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ABSOLUTE CONJUGATE FOURIER EFFECTIVE METHODS AND FUNCTIONS

A. KUMAR AND A. PEYERIMHOFF

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ABSTRACT. We consider matrix transformation of the conjugate series of a Fourier series. Necessary and sufficient conditions on the function, generating the Fourier series, as well as on the method of the transformation have been obtained for absolute summability of the transformed series.

1. INTRODUCTION AND MAIN RESULT

Let $\widehat{A} = (\hat{a}_{n,k})$ be the matrix of a series-to-series transformation, where $\hat{a}_{n,k}$ are real or complex numbers. A series $\sum_{n\geq 0} a_n$ is said to be summable $\widehat{A}$ if $\alpha_n = \sum_{k\geq 0} \hat{a}_{n,k} a_k$ exists for all n and $\sum_{n\geq 0} \alpha_n$ is convergent. If α_n exists for all n and $\sum_{n\geq 0} |\alpha_n| < \infty$, then the series $\sum_{n\geq 0} a_n$ is said to be absolutely summable $\widehat{A}$ or $|\widehat{A}|$ summable. The matrix $\widehat{A}$ is absolutely conservative, i.e., it transforms an absolutely convergent series to an absolutely convergent series if $\sum_{n\geq 0} |\hat{a}_{n,k}| = 0(1)$ for all k. Let $p = \{p_n\}$ be the sequence of constants, real or complex, such that $P_n = p_0 + p_1 + \cdots + p_n \neq 0$. The Nörlund matrix, denoted by $(\widehat{N}, p)$, is the matrix $\widehat{A}$ with $\hat{a}_{n,k} = P(n, k) = P_{n-k}/P_n - P_{n-k-1}/P_{n-1}$ for $0 < k \leq n$, $P(0, 0) = 0$, and $P(n, k) = 0$ for $k > n$. The matrix $(\widehat{N}, p)$ reduces to the (C, α) matrix if

$$p_n = A_n^{\alpha-1}, \quad \text{where } (1-z)^{-\alpha-1} = \sum_{n\geq 0} A_n^{\alpha} z^n \quad (|z| < 1, \alpha > -1).$$

In what follows we write $\Delta p_n = p_n - p_{n+1}$; S and T denote the class of sequences p for which $\sum_{n\geq k} 1/(n|P_n|) = O(1/P_k)$, $k = 1, 2, 3, \ldots$, and $\sum_{k=1}^{n} k|\Delta p_k| = O(P_n)$, respectively; $P_n^* = \sum_{k=0}^{n} |p_k|$; K denotes a positive constant which is not necessarily the same at each occurrence.

Let f be an L integrable, periodic function with period 2π. BV denotes the set of functions of bounded variation in the interval $(0, 2\pi)$. We say $f \in \mathrm{IC}$ if

$$\int_0^{\pi} |\psi(t)| t^{-1}\, dt < \infty, \qquad \psi(t) = f(x+t) - f(x-t).$$

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By $\widehat{A} \in (\mathrm{BV} \to |F|)$ we mean that the Fourier series $\sum_{n\geq 0} A_n(x)$ of $f(x)$, $f \in \mathrm{BV}$, is $|\widehat{A}|$ summable. Similarly, $\widehat{A} \in (\mathrm{BV} \to |\widetilde{F}|)$ means that the conjugate Fourier series $\sum_{n\geq 0} B_n(x)$ of $f(x)$, $f \in \mathrm{BV}$, is $|\widehat{A}|$ summable.

Concerning $(\widetilde{N}, p) \in (\mathrm{BV} \to |\widetilde{F}|)$, Pati [7] proved

Theorem A. *Let the positive sequence p be such that $\{(n+1)p_n/P_n\}$ and $\{\sum_{k=0}^{n} P_k/(kP_n)\}$ are of bounded variation. Then $(\widehat{N}, p) \in (BV \to |\widetilde{F}|)$, if $f \in IC$.*

For the Fourier series the following is known ([1, Theorem G]; see also [3, 4, 6]).

