FULL SUBALGEBRAS OF JORDAN-BANACH ALGEBRAS
AND ALGEBRA NORMS ON JB*-ALGEBRAS

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Abstract. We introduce normed Jordan Q-algebras, namely, normed Jordan algebras in which the set of quasi-invertible elements is open, and we prove that a normed Jordan algebra is a Q-algebra if and only if it is a full subalgebra of its completion. Homomorphisms from normed Jordan Q-algebras onto semisimple Jordan-Banach algebras with minimality of norm topology are continuous. As a consequence, the topology of the norm of a JB*-algebra is the smallest normable topology making the product continuous, and JB*-algebras have minimality of the norm. Some applications to (associative) C*-algebras are also given: (i) the associative normed algebras that are ranges of continuous (resp. contractive) Jordan homomorphisms from C*-algebras are bicontinuously (resp. isometrically) isomorphic to C*-algebras, and (ii) weakly compact Jordan homomorphisms from C*-algebras are of finite rank.

Introduction

Associative normed algebras in which the set of quasi-invertible elements is open were considered first by Kaplansky [15], who called them "normed Q-algebras". Since then, normed Q-algebras were seldom studied (exceptions are Yood's relevant papers [32, 33]) until the Wilansky conjecture [30], which states that associative normed Q-algebras are nothing but full subalgebras of Banach algebras. In fact, Palmer [20] set the bases for a systematic study of associative normed Q-algebras, providing, in particular, an affirmative answer to Wilansky's conjecture (see also [3] for an independent proof of this result). It must also be mentioned that full subalgebras of Banach algebras have played a relevant role in connection with the nonassociative extension of Johnson's uniqueness-of-norm theorem [25] and with the nonassociative extension of the Civin-Yood decomposition theorem [10].

The general theory of Jordan-Banach algebras began with the paper by Balachandran and Rema [2]; since then it has been fully developed in a complete analogy with the case of (associative) Banach algebras (see, e.g., [29, 16, 1, 9, 25, 10, 5, 11]), although in most of the cases new methods have been needed for such Jordan extensions of associative results. Noncomplete normed Jordan algebras whose sets of quasi-invertible elements are open (called, of course, "normed Jordan Q-algebras") were only germinally considered in [29].
It is the aim of this paper to develop the theory of normed Jordan $Q$-algebras, providing also the complete analogy with the associative case. With no additional effort we shall even consider normed noncommutative Jordan $Q$-algebras so that the associative (or even alternative) case will remain contained in our approach. In the first part of this paper we give several characterizations of normed noncommutative Jordan $Q$-algebras (Theorem 4 and Proposition 6), including the one asserting that normed noncommutative Jordan $Q$-algebras are nothing but full subalgebras of noncommutative Jordan-Banach algebras (the affirmative answer to Wilansky's conjecture in the Jordan setting). It must also be emphasized that the normed complexification of a normed noncommutative Jordan real $Q$-algebra is also a $Q$-algebra (Proposition 3), whose proof needs an intrinsic Jordan method as it is the Shirshov-Cohn theorem with inverses [18]. We end this section with a theorem on automatic continuity (Theorem 8) which is a Jordan extension of the main result in [27].

The second part of the paper is devoted to applying a part of the developed theory of normed Jordan $Q$-algebras in order to obtain new results on $JB^*$-algebras (hence on the Jordan structure of $C^*$-algebras). Thus in Theorem 10 we use the aforementioned result on automatic continuity to generalize Cleveland's theorem [8], which asserts that the topology of the norm of a $C^*$-algebra $A$ is the smallest algebra-normable topology on $A$, to noncommutative $JB^*$-algebras. (As a consequence, every norm on the vector space of a $C^*$-algebra that makes the Jordan product continuous defines a topology which is stronger than the topology of the $C^*$-norm—a result that improves the original Cleveland theorem.) The $JB^*$-extension of Cleveland's result was obtained almost at the same time and with essentially identical techniques by Bensebah [4]. With the nonassociative Vidav-Palmer Theorem [24], it is also proved that noncommutative $JB^*$-algebras have minimality of the norm (Proposition 11); i.e., $|\cdot| = \|\cdot\|$ whenever $|\cdot|$ is any algebra norm satisfying $|\cdot| \leq \|\cdot\|$. Finally, with the main result in [26], we determine the associative normed algebras that are ranges of continuous Jordan homomorphisms from $C^*$-algebras (Corollary 12), and we show that ranges of weakly compact Jordan homomorphisms from $C^*$-algebras are finite dimensional (Corollary 13).

