a \) in \( R \).

Let \( A \) be any ring and \( a \in A \). If \( a \) satisfies \( ab = ba = nb \) for some fixed integer \( n \) and all \( b \in A \), then \( a \) is said to be an \( n \)-plier of \( A \) and \( n \) is said to have an \( n \)-plier \( a \) in \( A \). Let \( K = \{n \in \mathbb{Z} : \text{n has } n\text{-pliers in } A \} \). Then \( K \) is an ideal in the ring \( \mathbb{Z} \) of integers. The ideal \( K \) is called the modal ideal of \( A \); its nonnegative generator \( k \) is called the mode of \( A \). If \( R \) is an \( f \)-ring with mode \( k > 0 \), then \( R \) has a unique \( k \)-plier \( x \geq 0 \) [3, III, Lemma 2.1].

Let \( R \) be an \( l \)-ring, and let \( S = S(R) = \{a \in R : |a| \geq d, \forall d \in T^+ \} \) and \( T^- = \{a \in R : |a| \wedge d = 0, \forall d \in T^+ \} \). It is clear that \( T^- \) is a convex \( l \)-subgroup of \( R \).

The referee has pointed out to us that the following lemma, on which our arguments are based, is a special case of [4, Lemma 6.2], and P. Conrad attributes it to A. H. Clifford.
Lemma 1. Let $R$ be an $l$-ring which contains nonzero $f$-elements and $T$ a totally ordered subring of $R$. Then $R = S \cup (T \oplus T^\perp)$, where the direct sum is regarded as the direct sum of $l$-subgroups and $S \cap (T \oplus T^\perp) = \varnothing$.

Proof. If $a \in R$ and $a \notin S$, then $|a| \wedge b < b$ for some $b \in T^+$. Let $a_1 = |a| - |a| \wedge b$ and $b_1 = b - |a| \wedge b$. Then $a_1 \wedge b_1 = 0$, and hence $(a_1 \wedge d) \wedge b_1 = 0$ for each $d \in T^+$. Since $(a_1 \wedge d) , b_1 \in T$, which is a totally ordered ring, and $b_1 > 0$, we have $a_1 \wedge d = 0$ for each $d \in T^+$, and hence $a_1 \in T^\perp$. Since $0 \leq a^+, a^- \leq |a| = a_1 + (|a| \wedge b)$, there are $a_2, a_3, b_2, b_3 \in R$ such that $0 \leq a_2, a_3 \leq a_1, 0 \leq b_2, b_3 \leq |a| \wedge b, a^+ = a_2 + b_2, a^- = a_3 + b_3$, and $a = a^+ - a^- = (b_2 - b_3) + (a_2 - a_3)$. But $T$ and $T^\perp$ are the convex $l$-subgroups of $R$, so $a \in T \oplus T^\perp$.

If $a \in S \cap (T \oplus T^\perp)$, then $a = a_1 + a_2$, where $a_1 \in T$ and $a_2 \in T^\perp$, and hence $2|a_1| \leq |a| \leq |a_1| + |a_2|$ since $a \in S$. Thus $a_1 = 0$ and $a = a_2$, and hence $T = \{0\}$. This contradicts $T \neq \{0\}$, so $S \cap (T \oplus T^\perp) = \varnothing$.

Let $R$ be an $l$-ring as in Lemma 1 and $a \in R$. If $a \in T \oplus T^\perp$, then $a$ can be uniquely represented as the sum of elements of $T$ and $T^\perp$, and we may write $a = a_T + a_T^\perp$, where $a_T \in T$ and $a_T^\perp \in T^\perp$ are respectively called the components of $a$ in $T$ and in $T^\perp$. It is clear that $a \vee 0 = (a_T \vee 0) + (a_T^\perp \vee 0)$.

Throughout this paper the following fact is used frequently. Let $R$ be an $l$-ring and $e$ an $f$-superunit of $R$. Then, for each $x \in R$, $x \geq 0$ if and only if $ex \geq 0$ or $xe \geq 0$. More generally, if $0 < a \in T$ is a regular element (or a non-zero-divisor), then, for each $x \in R$, $x \geq 0$ if and only if $ax \geq 0$ or $xa \geq 0$.