Theorem B. *Let $p \in S \cap T$. Then $(\widehat{N}, p) \in (BV \to |F|)$.*

From Theorems A and B we see that in the former case an additional condition on f, viz. $f \in \mathrm{IC}$, is assumed. The requirement of this additional condition for a particular case was shown by Kumar [5] in the following form.

Theorem C. *Let p be a nonnegative nonincreasing sequence $\in S$. Then $(\widehat{N}, p) \in (BV \to |\widetilde{F}|) \Leftrightarrow f \in IC$.*

A necessary condition on p such that $(\widehat{N}, p) \in (\mathrm{BV} \to |F|)$ has been investigated in [4, Theorem 8]. In the present paper we investigate what are necessary conditions on the sequence p and the function f for $(\widehat{N}, p) \in (\mathrm{BV} \to |\widetilde{F}|)$. We first consider a general series-to-series matrix transformation and obtain

Theorem 1. *If $\widehat{A} \in (BV \to |\widetilde{F}|)$, then*

$$\sum_{\substack{n \\ r(n)\geq m}} \left| \sum_{\substack{k \\ r(n)-m\leq k\leq r(n)+m}} \frac{\hat{a}_{n,k}}{k}\left(1 - \frac{|r(n)-k|}{m+1}\right)\right| = O(1), \qquad m \geq 1,$$

where $r(n)$ is any sequence contained in $\mathbb{N}_0$.

2. Lemmas

We make use of the following results in this paper.

Lemma 1. *Let $\psi(t)$ be a function of bounded variation in $[0, \pi]$, and let for $n > 1$*

$$E_n = \{t\colon |\psi(t)| \text{ is continuous at } t \in [\pi/n, \pi/(n-1))\}.$$

Then there exist θ_n and $\theta'_n \in E_n$ such that for all $x \in E_n$

$$|\psi(\theta_n)| - \tfrac{1}{n} \leq |\psi(x)| \leq |\psi(\theta'_n)| + \tfrac{1}{n},$$

and

$$\sum_{n>2} |\psi(\theta_n)| n^{-1} < \infty \Leftrightarrow \sum_{n>2} |\psi(\theta'_n)| n^{-1} < \infty \Leftrightarrow f \in IC.$$

Lemma 1 is contained in [5, Lemma 4].

Lemma 2. *Let* $p \in T$. *Then*

(i) $\sum_{n\geq k} |P(n, k)| \leq K, \ k > 0$;
(ii) $np_n = O(P_n)$;
(iii) $P_{2n} \sim P_n \sim P_n^*$;
(iv) $\sum_{n\geq \mu} |P(n, k)| \leq Kk(\mu - k + 1)^{-1}, \ \mu \geq k \geq 1$;
(v) $\sum_{k=1}^{n} |P(n, k)| = O(1)$.

Proof. That the results (i) and (ii) and the fact that $P_n^* = O(P_n)$ follow from $p \in T$ are contained in [1, Lemmas 4 and 1(b)]. In view of the result [2, Lemma 1], (iii) follows from (ii) and the fact that $P_n^* = O(P_n)$. In order to prove (iv), we write

$$\begin{aligned}\sum_{n\geq\mu} |P(n, k)| &\leq \sum_{n\geq\mu} \left| \frac{(P_n - P_{n-k}) - (P_{n-1} - P_{n-k-1})}{P_n} \right| \\ &\quad + \sum_{n\geq\mu} \left| \frac{p_n(P_{n-1} - P_{n-k-1})}{P_n P_{n-1}} \right| \\ &\leq \sum_{n\geq\mu} \sum_{r=n-k}^{n-1} \left| \frac{\Delta p_r}{P_n} \right| + K \sum_{n\geq\mu} \frac{1}{n|P_n|} \sum_{r=1}^{k} |p_{n-r}|.\end{aligned}$$