1. Preliminaries and notation

All the algebras we consider here are real or complex. A nonassociative algebra $A$ satisfying $x(yx) = (xy)x$ and $x^2(yx) = (x^2y)x$ for all $x$, $y$ in $A$ is called a noncommutative Jordan (in short, n.c.J.) algebra. As usual $A^+$ denotes the symmetrized algebra of $A$ with product $x \cdot y = \frac{1}{2}(xy + yx)$. Recall that $A^+$ is a Jordan algebra whenever $A$ is a n.c.J. algebra. For any element $a$ in a n.c.J. algebra $A$, $U_a$ denotes the linear operator on $A$ defined by

$$U_a(x) = a(ax + xa) - a^2x = (ax + xa)a - xa^2, \quad x \in A.$$  

Recall that $U_a = U_a^+$, where $U_a^+$ is the usual $U$-operator on the Jordan algebra $A^+$. An element $a$ in a n.c.J. algebra $A$ with unit $1$ is invertible with inverse $b$ if $ab = ba = 1$ and $a^2b = ba^2 = a$. This is equivalent to $a$ being invertible with inverse $b$ in the Jordan algebra $A^+$ [19], whence, if $\text{Inv}(A)$ denotes the set of invertible elements in $A$, then we have $\text{Inv}(A) = \text{Inv}(A^+)$. We recall the
following basic results (see [14, Theorem 13, p. 52]). For elements \( x, y \) in \( A \):
- \( x \) is invertible if and only if \( U_x \) is an invertible operator, and in that case \( x^{-1} = U_x^{-1}(x) \) and \( U_x^{-1} = U_{x^{-1}} \).
- \( x \) and \( y \) are invertible if and only if \( U_x(y) \) is invertible.
- \( x \) is invertible if and only if \( x^n \) (\( n \geq 1 \)) is invertible.

An element \( a \) in a n.c.J. algebra \( A \) is quasi-invertible with quasi-inverse \( b \) if \( 1 - a \) has inverse \( 1 - b \) in the n.c.J. algebra \( A_1 \) (the unitization of \( A \)) obtained by adjoining a unit to \( A \) in the usual way. Let \( q - \text{Inv}(A) \) denote the set of quasi-invertible elements in \( A \). Any real n.c.J. algebra \( A \) can be regarded as a real subalgebra of a complex n.c.J. algebra \( A_C \) which satisfies \( A_C = A \oplus iA \) and is called the complexification of \( A \) (see [6, Definition 3.1]). The spectrum of an element \( x \) in a n.c.J. algebra \( A \), denoted by \( sp(x, A) \), is defined as in the associative case (see [6, Definitions 5.1 and 13.6]). The "algebraic" spectral radius of \( x \) is defined by

\[
\rho(x, A) := \sup\{|\lambda| : \lambda \in sp(x, A)\}.
\]

We write \( sp(x) \) or \( \rho(x) \) instead of \( sp(x, A) \) or \( \rho(x, A) \) when no confusion can occur. A subalgebra \( B \) of a n.c.J. algebra \( A \) is called a full subalgebra of \( A \) if \( B \) contains the quasi inverses of its elements that are quasi-invertible in \( A \), that is, the equality \( q - \text{Inv}(B) = B \cap q - \text{Inv}(A) \) holds. Easy examples of full subalgebras are left or right ideals and strict inner ideals (see definition later).