Lemma 2. Let $R$ be an $l$-ring with an $f$-superunit and $T$ a totally ordered ring. Let $e$ be an $f$-superunit of $R$.

(a) If $0 < b \in S$, then $ne + be \geq 0$ for each $n \in Z$.

(b) If $b \in S$ and $me + be \geq 0$ for some $m \in Z$, then $b > 0$.

Proof. (a) Since $0 < b \in S$, we have $-ne \leq |n|e \leq |n|e^2 \leq be$ for each $n \in Z$.

(b) Let $me + be \geq 0$ for some $m \in Z$. If $m \leq 0$, then $be \geq -me \geq 0$, so $be = |be| = |b|e$; that is, $(|b| - b)e = 0$. But $e$ is an $f$-superunit, so $b = |b| > 0$. If $m > 0$, then $me \geq -be$, and hence $me \geq (be) \vee 0 = (-b \vee 0)e \geq -b \vee 0$, so $-b \vee 0 \in T$. Since $b \vee 0 = b + (-b \vee 0)$ and $b \in S$, we have $b \vee 0 \in S$ by Lemma 1. Thus $b + (-b \vee 0) = b \vee 0 > -b \vee 0$; that is, $b > 0$.

Lemma 3. Let $R$ be an $l$-ring with an $f$-superunit and $T$ a totally ordered ring. If the mode of $R$ is $n$ and the mode of $T$ is $k$, then $n = k$ and the $k$-fiber of $R$ equals the $k$-fiber of $T$.

Proof. Let $e$ be an $f$-superunit of $R$. If $x$ is a $k$-fiber of $T$, then $kd = dx = xd$ for each $d \in T$, especially $ke = ex = xe$. Thus $e(ka) = exa$ and $(ka)e = axe$ for each $a \in R$. Since $e$ is an $f$-superunit of $R$, $ka = xa = ax$ for each $a \in R$, so $nk$. Let $y$ be an $n$-fiber of $R$. Then $na = ay = ya$ for each $a \in R$. By Lemma 1 we have $R = S \cup (T \oplus T^\perp)$. If $y \in S$, then since $e$ is an $f$-superunit of $R$ and $ne = ey = ye$, we have $ne = |ne| = |ey| = e|y| \geq |x| \geq (n + 1)e$. This is a contradiction, so $y \in T \oplus T^\perp$. Let $y = y_T + y_T^\perp$. Then $ne = ey = ey_T + ey_T^\perp$, and hence $ne - ey_T = ey_T^\perp \in T \cap T^\perp = \{0\}$, so $y_T^\perp = 0$ and $y = y_T \in T$. Thus $k|n$, and we have $k = n$. 

Lemma 4. Let $R$ be an $l$-ring which has an $f$-superunit and $T$ a totally ordered ring. If $e$ is an $f$-superunit of $R$ and $n \in \mathbb{Z}$, $b \in T \oplus T^\perp$ satisfy $ne + be \geq 0$, then $b_{T^\perp} \geq 0$ and $ne + b_{T^\perp}e \geq 0$.

Proof. If $ne + be \geq 0$, then $(ne + b_{T^\perp}e) + b_{T^\perp}e \geq 0$. Thus $ne + b_{T^\perp}e \geq 0$ and $b_{T^\perp}e \geq 0$, so $b_{T^\perp} \geq 0$.

Lemma 5. Let $R$ be an $l$-ring with squares positive and $a \in T^\perp$ or $S$. Then $d|a| \leq a^2$ and $|a|d \leq a^2$ for each $d \in T^\perp$. If $R$ has an $f$-superunit, then $n|a| \leq a^2$ for each $n \in \mathbb{Z}^+$. 

