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ON THE FACTORIZATION OF A_p WEIGHTS

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ABSTRACT. It is shown that if $(u, v) \in A_p$ then there exist $(u_1, v_1) \in A_1$ and $(u_2, v_2) \in A_1$ such that $u = u_1^{p'} v_2^{-p}$, $v = v_1^{p'} u_2^{-p}$.

1. INTRODUCTION

Let M denote the Hardy-Littlewood maximal operator in R^n,

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f|,$$

where Q is always a cube in R^n and $|\cdot|$ denotes Lebesgue measure. Let $w(x)$, $u(x)$, and $v(x)$ be nonnegative measurable functions on R^n, and recall that [1]

$$W \in A_1 \quad \text{if and only if} \quad Mw(x) \le Cw(x) \text{ a.e. } x \in R^n;$$

$$w \in A_p \quad \text{if and only if} \quad \left(\int_Q w\right)\left(\int_Q w^{1-p'}\right)^{p-1} \le C|Q|^p;$$

$$(u, v) \in A_1 \quad \text{if and only if} \quad Mu(x) \le Cv(x) \text{ a.e. } x \in R^n;$$

$$(u, v) \in A_p \quad \text{if and only if} \quad \left(\int_Q u\right)\left(\int_Q v^{1-p'}\right)^{p-1} \le C|Q|^p;$$

$$(u, v) \in S_p \quad \text{if and only if} \quad \int |Mf|^p u \le C \int |f|^p v.$$

Throughout C denotes a constant and may vary from line to line, $1 < p < \infty$ (unless otherwise noted), and $\frac{1}{p} + \frac{1}{p'} = 1$.

If there exist $C_1 > 0$ and $C_2 > 0$ such that $C_1 u \le v \le C_2 u$ a.e. $x \in R^n$, then $u \sim v$.

In [2] Jones showed that $w \in A_p$ if and only if there exist weights w_1, $w_2 \in A_1$ with $w = w_1 w_2^{1-p}$. In this note we consider the factorization of the general A_p weights. Our results are as follows.

Theorem 1. *Let $(u, v) \in A_p$ and $0 < \delta < 1$. Then there exist (u_1, v_1) and $(u_2, v_2) \in A_1$ such that*

$$u^\delta = u_1 v_2^{1-p}, \qquad v^\delta = v_1 u_2^{1-p}.$$

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Moreover, if u, $v \in L^1_{\mathrm{loc}}(R^n)$ *and* $0 < u$, $v < \infty$ *a.e.* $x \in R^n$, *then* $u \sim v$ *if and only if* $u_1 \sim v_1$ *and* $u_2 \sim v_2$.

Theorem 2. *There exist* $(u_1, v_1) \in A_1$ *and* $(u_2, v_2) \in A_1$ *such that for any* $0 < \delta < 1$

$$(u, v) = ((u_1 v_2^{1-p})^{1/\delta}, (v_1 u_2^{1-p})^{1/\delta}) \notin A_p.$$

This shows that the converse of Theorem 1 is false.

Corollary 1. *If* $(u, v) \in A_p$, *then there exist* (u_1, v_1) *and* $(u_2, v_2) \in A_1$ *such that*

$$u = u_1^{p'} v_2^{-p}, \qquad v = v_1^{p'} v_2^{-p}.$$

Proof. Using Theorem 1 with $\delta = 1/p'$, we get the result.

Corollary 2. *If* $w \in A_p$, *then there exist* w_1, $w_2 \in A_1$ *such that* $w = w_1 w_2^{1-p}$.

Proof. We may assume that $0 < w < \infty$ a.e. so that $w \in L^1_{\mathrm{loc}}(R^n)$. Since $w \in A_p$, there exists $\tau > 1$ such that $w^\tau \in A_p$ [1]. Let $\delta = 1/\tau$. By Theorem 1 there exist (u_1, v_1), $(u_2, v_2) \in A_1$ with $u_1 \sim v_1$, $u_2 \sim v_2$, and therefore u_1, $u_2 \in A_1$, such that $(w^\tau)^{1/\tau} = u_1 v_2^{1-p}$, i.e., $w = u_1 v_2^{1-p}$. Let $u_1 = w_1$, $v_2 = w_2$; we get $w = w_1 w_2^{1-p}$ with w_1, $w_2 \in A_1$.

2. Proof of the theorems

We need the following lemmas.

Lemma 1 [3]. *If* $(u, v) \in A_p$, *and* $0 < \delta < 1$, *then* $(u^\delta, v^\delta) \in S_p$.

Lemma 2 [3]. *Assume that* $(u, v) \in S_p$ *and* $(v^{1-p'}, u^{1-p'}) \in S_{p'}$. *Then there are functions* $w_j \geq 0$ *such that*

$$u^{1/p} M w_j \leq C_j w_j v^{1/p}, \qquad j = 1, 2,$$

and

$$u^{1/p} v^{1/p'} = w_1 w_2^{1-p}.$$

Lemma 3. *If* $(u, v) \in A_p$ $(1 \leq p < \infty)$ *and* u, $v \in L^1_{\mathrm{loc}}(R^n)$, *then* $u(x) \leq Cv(x)$ *a.e.* $x \in R^n$.