It is clear that if \( B \) is a full subalgebra of a complex n.c.J. algebra \( A \), then

\[
sp(x, A) \cup \{0\} = sp(x, B) \cup \{0\} \quad (x \in B).
\]

A n.c.J. algebra \( A \) is said to be normed if an algebra norm (a norm \( \| \cdot \| \) on the vector space of \( A \) satisfying \( \|ab\| \leq \|a\| \|b\| \) for all \( a, b \) in \( A \)) is given on \( A \). In that case the "geometric" spectral radius of \( x \in A \) is the number

\[
r_{\|\cdot\|}(x) := \lim \|x^n\|^{1/n}.
\]

When no confusion can occur, we write \( r(x) \) to denote \( r_{\|\cdot\|}(x) \). The unitization \( A_1 \) of n.c.J. normed algebra \( A \) becomes normed by defining \( \|x + \alpha\| := \|x\| + \|\alpha\| \) for \( x + \alpha \) in \( A_1 \). Also the complexification of a real n.c.J. normed algebra can be normed as in [6, Proposition 13.3]. Since every element \( x \) in a n.c.J. normed algebra \( A \) can be immersed in a closed associative full subalgebra of \( A \) [5, Théorème 1], it follows that the properties of the spectrum and the classical functional calculus for a single element in (associative) Banach algebras remain valid for n.c.J. complete normed algebras. In particular, the Gelfand-Beurling formula, \( r(x) = \rho(x) \), holds for any element \( x \) in a n.c.J. complete normed algebra.

2. Noncommutative Jordan \( Q \)-algebras

A n.c.J. normed algebra \( A \) in which the set \( q - \text{Inv}(A) \) is open is called a n.c.J. \( Q \)-algebra. Taking into account that \( A^+ \) with the same norm as \( A \) is a Jordan normed algebra and \( q - \text{Inv}(A^+) = q - \text{Inv}(A) \), it is clear that \( A \) is a n.c.J. \( Q \)-algebra if and only if \( A^+ \) is a Jordan \( Q \)-algebra. This fact will be used without comment in what follows. Also note that when \( A \) has a unit, \( q - \text{Inv}(A) = \{1 - x : x \in \text{Inv}(A)\} \), so \( q - \text{Inv}(A) \) is open if and only if \( \text{Inv}(A) \) is open. If \( A \) is a n.c.J. complete normed algebra and \( \phi \) denotes the mapping \( x \to U_{1-x} \) from \( A \) into the Banach algebra \( BL(A_1) \) of bounded linear operators on \( A_1 \), then \( \phi \) is continuous and \( q - \text{Inv}(A) = \phi^{-1}((\text{Inv}(BL(A_1)))) \), so
Theorem 4: If the set of quasi-invertible elements of a n.c.J. normed algebra 
A has some interior point, then A is a n.c.J. Q-algebra.

Proof. Suppose first that A has a unit and the set Inv(A) has some interior
point, say x_o. Choose y £ Inv(A). Then the linear operator U_y is a homeo-
morphism on A, and it leaves invariant the set Inv(A), so U_y(x_o) is an interior
point of Inv(A). Since the mapping z —> U_z(x_o), z £ A, is continuous, it fol-
low that there is some number p > 0 such that U_z(x_o) £ Inv(A), (hence,
z £ Inv(A)) whenever ||z - y|| < p. Hence, Inv(A) is open.