Proof. Since $R$ is squares positive, $0 \leq (a \pm b)^2$ yields $|ad + da| \leq a^2 + d^2$. But $R$ is a $T$-$T$ $f$-bimodule, and $|ad| + |da| = |ad + da|$ holds in any totally ordered $T$-$T$ bimodule which is a homomorphic image of $R$ since $0 \leq d \in T$; so it also holds in $R$ [2, Lemma 4]. If $a \in T^\perp$, then $|a|d = |ad| \leq |ad| + |da| = |ad + da| \leq a^2 + d^2$ and $|a| \wedge d = 0$ imply $|a|d = |a|d \wedge (a^2 + d^2) \leq (|a|d \wedge a^2) + (|a|d \wedge d^2) = |a|d \wedge a^2 \leq a^2$. Similarly, $d|a| \leq a^2$. If $a \in S$, then $d \leq |a|$ for each $d \in T^+$, and hence $|ad + da| = |ad| + |da| = |ad + da| \leq a^2 + d^2 \leq a^2 + d|a|$. Thus $|ad| \leq a^2$. Similarly, $d|a| \leq a^2$. If $e$ is an $f$-superunit of $R$, then, as in the previous paragraph, $n|a| \leq ne|a| \leq a^2$ for each $n \in \mathbb{Z}^+$. 

Theorem 1. An $l$-prime $l$-ring $R$ with squares positive and an $f$-superunit can be embedded in a unit $l$-prime $l$-ring with squares positive.

Proof. We first note that $R$ is a domain [2, Theorem 1], and hence $T$ is a totally ordered ring. Thus $R = S \cup (T \oplus T^\perp)$ by Lemma 1. Let $e$ be an $f$-superunit of $R$.

Let $R$ be the ring obtained by freely adjoining the integers to $R$, and let $k$ be the mode of $T$ and $x$ the unique $k$-fier of $T$. Then $k$ is the mode of $R$ and $x$ the unique $k$-fier of $R$ by Lemma 3. Let $I(k, x)$ be the set of all integral multiples of $(k, -x) \in R$. Then $I(k, x)$ is an ideal of $R$. Denote the ring $R/I(k, x)$ by $R'$ which has identity $(1, 0)$. Since $R$ is a domain, it is well known that $R'$ is a domain. Consider the subset $(R')^+$ of $R'$ defined by $(\overline{n}, a) \in (R')^+$ if and only if $nc + ac \geq 0$ for all $c \in R^+$, which is equivalent to $ne + ae \geq 0$. Since $nc + ac \geq 0$ if and only if $(n + mk)c + (a - mx)c \geq 0$ for each $m \in \mathbb{Z}$, $(R')^+$ is well defined. It is obvious that $(R')^+$ has the following properties:

(i) $(0, 0) \in (R')^+$,
(ii) $(R')^+ \cap [-(R')^+] = \{0\}$,
(iii) $(R')^+ + (R')^+ \subseteq (R')^+$,
(iv) $(R')^+(R')^+ \subseteq (R')^+$. 

Hence, the ring $R'$ is a partially ordered ring, and $a' \leq b'$ if and only if $(b' - a') \in (R')^+$ for all $a', b' \in R'$. It is clear that, for each $x' \in R'$, $x' \geq 0$ if and only if $x'(0, e) \geq 0$.

We show below that $R'$ is a lattice-ordered ring under the partial order defined above. In fact, if $a \in S$, then

$$(\overline{n}, a) \vee 0 = \begin{cases} (\overline{n}, a \vee 0) & \text{if } a \vee 0 \in S, \\ (0, a \vee 0) & \text{if } a \vee 0 \in T \oplus T^\perp. \end{cases}$$

If $a \in T \oplus T^\perp$, then since $T$ is a totally ordered ring and $ne + a_{T^\perp}e \in T$, 

\begin{align*}
\end{align*}
ne + at ≥ 0 or ne + at ≤ 0. Thus

\[(\overline{n, a}) \vee 0 = \begin{cases} (n, aT + aT^\perp \vee 0) & \text{if } ne + at \ge 0, \\ (0, aT^\perp \vee 0) & \text{if } ne + at \le 0. \end{cases}\]

We show the above results according to the following four cases.