Proof. For $p = 1$ it is obvious. For $1 < p < \infty$, since $(u, v) \in A_p$ if and only if [1]

$$\left(1/|Q| \int_Q f\right)^p \left(\int_Q u\right) \leq C \int_Q f^p v$$

for any $f \geq 0$, $Q \subset R^n$, let $f \equiv 1$. Then $\int_Q u \leq C \int_Q v$, i.e.,

$$1/|Q| \int_Q u \leq C 1/|Q| \int_Q v.$$

Let $|Q| \to 0$. Since u, $v \in L^1_{\mathrm{loc}}(R^n)$ we get

$$u(x) \leq Cv(x) \quad \text{a.e. } x \in R^n.$$

Lemma 4. *If* $(u, v) \in S_p$ *and* $(v^{1-p'}, u^{1-p'}) \in S_{p'}$, *then there exist* $(u_j, v_j) \in A_1$, $j = 1, 2$, *such that*

$$u = u_1 v_2^{1-p}, \qquad v = v_1 u_2^{1-p}.$$

Proof. Let $u_j = w_j$ and $v_j = w_j v^{1/p} u^{-1/p}$ by Lemma 2. Then $(u_j, v_j) \in A_1$, $j = 1, 2$.

Since $u^{1/p} v^{1/p'} = w_1 w_2^{1-p}$, we get

$$u = w_1^p w_2^{p(1-p)} v^{1-p}, \qquad v = w_1^{p'} w_2^{-p} u^{1-p'}.$$

Thus,

$$\begin{aligned} u_1 v_2^{1-p} &= w_1 (w_2 v^{1/p} u^{-1/p})^{1-p} \\ &= w_1 w_2^{1-p} v^{-1/p'} (w_1^p w_2^{p(1-p)} v^{1-p})^{1/p'} = w_1^p w_2^{p(1-p)} v^{1-p} = u \end{aligned}$$

and

$$\begin{aligned} v_1 u_2^{1-p} &= (w_1 v^{1/p} u^{-1/p}) w_2^{1-p} \\ &= w_1 u^{-1/p} (w_1^{p'} w_2^{-p} u^{1-p'})^{1/p} w_2^{1-p} = w_1^{p'} w_2^{-p} u^{1-p'} = v. \end{aligned}$$

Proof of Theorem 1. Since $(u, v) \in A_p$ if and only if $(v^{1-p'}, u^{1-p'}) \in A_{p'}$. Lemma 1 gives $(u^\delta, v^\delta) \in S_p$ and $(v^{\delta(1-p')}, u^{\delta(1-p')}) \in S_{p'}$. From Lemma 4 we get $(u_j, v_j) \in A_1$, $j = 1, 2$, such that

$$u^\delta = u_1 v_2^{1-p}, \qquad v^\delta = v_1 u_2^{1-p}.$$

For u, $v \in L^1_{\mathrm{loc}}(R^n)$ and $0 < u$, $v < \infty$ a.e. if $u_1 \sim v_1$, $u_2 \sim v_2$, then $u_1 v_2^{1-p} \sim v_1 u_2^{1-p}$. Hence $u^\delta \sim v^\delta$, i.e., $u \sim v$.

Conversely, if $u \sim v$, then there exist $C_1 > 0$, $C_2 > 0$ such that

$$C_1 v_1 u_2^{1-p} \le u_1 v_2^{1-p} \le C_2 v_1 u_2^{1-p}.$$

From the left inequality, we have

$$(v_2/u_2)^{p-1} \le C u_1/v_1.$$

Note that $u_j \le C v_j$, $j = 1, 2$. We get $u_2 \sim v_2$ and also $u_1 \sim v_1$.

Proof of Theorem 2. Let $u_1 = v_1 \equiv 1$, $u_2 = \varphi(x) = \chi_{[0,1]}$, and $v_2 = M\varphi$. Then $(u_j, v_j) \in A_1$, $j = 1, 2$, but $(u_1 v_2^{1-p}, v_1 u_2^{1-p}) = ((M\varphi)^{1-p}, \varphi^{1-p}) \notin S_p$. In fact, let $f = \varphi = \chi_{[0,1]}$. Then we have

$$\int (Mf)^p (M\varphi)^{1-p} = \int M\varphi = +\infty,$$

but $\int |f|^p \varphi^{1-p} = \int \varphi = 1$.

From Lemma 1, $(u, v) = ((u_1 v_2^{1-p})^{1/\delta}, (v_1 u_2^{1-p})^{1/\delta}) \notin A_p$ for any $0 < \delta < 1$.

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