Suppose now that the set q - Inv(A) has some interior point, say a, and
let p > 0 be such that u £ q - Inv(A) whenever ||u - u_o|| < p. Put \( \delta = \rho/(1+p+||u_o||) \) and x_o = 1 - u_u_o £ Inv(A_1). Then for z = a + u in A_1 such that
||z - x_o|| < \( \delta \) we have \( |1-a| < \delta \) and \( \|au+u\| < \|1+a\|\|u\| \), which implies \( a \neq 0 \) and \( \|u_o-(-u/\alpha)\| < \delta(1+||u_o||)/(1-\delta) < \rho \), so \(-u/\alpha \in q - Inv(A)\); that
is, \( z = a + u \in Inv(A_1) \), and, as we noted in the beginning, this implies that the
set Inv(A_1) is open. Since A is an ideal of A_1, it is also a full subalgebra of
A_1; hence, \( q - Inv(A) = A \cap q - Inv(A_1) \), which shows that the set \( q - Inv(A) \)
is open. □

As a consequence of Proposition 1 and its proof we obtain

Proposition 2. Let A be a n.c.J. normed algebra and A_1 its normed unitization.
Then A is a n.c.J. Q-algebra if and only if the same is true for A_1.

Proposition 3. Let A be a n.c.J. real normed algebra and A_C its normed com-
plexification. Then A is a n.c.J. Q-algebra if and only if the same is true for
A_C.

Proof. Assume that A is a real n.c.J. Q-algebra. We can suppose that A actu-
ally is a Jordan algebra and, by Proposition 2, that A has a unit. Let p denote
the algebra norm on A_C defined as in [6, Proposition 13.3]. Choose 0 < \( \alpha < 1 \)
such that \( x \in Inv(A) \) whenever \( ||1-x|| < \alpha \). Put \( \delta = \frac{\rho}{2} \). For \( a+ib \in A_C \) such that
\( p(1-(a+ib)) < \delta \) we have max\( \{||1-a||, ||b||\} < \delta \), so \( ||1-a|| < \delta < \alpha \);therefore, \( a \in Inv(A) \). Now
\[
||1 - (a + U_b(a^{-1}))|| \leq ||1 - a|| + ||U_b(a^{-1})|| \leq ||1 - a|| + 3||b||^2||a^{-1}||
\]
\[
\leq ||1 - a|| + \frac{3||b||^2}{1-||1-a||} < \delta + \frac{3\delta^2}{1-\delta} < \alpha ,
\]
which implies that \( a + U_b(a^{-1}) \in Inv(A) \). Next we shall prove
\[
U_{a+ib}(U_{a-b}(a-ib)^2)) = (a + U_b(a^{-1}))^2 .
\]
To this end note that if \( c = 1 + b \) then \( ||1 - c|| < \alpha \), so \( c \in Inv(A) \). Also note that the above equality can be localized to the subalgebra B of A_C generated
by c, a, c^{-1}, and a^{-1}. By the Shirshov-Cohn theorem with inverses [18],
B is a special Jordan algebra. Now in terms of the associative product of any
associative envelop of B our equality is
\[
(a + ib)a^{-1}(a - ib)(a - ib)a^{-1}(a + ib) = (a + ba^{-1}b)^2 ,
\]
which can be easily verified. The equality just proved together with the fact that \( a + U_b(a^{-1}) \in \text{Inv}(A) \subseteq \text{Inv}(A_C) \) gives that \( a + bi \in \text{Inv}(A_C) \). Hence the unit is an interior point of \( \text{Inv}(A_C) \) and, by Proposition 1, \( A_C \) is a n.c.J. \( Q \)-algebra. The converse is an easy consequence of the fact that \( A \) is a full real subalgebra of \( A_C \).

Theorem 4. Let \( A \) be a n.c.J. normed algebra. The following are equivalent:

(i) \( A \) is a n.c.J. \( Q \)-algebra.

(ii) \( \rho (x) = r(x) \) for all \( x \) in \( A \).

(iii) \( \rho (x) \leq \|x\| \) for all \( x \) in \( A \).

(iv) \( A \) is a full subalgebra of its normed completion.

(v) \( A \) is a full subalgebra of some n.c.J. complete normed algebra.

(vi) Every element \( x \) in \( A \) with \( \|x\| < 1 \) is quasi-invertible in \( A \).

