ON THE FACTORIZATION OF $A_p$ WEIGHTS

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Abstract. It is shown that if $(u, v) \in A_p$ then there exist $(u_1, v_1) \in A_1$ and $(u_2, v_2) \in A_1$ such that $u = u_1^p v_2^{-p}$, $v = v_1^p u_2^{-p}$.

1. Introduction

Let $M$ denote the Hardy-Littlewood maximal operator in $\mathbb{R}^n$,
\[ Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f|, \]
where $Q$ is always a cube in $\mathbb{R}^n$ and $|\cdot|$ denotes Lebesgue measure. Let $w(x)$, $u(x)$, and $v(x)$ be nonnegative measurable functions on $\mathbb{R}^n$, and recall that
\[ W \in A_1 \quad \text{if and only if} \quad Mw(x) \leq Cw(x) \text{ a.e. } x \in \mathbb{R}^n; \]
\[ w \in A_p \quad \text{if and only if} \quad \left( \int_Q w \right) \left( \int_Q w^{1-p'} \right)^{p-1} \leq C|Q|^p; \]
\[ (u, v) \in A_1 \quad \text{if and only if} \quad Mu(x) \leq Cv(x) \text{ a.e. } x \in \mathbb{R}^n; \]
\[ (u, v) \in A_p \quad \text{if and only if} \quad \left( \int_Q u \right) \left( \int_Q v^{1-p'} \right)^{p-1} \leq C|Q|^p; \]
\[ (u, v) \in S_p \quad \text{if and only if} \quad \int |Mf|^p u \leq C \int |f|^p v. \]
Throughout $C$ denotes a constant and may vary from line to line, $1 < p < \infty$ (unless otherwise noted), and $\frac{1}{p} + \frac{1}{p'} = 1$.

If there exist $C_1 > 0$ and $C_2 > 0$ such that $C_1 u \leq v \leq C_2 u$ a.e. $x \in \mathbb{R}^n$, then $u \sim v$.

In [2] Jones showed that $w \in A_p$ if and only if there exist weights $w_1$, $w_2 \in A_1$ with $w = w_1 w_2^{1-p}$. In this note we consider the factorization of the general $A_p$ weights. Our results are as follows.

Theorem 1. Let $(u, v) \in A_p$ and $0 < \delta < 1$. Then there exist $(u_1, v_1)$ and $(u_2, v_2) \in A_1$ such that
\[ u^\delta = u_1 v_2^{1-p}, \quad v^\delta = v_1 u_2^{1-p}. \]
Moreover, if \( u, v \in L^1_{\text{loc}}(\mathbb{R}^n) \) and \( 0 < u, v < \infty \) a.e. \( x \in \mathbb{R}^n \), then \( u \sim v \) if and only if \( u_1 \sim v_1 \) and \( u_2 \sim v_2 \).

**Theorem 2.** There exist \( (u_1, v_1) \in A_1 \) and \( (u_2, v_2) \in A_1 \) such that for any \( 0 < \delta < 1 \)
\[
(u, v) = ((u_1v_2^{1-p})^{1/\delta}, (v_1u_2^{1-p})^{1/\delta}) \notin A_\rho.
\]

This shows that the converse of Theorem 1 is false.

**Corollary 1.** If \( (u, v) \in A_\rho \), then there exist \( (u_1, v_1) \) and \( (u_2, v_2) \in A_1 \) such that
\[
u = u_1^{p'}v_2^{-p}, \quad v = v_1^{p'}v_2^{-p}.
\]

**Proof.** Using Theorem 1 with \( \delta = 1/p' \), we get the result.

**Corollary 2.** If \( w \in A_\rho \), then there exist \( w_1, w_2 \in A_1 \) such that \( w = w_1w_2^{1-p} \).

**Proof.** We may assume that \( 0 < w < \infty \) a.e. so that \( w \in L^1_{\text{loc}}(\mathbb{R}^n) \). Since \( w \in A_\rho \), there exists \( \tau > 1 \) such that \( w^\tau \in A_\rho \) [1]. Let \( \delta = 1/\tau \). By Theorem
\[
(ux, vx) \in A_1 \text{ with } u_1 \sim v_1, u_2 \sim v_2, \text{ and therefore}
\]
\[
u_1, u_2 \in A_1, \text{ such that } (w^\tau)^{1/\tau} = u_1v_2^{1-p}, \text{ i.e., } w = u_1v_2^{1-p}. \]

2. **Proof of the theorems**

We need the following lemmas.

**Lemma 1** [3]. If \( (u, v) \in A_\rho \), and \( 0 < \delta < 1 \), then \( (u^\delta, v^\delta) \in S_\rho \).

**Lemma 2** [3]. Assume that \( (u, v) \in S_\rho \) and \( (u_1^{1-p'}, u_2^{1-p'}) \in S_\rho \). Then there are functions \( w_j \geq 0 \) such that
\[
u^{1/p}M_{w_j} \leq C_jw_j^{1/p}, \quad j = 1, 2,
\]
and
\[
u^{1/p}v^{1/p'} = w_1w_2^{1-p}.
\]

**Lemma 3.** If \( (u, v) \in A_\rho \) \((1 \leq p < \infty)\) and \( u, v \in L^1_{\text{loc}}(\mathbb{R}^n) \), then \( u(x) \leq Cv(x) \) a.e. \( x \in \mathbb{R}^n \).