(a) \(a \in S\) and \(a \vee 0 \in S\). By Lemma 2(a) we have \((\overline{n, a}) \vee 0 \ge (\overline{n, a}),\) 0. If \(b' = (m, b) \in R'\) and \(b' \ge (\overline{n, a}),\) 0, then \(me + be \ge 0\) and \((m - n)e + (b - a)e \ge 0\). If \(b \in S\), then \(b > 0\) by Lemma 2(b), and hence \((m - n)e + be \ge 0\) by Lemma 2(a). Thus \((m - n)e + be \ge ae \vee 0 = (a \vee 0)e\); that is, \(b' \ge (\overline{n, a} \vee 0)\).

If \(b \in T \oplus T^\perp\), then \(bT^\perp \ge 0\) by Lemma 4. Since \((b - a) \in S\), \(b - a > 0\) by Lemma 2(b), and hence \(bT^\perp > a\), so \(bT^\perp > a \vee 0\). Since \(bT^\perp > a \vee 0\), \(bT^\perp\) is in \(S\), a contradiction.

(b) \(a \in S\) and \(a \vee 0 \in T \oplus T^\perp\). Since \(a = (a \vee 0) - (-a \vee 0)\), we have \((-a \vee 0) \in S\). It follows from (a) that \((-n, -a) \vee 0 = (-n, -a \vee 0)\), so \((\overline{n, a}) \vee 0 = (0, a \vee 0)\).

(c) \(a \in T \oplus T^\perp\) and \(ne + at \ge 0\). It is evident that \((\overline{n, aT + (aT^\perp \vee 0)}) \ge (\overline{n, a}),\) 0. Let \(b' = (m, b) \in R'\) and \(b' \ge (\overline{n, a}),\) 0. Then \(me + be \ge 0\) and \((m - n)e + (b - a)e \ge 0\). If \(b \in S\), then \(b > 0\), and hence \(0 < b - aT \in S\).

Since \((m - n)e + (b - aT)e \ge 0\) by Lemma 2, we have \((m - n)e + (b - aT)e \ge \) by Lemma 2(a); that is, \(b' \ge (\overline{n, aT + (aT^\perp \vee 0)})\). If \(b \in T \oplus T^\perp\), then \(bT^\perp \ge 0\) and \((m - n)e + (b - aT)e \ge 0\) by Lemma 4. Thus \((m - n)e + (b - aT - aT^\perp \vee 0)e \ge 0\). Again we have \(b' \ge (\overline{n, aT + (aT^\perp \vee 0)})\).

(d) \(a \in T \oplus T^\perp\) and \(ne + at \le 0\). It is clear that \((0, aT^\perp \vee 0) \ge (\overline{n, a}),\) 0. If \(b' = (m, b) \in R'\) and \(b' \ge (\overline{n, a}),\) 0, then \((me + be) \ge 0\) and \((m - n)e + (b - a)e \ge 0\). If \(b \in S\), then since \(b > 0\) and \(b > a\), \(0 < (b - aT^\perp \vee 0) \in S\), and hence \(me + (b - aT^\perp \vee 0)e \ge 0\) by Lemma 2(a); that is, \(b' \ge (0, aT^\perp \vee 0)\).

If \(b \in T \oplus T^\perp\), then \(bT^\perp \ge 0\) and \(me + bT^\perp e \ge 0\) by Lemma 4. Hence \(me + (b - aT^\perp \vee 0)e \ge 0\). Thus \(b' \ge (0, aT^\perp \vee 0)\).

So far we have shown that \(R'\) is an \(l\)-ring.

Finally, we show that \(R'\) is squares positive.

Let \(a' = (\overline{n, a}) \in R'\), \(a \in R\). Then \((a')^2 = (n^2, 2na + a^2)\). If \(a \in S\), by Lemma 5 we have \(2na + a^2 \ge 0\), and hence \((a')^2 \ge 0\). If \(a \in T \oplus T^\perp\) and \(a = aT + aT^\perp\), then \(2na + a^2 = 2naT + 2naT^\perp + a^2 + aT^2 + aT^\perp aT + aT^\perp + aT \ge (aT^\perp)^2\), and hence \((a')^2 \ge (\overline{n, aT} + (0, 2naT^\perp + aT^2 + aT^\perp aT + (aT^\perp)^2) \ge 0\). Since \(T\) is a totally ordered ring, \((\overline{n, aT})^2 \ge 0\), and, from Lemma 5 and \(aT^\perp \in T^\perp\), \(2naT^\perp + aT^\perp aT + aT^\perp aT + (aT^\perp)^2 \ge 0\), so \((a')^2 \ge 0\); that is, \(R'\) is squares positive.