Proof. Suppose (i). Then there is some number \( \alpha > 0 \) such that \( x \in q - \text{Inv}(A) \) whenever \( \|x\| < \alpha \). By Propositions 2 and 3 we can assume that \( A \) is a complex Jordan \( Q \)-algebra with unit. Given \( x \) in \( A \) choose \( \lambda \in \mathbb{C} \) such that \( \|x\|/\alpha < |\lambda| \). Then \( \|x/\lambda\| < \alpha \), so \( 1 - x/\lambda \in \text{Inv}(A) \); that is, \( \lambda \notin \text{sp}(x) \). This shows that \( \rho (x) \leq \|x\|/\alpha \). Repeating with \( x \) replaced by \( x^n \) \((n \geq 1)\), we obtain \( \rho (x^n) \leq \|x^n\|/\alpha \). Since \( \text{sp}(x^n) = \{\lambda^n : \lambda \in \text{sp}(x)\} \) [16, Theorem 1.1] it follows that \( \rho (x^n) = \rho (x)^n \). Now taking \( n \)th roots in the above inequality and letting \( n \to \infty \), we see that \( \rho (x) \leq r(x) \). Now if \( \hat{A} \) denotes the normed completion of \( A \), we have \( r(x) = \rho (x, \hat{A}) \). Since \( \rho (x, \hat{A}) \leq \rho (x) \), it follows that \( \rho (x) = r(x) \), so (ii) is obtained. Clearly (ii) implies (iii). Next suppose (iii). Since for \( z = a + x \) in \( A_1 \) we have \( \rho (z, A_1) \leq \rho (x) + |a| \), (iii) is valid for both \( A \) and \( A_1 \), so we can assume that \( A \) has a unit. Let \( \hat{A} \) denote the normed completion of \( A \) and choose \( a \in A \cap \text{Inv}(\hat{A}) \). Then \( U_a \) is a linear homeomorphism on \( \hat{A} \), and, in particular, \( U_a(A) \) is dense in \( A \). Therefore, there is \( b \in A \) such that \( \|1 - U_a(b)\| < 1 \); whence, \( \rho (1 - U_a(b)) < 1 \), so \( U_a(b) \in \text{Inv}(A) \), which implies that \( a \in \text{Inv}(A) \). We have proved that \( A \cap \text{Inv}(\hat{A}) \subseteq \text{Inv}(A) \). Since the opposite inclusion is always true, we have \( A \cap \text{Inv}(\hat{A}) = \text{Inv}(A) \) and (iv) follows. Clearly (iv) implies (v). Suppose now that \( A \) is a full subalgebra of a n.c.J. complete normed algebra \( J \). Then \( x \in q - \text{Inv}(J) \) whenever \( x \in J \) with \( \|x\| < 1 \), because \( J \) is complete. In particular, if \( x \in A \) and \( \|x\| < 1 \), then \( x \in A \cap q - \text{Inv}(J) = q - \text{Inv}(A) \) and (vi) follows. Finally, by Proposition 1, (vi) implies (i). □

As a clear consequence of (v) the spectrum of an element in a n.c.J. \( Q \)-algebra is a compact (nonempty) subset of \( \mathbb{C} \). For associative \( Q \)-algebras the equivalence of (i), (ii), and (iii) of Theorem 4 was proved by Yood [32, Lemma 2.1]. Also Palmer in [20, Theorem 3.1 and Proposition 5.10] states the associative version of Theorem 4. Next we are going to give a characterization of n.c.J. \( Q \)-algebras as those n.c.J. normed algebras in which the maximal modular inner ideals are closed.