**Proof.** For \( p = 1 \) it is obvious. For \( 1 < p < \infty \), since \( (u, v) \in A_\rho \) if and only if [1]
\[
\left( \frac{1}{|Q|} \int_Q f \right)^p \left( \int_Q u \right) \leq C \int_Q f^{p}v
\]
for any \( f \geq 0, Q \subset \mathbb{R}^n \), let \( f \equiv 1 \). Then \( \int_Q u \leq C \int_Q v \), i.e.,
\[
\frac{1}{|Q|} \int_Q u \leq C \frac{1}{|Q|} \int_Q v.
\]
Let \( |Q| \to 0 \). Since \( u, v \in L^1_{\text{loc}}(\mathbb{R}^n) \) we get
\[
u(x) \leq Cv(x) \text{ a.e. } x \in \mathbb{R}^n.
\]

**Lemma 4.** If \( (u, v) \in S_\rho \) and \( (u_1^{1-p'}, u_1^{1-p'}) \in S_\rho \), then there exist \( (u_j, v_j) \in A_1, j = 1, 2 \), such that
\[
u = u_1v_2^{1-p}, \quad v = v_1u_2^{1-p}.
\]
Proof. Let \( u_j = w_j \) and \( v_j = w_j v_j^{1/p} u_j^{-1/p} \) by Lemma 2. Then \((u_j, v_j) \in A_1, \)
\( j = 1, 2. \)

Since \( u_j^{1/p} v_j^{1/p} = w_1 w_2^{1-p} \), we get
\[
u = w_1^{p} w_2^{1-p} u_1^{-p'}, \quad v = w_1^{p} w_2^{1-p} u_1^{-p'}.
\]

Thus,
\[
u_1 v_2^{1-p} = w_1 (w_2 v_1^{1/p} u_1^{-1/p})^{1-p} = w_1 w_2^{1-p} v_2^{1/p} (w_1^{p} w_2^{(1-p)} v_1^{-p})^{1/p'} = w_1^{p} w_2^{p(1-p)} u_1^{1-p} = u
\]
and
\[
u_1 v_2^{1-p} = (w_1 v_1^{1/p} u_1^{-1/p}) w_2^{1-p} = w_1 u_1^{1-p} (w_1^{p} w_2^{1-p} u_1^{1-p'})^{1/p} w_2^{1-p} = w_1^{p} w_2^{1-p} u_1^{1-p'} = v.
\]

Proof of Theorem 1. Since \((u, v) \in A_\rho\) if and only if \((v_1^{1-p'}, u_1^{-1-p'}) \in A_{\rho'}\). Lemma 1 gives \((u^\delta, v^\delta) \in S_\rho\) and \((v_2^{1-p'}, u_2^{-1-p'}) \in S_{\rho'}\). From Lemma 4 we get \((u_j, v_j) \in A_1, \ j = 1, 2, \) such that
\[
u^\delta = u_1 v_2^{1-p}, \quad v^\delta = v_1 u_2^{1-p}.
\]

For \( u, v \in L^{1}_{\text{loc}}(R^n) \) and \( 0 < u, v < \infty \) a.e. if \( u_1 \sim v_1, u_2 \sim v_2, \) then \( u_1 v_2^{1-p} \sim v_1 u_2^{1-p}. \) Hence \( u^\delta \sim v^\delta, \) i.e., \( u \sim v. \)

Conversely, if \( u \sim v \), then there exist \( C_1 > 0, C_2 > 0 \) such that
\[
C_1 v_1 u_2^{1-p} \leq u_1 v_2^{1-p} \leq C_2 v_1 u_2^{1-p}.
\]

From the left inequality, we have
\[
(v_2/u_2)^{p-1} \leq Cu_1/v_1.
\]

Note that \( u_j \leq Cv_j, \ j = 1, 2. \) We get \( u_2 \sim v_2 \) and also \( u_1 \sim v_1. \)

Proof of Theorem 2. Let \( u_1 = v_1 \equiv 1, u_2 = \varphi(x) = \chi_{[0, 1]}, \) and \( v_2 = M\varphi. \) Then \((u_j, v_j) \in A_1, \ j = 1, 2, \) but \((u_1 v_2^{1-p}, v_1 u_2^{1-p}) = (M\varphi)^{1-p}, \varphi^{1-p} \notin S_\rho. \) In fact, let \( f = \varphi = \chi_{[0, 1]} \). Then we have
\[
\int (Mf)^p (M\varphi)^{1-p} = \int M\varphi = +\infty,
\]
but \( \int |f|^p \varphi^{1-p} = \int \varphi = 1. \)

From Lemma 1, \((u, v) = ((u_1 v_2^{1-p})^{1/\delta}, (v_1 u_2^{1-p})^{1/\delta}) \notin A_\rho\) for any \( 0 < \delta < 1. \)

References


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