We consider a typical term, i.e., the term corresponding to $r = n-j$, $1 \leq j \leq k$, of the first sum. In virtue of Lemma 2(iii) and (ii), we have $|\Delta[1/((n+1)P_n^*)]| \leq K/|(n+1)^2 P_n|$ and $P_{n+j} \sim P_{n+j}^* \sim P_n^*$. For $N \to \infty$, we obtain by Abel's transformation

$$\begin{aligned}\sum_{n=\mu}^{N} \left| \frac{\Delta p_{n-j}}{P_n} \right| &\leq K \sum_{n=\mu-j}^{N-j} \frac{|\Delta p_n|}{P_n^*} \\ &\leq K \sum_{n=\mu-j}^{N-j} \frac{1}{(n+1)^2 |P_n|} \sum_{k=1}^{n} (k+1)|\Delta p_k| \\ &\quad + \frac{1}{(N-j+2)P_{N-j+1}^*} \sum_{k=1}^{N-j} (k+1)|\Delta p_k| \\ &\quad + \frac{1}{(\mu-j+1)P_{\mu-j}^*} \sum_{k=1}^{\mu-j-1} (k+1)|\Delta p_k| \\ &\leq \frac{K}{(\mu-j+1)},\end{aligned}$$

since $p \in T$. In view of this and Lemma 2(ii), we get

$$\sum_{n\geq\mu} |P(n, k)| \leq Kk/(\mu - k + 1) + Kk \sum_{n\geq\mu} 1/(n(n-k+1)) \leq Kk/(\mu - k + 1),$$

since $\sum_{n\geq\mu} 1/(n(n-k+1)) \leq 2\sum_{n\geq\mu} 1/((n-k+2)(n-k+1)) = 2/(\mu-k+1)$.

The last part follows from Lemma 2(ii) and (iii) and the identity $P(n, k) = p_{n-k}/P_n - p_n P_{n-k-1}/(PnPn-1)$. This completes the proof of the lemma.

3. Proof of Theorem 1

Writing

$$B_k(x) = \frac{2}{\pi}\int_0^\pi \psi(t)\sin kt\,dt = -\frac{2}{k\pi}\int_0^\pi (1-\cos kt)\,d\psi(t),$$

we see that $\widehat{A} \in (\mathrm{BV} \to |\widetilde{F}|)$ if and only if

$$\sum_{n\geq 1}\left|\sum_{k\geq 1}\frac{\hat{a}_{n,k}}{2k}\int_0^\pi d\psi(t)\sin^2\frac{kt}{2}\right| < \infty. \tag{1}$$

Define $\phi_{n,m}(f) = \sum_{k=1}^m \hat{a}_{n,k}B_k(0)$,

$$f_t(x) = \sum_{k=1}^\infty \frac{1-\cos kt}{2k}\sin kx = \sum_{k=1}^\infty \frac{\sin^2(kt/2)}{k}\sin kx.$$

Then (cf. [4, p. 102])

$$|\phi_{n,m}(f)| \leq C_n \operatorname{Var} f$$

and hence

$$\left|\sum_{k=1}^m \frac{\hat{a}_{n,k}}{k}\sin^2\frac{kt}{2}\right| \leq cC_n,$$

where $c = \sup_t \operatorname{Var} f_t < \infty$. Since $\widehat{A} \in (\mathrm{BV} \to |\widetilde{F}|)$, we have

$$\sum_{n\geq 1}\left|\lim_m \phi_{n,m}(f)\right| < \infty$$

for every 2π periodic $f \in \mathrm{BV}$. Hence, by the Banach-Steinhaus theorem, $\sum_n \varepsilon_n \lim_m \phi_{n,m}(f)$, where $\varepsilon_n \in \{1, -1\}$, is a bounded linear functional in BV, so

$$\sum_{n\geq 1}\left|\lim_m \phi_{n,m}(f)\right| \leq D \operatorname{Var} f \quad \text{for some } D > 0.$$