A vector subspace \( M \) of a Jordan algebra \( A \) such that \( U_m(A) \subseteq M \) for all \( m \in M \) is called an inner ideal of \( A \). If, in addition, \( M \) is also a subalgebra of \( A \), then it is called a strict inner ideal of \( A \). Recall that for \( a, b \) in \( A \) the operator \( U_{a,b} \) is defined by \( U_{a,b} = (U_{a+b} - U_a - U_b)/2 \). The element \( U_{a,b}(x) \) is usually written as \( \{a, x, b\} \). A strict inner ideal \( M \) of \( A \) is called
x-modular for some \( x \in A \) when the following three conditions are satisfied:

(i) \( U_{1-x}(A) \subset M \).

(ii) \( \{1 - x, z, m\} \in M \) for all \( z \in A_1 \) and all \( m \in M \).

(iii) \( x^2 - x \in M \).

This concept of modularity in Jordan algebras is due to Hogben and McCrimmon \cite{13}. The next result has been used in \cite{9}, giving the clue for its proof in the case of Jordan-Banach algebras, although it has not been explicitly stated.

**Proposition 5.** The closure \( \overline{M} \) of a proper x-modular strict inner ideal \( M \) of a Jordan \( Q \)-algebra \( A \) is a proper x-modular strict inner ideal of \( A \).

**Proof.** Using the continuity of the product of \( A \), it is easily obtained that \( \overline{M} \) is an x-modular strict inner ideal of \( A \). Let us show it is proper. Choose \( m \in M \), and let \( z = x - m \). If \( \|z\| < 1 \), then by Theorem 4 we know that \( z \in q - \text{Inv}(A) \). If \( w \) is the quasi inverse of \( z \), then \( 1 - z = U_{1-z}(1 - w) = U_{1-z}(1 - w^2) - U_{1-z}(w^2 - w) = 1 - U_{1-z}(w^2 - w) \), so \( z = U_{1-z}(t) \), where \( t = w^2 - w \in A \). Now

\[
z = U_{1-z}(t) = U_{1-x-m}(t) = U_{1-x}(t) + U_m(t) + 2U_{1-x,m}(t),
\]

and it follows that \( z \in M \), but then \( x \in M \), and this implies that \( M = A \) \cite[Proposition 3.1]{13}, which contradicts the assumption that \( M \) is proper. Hence it must be \( \|x - m\| \geq 1 \) for every \( m \in M \), so \( x \notin \overline{M} \). Thus \( \overline{M} \) is proper. \( \square \)

A maximal modular inner ideal of a Jordan algebra \( A \) is a strict inner ideal which is x-modular for some \( x \in A \) and maximal among all proper x-modular strict inner ideals of \( A \) (for \( x \) fixed). The maximal modular inner ideals of a n.c.J. algebra \( A \) are, by definition, the maximal modular inner ideals of the Jordan algebra \( A^+ \).

**Proposition 6.** Let \( A \) be a n.c.J. normed algebra. The following are equivalent:

(i) \( A \) is a n.c.J. \( Q \)-algebra.

(ii) The maximal modular inner ideals of \( A \) are closed.

**Proof.** As a consequence of Proposition 5 we have that (i) implies (ii). To prove the converse we can suppose that \( A \) is a Jordan algebra. Let \( \hat{A} \) denote the normed completion of \( A \). Choose \( x \in A \cap q - \text{Inv}(\hat{A}) \). Then \( 1 - x \) is invertible in \( \hat{A}_1 \), so \( U_{1-x} \) is a homeomorphism on \( \hat{A}_1 \); in particular, \( U_{1-x}(A_1) \) is dense in \( A_1 \). Therefore, if \( z \in A \), there is a sequence \( \{\alpha_n + z_n\} \) in \( A_1 \) such that \( \lim\{U_{1-x}(\alpha_n + z_n)\} = z \). Since \( U_{1-x}(\alpha_n + z_n) \) can be written in the form \( \alpha_n + w_n \) with \( w_n \in A_1 \), it follows that \( \lim\{\alpha_n\} = 0 \), and we deduce that \( \lim\{U_{1-x}(z_n)\} = z \). Hence \( U_{1-x}(A) \) is dense in \( A \). Note that \( U_{1-x}(A) \subset A_i \), since \( A_i \) is an ideal of \( A_1 \). If \( U_{1-x}(A) \neq A \), then it follows from \cite[Remark 2.8]{13} that there is a maximal modular inner ideal \( M \) of \( A \) such that \( U_{1-x}(A) \subset M \). Since, by assumption, \( M \) is closed, we have a contradiction with the density of \( U_{1-x}(A) \) in \( A \). Hence \( U_{1-x}(A) = A \). It has been seen in the proof of Proposition 5 that the quasi inverse \( y \) of \( x \) is given by \( y = U_{1-y}(x^2 - x) = U_{1-x}^{-1}(x^2 - x) \), so it follows that \( y \) lies in \( A \). We have proved that \( A \) is a full subalgebra of \( \hat{A} \), and therefore \( A \) is a Jordan \( Q \)-algebra. \( \square \)