Let \(\psi\) be the map given by \(a \rightarrow (0, a)\) for every \(a \in R\). It is evident that \(\psi\) is a ring monomorphism from \(R\) into \(R'\) and \(\psi(a \vee 0) = (0, a \vee 0) = (\overline{0, a}) \vee 0 = \psi(a) \vee 0\) for each \(a \in R\); that is, \(\psi\) is an \(l\)-ring monomorphism.

The proof of Theorem 1 is now complete.

**Corollary.** An \(l\)-semiprime \(l\)-ring with squares positive and an \(f\)-superunit can be embedded in a unital \(l\)-semiprime \(l\)-ring with squares positive.

**Proof.** Since an \(l\)-semiprime \(l\)-ring with squares positive and an \(f\)-superunit is a subdirect product of its \(l\)-prime homomorphic images which are squares positive and contain \(f\)-superunits, the proof is completed by Theorem 1.
Lemma 6. Let $R$ be an l-prime l-ring with squares positive and an f-superunit, and let $R'$ be as in Theorem 1. If $e$ is an f-superunit of $R$, then $(0, e)$ is an f-superunit of $R'$.

Proof. Since $e$ is an f-superunit of $R$, $(1, 0) \leq (0, e)$, and hence $(0, e)$ is a superunit of $R'$. We show $(0, e)$ is an f-element of $R'$. Suppose that $(n, x) \wedge (m, y) = 0$. First we show that $(ne + xe) \wedge (me + ye) = 0$ in $R$. Note that $x$ and $y$ are not both in $S'$; otherwise, we have $(1, 0) \leq (\bar{n}, \bar{x})$, $(\bar{m}, \bar{y})$ by Lemma 2. If $x \in S$ and $y \in T \oplus T'$, then $(\bar{m}, \bar{y}) = 0$ and $(\bar{n}, \bar{x}) \wedge (0, y_{T'}) = 0$, so $me + ye = 0$ and $x \wedge y_{T'} = 0$ since $0 \leq (0, x) \leq 2(n, \bar{x})$. Thus $(ne + xe) \wedge (me + ye) = 0$. Similarly, we also have $(ne + ex) \wedge (me + ey) = 0$.

Let $R$ be an l-prime l-ring with squares positive and an f-superunit, and let $R'$ be as in Theorem 1. It is easy to verify $T(R') = \{(\bar{n}, a) : a \in T\}$ by Lemma 6. Thus $R' = S(R') \cup \{T(R') \cup T(R') \}$ by Lemma 1, where $S(R') = \{(\bar{n}, a) : a \in S\}$ and $T(R') = \{(0, a) : a \in T\}$.

Theorem 2. Let $R$ be an l-prime l-ring with squares positive and an f-superunit, and let $R'$ be as in Theorem 1. If $R$ can be embedded as an l-subring in a unital l-ring $B$ and some f-superunit of $R$ is an f-superunit of $B$, then there are an l-subring $R_1$ of $B$ containing $R$ and an l-isomorphism $\psi$ from l-ring $R'$ to l-ring $R_1$, which satisfies $\psi(a) = a$ for each $a \in R$.