Replacing f by f_t in this inequality yields

$$\sup_t \sum_{n\geq 1}\left|\sum_{k\geq 1}\frac{\hat{a}_{n,k}}{k}\sin^2\frac{kt}{2}\right| < \infty. \tag{2}$$

Let $K_m(x) = \frac{1}{2} + \sum_{i=1}^m (1 - \frac{i}{m+1})\cos ix$. We have

$$\cos(r(n)x)K_m(x) = \sum_{i=r(n)-m}^{r(n)+m}\left(1 - \frac{|r(n)-i|}{m+1}\right)\frac{\cos ix}{2}.$$

For every sequence $\{\varepsilon_n\}$ and for all large values of N we write

$$\begin{aligned} I &= \int_0^\pi K_m(t)\left(\sum_{n=1}^N \varepsilon_n \cos(r(n)t)\sum_{k\geq 1}\frac{\hat{a}_{n,k}}{k}(1-\cos kt)\right)dt \\ &= \frac{\pi}{8}\sum_{n=1}^N \varepsilon_n \sum_{k=r(n)-m}^{r(n)+m}\frac{\hat{a}_{n,k}}{k}\left(1 - \frac{|k-r(n)|}{m+1}\right), \end{aligned}$$

since $\int_0^\pi \cos it \sin^2(kt/2)\,dt = \frac{\pi}{4}$ for $i = k$, and zero otherwise. By the fact that the integral of $K_m(t)$ over $(-\pi, \pi) = \pi$, we see that (2) implies that $I = O(1)$. The theorem follows as $I = O(1)$ for every choice of $\{\varepsilon_n\}$.

4. Theorem 2 and its proof

Using the above result we prove the following for the Nörlund matrix. For the sake of convenience we say $p \in S'$ if

$$\sum_{n \geq 2m} \frac{1}{m|P_n|} \left| \sum_{k=0}^{m} \frac{P_k}{n-k} \right| = O(1), \qquad m = 1, 2, \ldots. \tag{3}$$

Theorem 2. *Let $p \in T$. Then*

(i) $(\widehat{N}, p) \in (BV \to |\widetilde{F}|) \Rightarrow p \in S'$;
(ii) $(\widehat{N}, p) \in (BV \to |\widetilde{F}|) \Rightarrow f \in IC$ *if* $p \in S$;
(iii) $f \in IC$ *and* $p \in S \Rightarrow (\widehat{N}, p) \in (BV \to |\widetilde{F}|)$.

Remark. If p is real, $p \in S'$ is equivalent to $p \in S$ for $p \in T$, because (cf. [4, p. 112]) the sequence $\{P_n\}$, for sufficiently large n, has a fixed sign. In view of Lemma 2(iii), (3) implies

$$\sum_{n \geq 2m} \frac{1}{m|P_n|} \sum_{k=0}^{m} \frac{P_k^*}{n-k} = O(1),$$

and this gives that

$$\sum_{n \geq 2m} \frac{P_{[m/2]}^*}{mn|P_n|} \sum_{k=[m/2]}^{m} 1 = O(1).$$

This is equivalent to $p \in S$ by Lemma 2(iii). It is direct to see that, even if p is complex, $p \in S$ implies $p \in S'$. Thus, if p is real and $\in T$, then

$$(\widehat{N}, p) \in (\mathrm{BV} \to |\widetilde{F}|) \Leftrightarrow f \in \mathrm{IC} \text{ and } p \in S. \tag{4}$$

Proof of Theorem 2. Letting $r(n) = n$ and $m = \tau = [\pi/t]$ in Theorem 1, we find that a necessary condition for $(\widehat{N}, p) \in (\mathrm{BV} \to |\widetilde{F}|)$ is

$$\sum = \sum_{n \geq 2\tau} \left| \sum_{k=n-\tau}^{n} \frac{P(n,k)}{k} \left(1 - \frac{n-k}{\tau+1}\right) \right| = O(1).$$

Writing $P(n,k) = p_{n-k}/P_n - P_{n-k-1}p_n/(P_nP_{n-1})$ and using that $np_n = O(P_n)$, we get

$$\sum \geq \sum_{n \geq 2\tau} \frac{1}{(\tau+1)|P_n|} \left| \sum_{k=0}^{\tau} \frac{(\tau-k+1)p_k}{n-k} \right| - \sum_{n \geq 2\tau} \frac{K}{n|P_n|} \sum_{k=0}^{\tau} \frac{(\tau-k+1)|P_{k-1}|}{(n-k)(\tau+1)}.$$

By Lemma 2(iii), the second sum $\leq K \sum_{n \geq 2\tau} \tau/((n-\tau)n) = O(1)$. Since

$$\sum_{k=0}^{\tau} \frac{\tau-k+1}{n-k} p_k = \sum_{k=0}^{\tau} \left(\frac{\tau-k+1}{n-k} - \frac{\tau-k}{n-k-1} \right) P_k,$$

we get

$$\sum \geq \sum_{n\geq 2\tau} \frac{1}{(\tau+1)|P_n|}\left|\sum_{k=0}^{\tau}\frac{P_k}{n-k}\right| - \sum_{n\geq 2\tau}\frac{1}{(\tau+1)|P_n|}\sum_{k=0}^{\tau}\frac{(\tau-k)|P_k|}{(n-k)(n-k-1)} - O(1).$$

Again the second sum is bounded by virtue that $P_n^* = O(P_n)$. Since $\sum = O(1)$, we get that $p \in S'$.

In order to prove the necessity of $f \in IC$, we write

$$B_k(x) = -\frac{4}{k\pi}\int_0^{\theta_n}\sin^2\frac{kt}{2}\,d\psi(t) + \frac{2\psi(\theta_n)}{k\pi} + \frac{2}{k\pi}\int_{\theta_n}^{\pi}\cos kt\,d\psi(t), \tag{5}$$

where the θ_n's are numbers considered in Lemma 1.

We proceed to estimate

$$\begin{aligned}
\Sigma_1 &= \sum_{n>1}\left|\int_0^{\theta_n}\left\{\sum_{k=1}^{n}\frac{P(n,k)}{k}\sin^2\frac{kt}{2}\right\}d\psi(t)\right|,\\
\Sigma_2 &= \sum_{n>1}|\psi(\theta_n)|\left|\sum_{k=1}^{n}\frac{P(n,k)}{k}\right|,\\
\Sigma_3 &= \sum_{n>1}\left|\int_{\theta_n}^{\pi}\left\{\sum_{k=1}^{n}\frac{P(n,k)}{k}\cos kt\right\}d\psi(t)\right|.
\end{aligned} \tag{6}$$

Since $\psi(t) \in \mathrm{BV}$, $t < \pi/(n-1)$, and $|\sin\theta| \leq K\theta$, we get by Lemma 2(i)

$$\begin{aligned}
\Sigma_1 &\leq \int_0^{\pi}|d\psi(t)|\sum_{n=2}^{\tau+1}\left|\sum_{k=1}^{n}\frac{P(n,k)}{k}\sin^2\frac{kt}{2}\right|\\
&\leq K\int_0^{\pi}|d\psi(t)|t^2\sum_{k=1}^{\tau+1}k\sum_{n=k}^{\tau+1}|P(n,k)| \leq K.
\end{aligned} \tag{7}$$