Proof. Let $1_B$ be the identity element of $B$ and $R_1$ the subring of $B$ generated by $1_B$ and $R$. Let $\psi$ be the map from $R'$ into $R_1$ given by $(\bar{n}, a) \rightarrow n_1 + a$, and let $e$ be an f-superunit of $R$ which is also an f-superunit of $B$. If $(\bar{n}, a) = (m, b)$, then $(n-m)e + (a-b)e = 0$, and hence $[(n-m)1_B + (a-b)]e = 0$ in $B$. Since $e$ is an f-superunit of $B$, we have $(n-m)1_B + (a-b) = 0$, and hence $\psi((\bar{n}, a)) = \psi((m, b))$; that is, $\psi$ is well defined. It is evident that $\psi$ is an isomorphism of rings and $\psi((0, a)) = a$. If $a \in R$ and $(n_1 + a) \vee 0 = b$ in $B$, then $be = (n_1 + a)e \vee 0 = (ne + ae) \vee 0$ since $e$ is an f-element of $B$. Let $(\bar{n}, a) \vee 0 = (\bar{n}_1, a_1)$. Since $(0, e)$ is an f-element of $R'$ by Lemma 6, we get $(\bar{n}, a)(0, e) \vee 0 = (\bar{n}_1, a_1)(0, e)$ and $(0, ne + ae) \vee 0 = (0, n_1e + a_1e)$, and hence $(ne + ae) \vee 0 = n_1e + a_1e$. Since $R$ is an l-subring of $B$, $be = n_1e + a_1e = (n_11_B + a_1)e$, and hence $b = n_11_B + a_1 \in R_1$ since $e$ is an f-superunit of $B$; that is, $R_1$ is an l-subring of $B$. Finally, it is easy to verify that $\psi$ is also an l-isomorphism from $R'$ to $R_1$.

Let $R$ be an l-prime l-ring with squares positive and an f-superunit. Theorem 2 shows that the $R'$ in Theorem 1 is the unique smallest unital l-ring which contains $R$ and has an f-superunit that is an f-superunit of $R$. In general, no such uniqueness holds [1, p. 331].

Now we determine the conditions which ensure that an l-prime l-ring with squares positive which has nonzero f-elements can be embedded as a convex l-subring in a unital l-prime l-ring with squares positive. If $R$ is an l-prime
$l$-ring with squares positive which has nonzero $f$-elements, then $R$ is a domain [5, Theorem 1], and hence $T$ is a totally ordered ring. Thus $R = S \cup (T \oplus T_\perp)$ by Lemma 1. In [6, Remark, p. 367] Steinberg defined an $l$-ring $R$ to be supertessimal if $n|x| \leq |x^2|$ for each $n \in \mathbb{Z}^+$ and $x \in R$.

**Theorem 3.** Let $R$ be an $l$-prime $l$-ring with squares positive which has nonzero $f$-elements, and suppose $R$ has no identity element. $R$ can be embedded as a convex $l$-subring in a unital $l$-prime $l$-ring with squares positive if and only if $R$ is supertessimal; or $S = \emptyset$, $T$ is infinitesimal, and $n|a| \leq a^2$ for each $n \in \mathbb{Z}^+$ and $a \in T_\perp$.

**Proof.** Let $B$ be a unital $l$-prime $l$-ring with squares positive and $R$ a convex $l$-subring of $B$. Since $R$ is a convex $l$-subring of $B$, $T_\perp \subseteq T(B)_\perp$, and, also, since $R$ does not have an identity element, $R \subseteq T(B) + T(B)_\perp$ by Lemma 1. If $S \neq \emptyset$, then, for any $a \in S$, we have $|a| \in T(B)_\perp$. This yields $S \subseteq T(B)_\perp$, and hence $T \subseteq T(B)_\perp$, so $R \subseteq T(B)_\perp$ by Lemma 1. From Lemma 5 we get $R$ to be supertessimal. If $S = \emptyset$, but $T$ is not infinitesimal, then there is $d \in T$ which satisfies $0 < d < d^2$. Let $d = d_1 + d_2$, where $0 \leq d_1 \in T(B)$ and $0 \leq d_2 \in T(B)_\perp$. Since $R$ is a convex $l$-subring of $B$, $d_1$, $d_2 \in R$, and hence $d_1$, $d_2 \in T$. But $T$ is a totally ordered ring and $d_1 \wedge d_2 = 0$, so $d_1 = 0$ or $d_2 = 0$. If $d_2 = 0$, then $d \in T(B)$, and hence $d > 1_B$ (identity element of $B$) since $d$ is comparable with $1_B$ and $d < d^2$. This contradicts the fact that $R$ has no identity element, so $d_1 = 0$ and $d \in T(B)_\perp$. This yields $T \subseteq T(B)_\perp$, and hence $R \subseteq T(B)_\perp$. Thus, again, we get $R$ to be supertessimal by Lemma 5.