Suppose N is any arbitrary large number. In view of Lemma 2(i), we find that

$$\begin{aligned}
\Sigma_2 &\geq \left|\sum_{n=2}^{N}|\psi(\theta_n)|\sum_{k=2}^{n}\frac{P(n,k)}{k}\right| - \sum_{n=2}^{N}|\psi(\theta_n)|\,|P(n,1)|\\
&= \left|\sum_{k=2}^{N}\frac{1}{k}\sum_{n=k}^{N}|\psi(\theta_n)|P(n,k)\right| - O(1)\\
&= \left|\sum_{k=2}^{N}\frac{1}{k}\sum_{n=k}^{N}\Delta|\psi(\theta_n)|\frac{P_{n-k}}{P_n} + \sum_{k=2}^{N}\frac{|\psi(\theta_{N+1})|P_{N-k}}{kP_N}\right| - O(1)\\
&= \left|\sum_{k=2}^{N}\frac{|\psi(\theta_k)|-|\psi(\theta_{N+1})|}{k} - \sum_{k=2}^{N}\frac{1}{k}\sum_{n=k}^{N}\Delta|\psi(\theta_n)|\frac{(P_n-P_{n-k})}{P_n}\right.\\
&\qquad\left. + \sum_{k=2}^{N}\frac{|\psi(\theta_{N+1})|P_{N-k}}{kP_N}\right| - O(1)
\end{aligned}$$

$$\geq \sum_{k=2}^{N} \frac{|\psi(\theta_k)|}{k} - \sum_{n=2}^{N} |\Delta\psi(\theta_n)| \sum_{k=2}^{n} \left| \frac{P_n - P_{n-k}}{kP_n} \right|$$
$$- \sum_{k=2}^{N} |\psi(\theta_{N+1})| \left| \frac{P_N - P_{N-K}}{kP_N} \right| - O(1).$$

From Lemma 2(ii) and (iii) we see that

$$\sum_{k=1}^{[n/2]} \left| \frac{P_n - P_{n-k}}{kP_n} \right| \leq \sum_{k=1}^{[n/2]} \frac{1}{k} \sum_{r=0}^{k-1} \left| \frac{p_{n-r}}{P_n} \right| \leq K \sum_{k=1}^{[n/2]} \frac{1}{n-k} \leq K$$

and

$$\sum_{k=[n/2]}^{n} \frac{|P_n - P_{n-k}|}{k|P_n|} \leq K \sum_{k=[n/2]}^{n} \frac{1}{k} \leq K.$$

Consequently, $\sum_2 \geq \sum_{k=2}^{N} \frac{|\psi(\theta_k)|}{k} - O(1)$, since $\psi(t) \in \mathrm{BV}$. Hence from Lemma 1 we get

$$\sum_2 < \infty \Rightarrow f \in \mathrm{IC}. \tag{8}$$

Let N be a number tending to ∞, and assume $f \in \mathrm{IC}$. We have

$$\sum_2 \leq \sum_{n=2}^{N} |\psi(\theta_n)| \sum_{k=1}^{[n/2]} \frac{|P(n, k)|}{k} + \sum_{n=2}^{N} \frac{|\psi(\theta_n)|}{[n/2]} \sum_{k=[n/2]}^{n} |P(n, k)|$$
$$\leq \sum_{k=1}^{[N/2]} \frac{1}{k} \sum_{n=2k}^{N} |\psi(\theta_n)| \, |P(n, k)| + O(1)$$

by Lemmas 2(v) and 1. Writing the inner sum of the first term as

$$\sum_{n=2k}^{N} |\psi(\theta_n) P(n, k)| = - \sum_{n=2k}^{N} \Delta|\psi(\theta_n)| \sum_{r=n+1}^{N+1} |P(r, k)|$$
$$+ |\psi(\theta_{2k})| \sum_{r=2k}^{N+1} |P(r, k)| - |\psi(\theta_{N+1}) P(N+1, k)|$$

and using Lemma 2(i), (iv), and (v), we obtain

$$\begin{aligned} \sum_2 &\leq K \sum_{k=1}^{[N/2]} \frac{1}{k} \sum_{n=2k}^{N} |\Delta\psi(\theta_n)| \frac{k}{n-k} + K \sum_{k=1}^{N} \frac{|\psi(\theta_{2k})|}{k} + O(1) \\ &\leq K \sum_{n=1}^{N} |\Delta\psi(\theta_n)| \sum_{k=1}^{[n/2]} \frac{1}{n-k} + O(1) = O(1). \end{aligned} \tag{9}$$

To prove that $\sum_3 < \infty$, it is sufficient to show that for $0 < t \leq \pi$

$$\sum_{n \geq \tau} \left| \sum_{k=1}^{n} P(n, k) k^{-1} \cos kt \right| = O(1). \tag{10}$$

We estimate the sum in (10) in parts. The following sum with $\tau = 1$ is bounded by Lemma 2(i); for $\tau > 1$ we get by Lemma 2(i) and (iv)

$$\begin{aligned}\sum_{n\geq\tau}\left|\sum_{k=1}^{\tau}\frac{P(n,k)\cos kt}{k}\right| &\leq \left(\sum_{k=1}^{[\tau/2]}+\sum_{k=[\tau/2]}^{\tau}\right)\frac{1}{k}\sum_{n\geq\tau}|P(n,k)| \\ &\leq \sum_{k=1}^{[\tau/2]}\frac{1}{\tau-k}+K = O(1).\end{aligned} \tag{11}$$

Also,

$$\sum_{n=\tau}^{2\tau}\left|\sum_{k=\tau+1}^{n}P(n,k)\frac{\cos kt}{k}\right| \leq \sum_{k=\tau}^{2\tau}\frac{1}{k}\sum_{n=k}^{2\tau}|P(n,k)| = O(1). \tag{12}$$

By Abel's transformation and Lemma 2(i), we have

$$\begin{aligned}S &= \sum_{n>2\tau}\left|\sum_{k=\tau+1}^{n-\tau}P(n,k)\frac{\cos kt}{k}\right| \\ &\leq \frac{K}{t}\left\{\sum_{n>2\tau}\sum_{k=\tau}^{n-\tau-1}\frac{1}{k}\left|\frac{p_{n-k}}{P_n}-\frac{p_{n-k-1}}{P_{n-1}}\right|+\sum_{n>2\tau}\sum_{k=\tau}^{n-\tau}\frac{|P(n,k)|}{k^2}\right. \\ &\qquad\left.+\sum_{n>2\tau}\frac{|P(n,\tau+1)|}{\tau}+\sum_{n>2\tau}\frac{|P(n,n-\tau)|}{n-\tau}\right\} \\ &\leq \frac{K}{t}\left\{\sum_{n>2\tau}\sum_{k=\tau}^{n-\tau}\left|\frac{p_{n-k}-p_{n-k-1}}{kP_n}\right|+\sum_{n>2\tau}\sum_{k=\tau}^{n}\left|\frac{p_np_{n-k-1}}{kP_nP_{n-1}}\right|\right. \\ &\qquad+\sum_{k\geq\tau}\frac{1}{k^2}\sum_{n\geq k}|P(n,k)|+\sum_{n>2\tau}\left|\frac{p_\tau}{(n-\tau)P_n}\right| \\ &\qquad\left.+\sum_{n>2\tau}\left|\frac{P_{\tau-1}p_n}{P_nP_{n-1}(n-\tau)}\right|\right\}+O(1) \\ &\leq \frac{K}{t}\left\{\sum_{n>2\tau}\sum_{k=\tau}^{[n/2]}\left|\frac{p_{n-k}-p_{n-k-1}}{kP_n}\right|\right. \\ &\qquad+\sum_{n>2\tau}\frac{1}{n|P_n|}\sum_{k=[n/2]}^{n-\tau}|p_{n-k}-p_{n-k-1}| \\ &\qquad\left.+\sum_{n>2\tau}\sum_{k=\tau}^{[n/2]}\left|\frac{p_np_{n-k-1}}{kP_nP_{n-1}}\right|+\sum_{n>2\tau}\sum_{k=[n/2]}^{n}\left|\frac{p_np_{n-k-1}}{kP_nP_{n-1}}\right|\right\} \\ &\quad+K|P_\tau|\sum_{n>\tau}\frac{1}{n|P_n|}+O(1)\end{aligned}$$

by Lemma 2(ii) and (iii) and the fact that $\sum_{k\geq m>0}k^{-2}\leq 2m^{-1}$. Applying change of the order of summation in the first term and the facts that $np_n =$