Conversely, first suppose $R$ is supertessimal. Let $Z \oplus R$ be as an $l$-group with multiplication defined by $(n, a)(m, b) = (nm, nb + ma + ab)$. Then $Z \oplus R$ is a unital $l$-ring with squares positive [6, p. 367]. If $(n, a)(m, b) = 0$, then $nm = 0$ and $nb + ma + ab = 0$. We may assume $n = 0$. If $a \neq 0$, then, since $R$ is domain and $ma + ab = 0$, we have $mb + b^2 = 0$, and hence $|\langle m \rangle + 1| \leq b^2 = -mb = |mb| = |m||b|$ since $R$ is supertessimal. This yields $b = m = 0$, and hence $(m, b) = 0$; that is, $Z \oplus R$ is a domain. Let $\psi$ be the map given by $a \rightarrow (0, a)$. It is evident that $\psi$ is an $l$-monomorphism and $\psi(R)$ is a convex $l$-subring of $Z \oplus R$.

Finally, let $R$ satisfy the following conditions: $S = \emptyset$, $T$ is infinitesimal, and $n|a| \leq a^2$ for each $a \in T_\perp$ and $n \in \mathbb{Z}^+$. From Lemma 1 we have $R = T \oplus T_\perp$.

Let $T'$ be the ring obtained by freely adjoining the $Z$ to $T$ with partial order defined by $(n, a) \geq 0$ if and only if $n > 0$ or $n = 0$ and $a \geq 0$. Then $T'$ is a totally ordered domain ring with identity $(1, 0)$ [7, 1.4].

Let $R'' = T' \oplus T_\perp$ as an $l$-group. If $a'' = ((n, a_1), a_2)$ and $b'' = ((m, b_1), b_2) \in R''$, where $a_1, b_1 \in T$ and $a_2, b_2 \in T_\perp$, we define the multiplication in $R''$ as

$$a''b'' = ((n, a_1)(m, b_1) + 0, (a_2 b_2)_T), nb_2 + a_1 b_2 + ma_2 + a_2 b_1 + (a_2 b_2)_T_\perp).$$

It is easy to verify $R''$ to be a ring with identity $((1, 0), 0)$. Since $T$ has mode $0$ and $R = T \oplus T_\perp$, $R$ has mode $0$, so $R''$ is a domain. Also, the product of positive elements of $R''$ is clearly positive by the definition of the multiplication in $R''$; that is, $R''$ is an $l$-ring.

Let $a'' = ((n, a_1), a_2) \in R''$, where $a_1 \in T$ and $a_2 \in T_\perp$. Then $(a'')^2 = ((n, a_1)^2 + 0, (a_2^2)_T), 2n a_2 + a_1 a_2 + a_2 a_1 + (a_2^2)_T_\perp)$. From the hypothesis and
Lemma 5, we get $2na_2 + a_1a_2 + a_2a_1 + a_2^2 \geq 0$, and hence $2na_2 + a_1a_2 + a_2a_1 + (a_2^2)_T \geq 0$. Thus $(a'')^2 \geq 0$.

So far we have shown that $R''$ is a unital $l$-prime $l$-ring with squares positive. Let $\psi$ be the map from $R$ into $R''$ given by $\psi(a) = ((0, a_T), a_{T\perp})$ for each $a \in R$. It is clear that $\psi$ is an $l$-monomorphism. If $((n, a_1), a_2)$ and $((0, b_1), b_2) \in R''$ satisfy $0 \leq ((n, a_1), a_2) \leq ((0, b_1), b_2)$, then $n = 0$ by the definition of the partial order of $R''$, and hence $\psi(R)$ is a convex $l$-subring of $R''$.

The proof of Theorem 3 is now completed.

ACKNOWLEDGMENT

The author thanks the referee for simplifying the proofs of Theorems 1 and 3.

REFERENCES

1. S. A. Steinberg, On the unitability of a class of partially ordered rings that have squares positive, J. Algebra 100 (1986), 325–343.