$O(P_n)$ and $p \in S$, we obtain

$$S \le \frac{K}{t}\left\{\sum_{k\ge\tau}\frac{1}{k}\sum_{n\ge 2k}\left|\frac{p_{n-k}-p_{n-k-1}}{P_n}\right| + \sum_{n>2\tau}\frac{1}{n|P_n|}\sum_{k=\tau}^{[n/2]+1}|\Delta p_{k-1}| \right.$$
$$\left. + \sum_{k\ge\tau}\frac{1}{k}\sum_{n\ge 2k}\frac{1}{n(n-k)} + \sum_{n\ge 2\tau}\frac{1}{n^2}\sum_{k=[n/2]}^{n}\left|\frac{p_{n-k-1}}{P_{n-1}}\right|\right\} + O(1).$$

Since $P_n^* = O(P_n)$, we have

$$S \le \frac{K}{t}\left\{\sum_{k\ge\tau}\frac{1}{k}\sum_{n\ge k}\left|\frac{\Delta p_{n-1}}{P_n}\right| + \sum_{k\ge\tau}|\Delta p_{k-1}|\sum_{n\ge k}\frac{1}{n|P_n|} + \sum_{k\ge\tau}\frac{1}{k^2}\right\} + O(1) \tag{13}$$
$$= O(1)$$

by the following inequality obtained in the proof of Lemma 2 and the fact that $p \in S$

$$\sum_{n\ge\mu}\frac{|\Delta p_{n-1}|}{|P_n|} \le \frac{K}{\mu}, \qquad \mu \ge 1.$$

Lastly,

$$\sum_{n>2\tau}\left|\sum_{k=n-\tau+1}^{n} P(n,k)\frac{\cos kt}{k}\right| \le K\sum_{n\ge 2\tau}\frac{1}{n}\sum_{k=n-\tau}^{n}|P(n,k)|$$
$$\le K\sum_{n\ge 2\tau}\frac{1}{n}\sum_{k=n-\tau}^{n}\left|\frac{p_{n-k}}{P_n}\right| + K\sum_{n\ge 2\tau}\frac{1}{n}\sum_{k=n-\tau}^{n}\left|\frac{p_n P_{n-k-1}}{P_n P_{n-1}}\right| \tag{14}$$
$$\le KP_{\tau^*}\sum_{n\ge 2\tau}\frac{1}{(n|P_n|)} + K\sum_{n\ge 2\tau}\frac{(\tau+1)}{n^2} = O(1),$$

by virtue of $p \in S$. Combining (11)–(14), we get (10). Hence

$$\textstyle\sum_3 < \infty. \tag{15}$$

From (5) and (6) we can see that the expression in (1) lies between $|\sum_2| - |\sum_1| - |\sum_3|$ and $|\sum_1| + |\sum_2| + |\sum_3|$. Further, (7) and (15) give that $\sum_1 + \sum_3 < \infty$ if $p \in T \cap S$, while (8) and (9) imply that $\sum_2 < \infty$ if and only if $f \in IC$ whenever $p \in T \cap S$. Hence we get (ii) and (iii) of the theorem.

This completes the proof of the theorem.

Combining [4, Theorem 8 and Remarks, p. 112] and (4), we obtain

Theorem 3. *Let p be real and $\in T$. Then*

$$(\widehat{N}, p) \in (BV \to |\widetilde{F}|) \Leftrightarrow (\widehat{N}, p) \in (BV \to |F|) \text{ and } f \in IC.$$

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DEPARTMENT OF MATHEMATICS, RANI DURGAVATI UNIVERSITY, JABALPUR, INDIA

DEPARTMENT OF MATHEMATICS, ULM UNIVERSITY, ULM, GERMANY