The maximal modular left or right ideals in associative algebras are also maximal modular inner ideals \cite[Example 3.3]{13}. In this respect the above
proof can be easily modified to show that, if $A$ is a normed associative algebra and the maximal modular left ideals of $A$ are closed, then $A$ is an associative $Q$-algebra (see also [33, Theorem 2.9]).

Since for any element $x$ in a n.c.J. $Q$-algebra we have $\rho(x) = r(x)$, it follows that homomorphisms of n.c.J. $Q$-algebras decrease the (geometric) spectral radius. Moreover, if $r(x) = 0$ then $\text{sp}(x) = \{0\}$, so $x$ is quasi-invertible. Taking this into account, it is easily seen that the proof given by Aupetit [1] and the recent and more simple proof given by Ransford [23] of Johnson’s uniqueness of norm theorem yield immediately to the following result (see also [25, Proposition 3.1]). If $X$ and $Y$ are normed spaces and $F$ is a linear mapping from $X$ into $Y$, we denote by $S(F)$ (the separating subspace of $F$) the set of those $y$ in $Y$ for which there is a sequence $\{x_n\}$ in $X$ such that $\lim{x_n} = 0$ and $\lim{F(x_n)} = y$. If $A$ is a n.c.J. algebra, $\text{Rad}(A)$ means the Jacobson radical of $A$ [19]; namely, $\text{Rad}(A)$ is the largest quasi-invertible ideal of $A$. If $\text{Rad}(A) = \{0\}$, $A$ is called semisimple.

**Proposition 7** [1, 23]. Let $A$ and $B$ be n.c.J. complex $Q$-algebras, and let $F$ be a homomorphism from $A$ into $B$. Then $r(b) = 0$ for every $b$ in $S(F) \cap F(A)$. Moreover, if $F$ is a surjective homomorphism, then $S(F) \subseteq \text{Rad}(B)$.

Suppose $A$ is a n.c.J. $Q$-algebra, and let $M$ be a closed ideal of $A$. Then the algebra $A/M$ is a n.c.J. $Q$-algebra. (Indeed, if $\pi$ denotes the canonical projection of $A$ onto $A/M$, then $\pi$ is open and $\pi(q - \text{Inv}(A)) \subseteq q - \text{Inv}(A/M)$. Hence we may apply Proposition 1 to $A/M$.) Moreover, if $B$ is a semisimple n.c.J. algebra and $\phi$ is a homomorphism from $A$ onto $B$, then $\text{Ker}(\phi)$ is closed (just use Theorem 4(vi) to obtain in the usual way that $\phi(\text{Ker}(\phi))$ is a quasi-invertible ideal of $B$). With Proposition 7 and these considerations the proof of the main result in [27] yields directly to the following result. Recall that a normed algebra $(A, \| \cdot \|)$ is said to have *minimality of norm topology* if any algebra norm on $A$, $| \cdot |$, minorizing $\| \cdot \|$, i.e., $| \cdot | \leq \alpha \| \cdot \|$ for some $\alpha > 0$, is actually equivalent to $\| \cdot \|$.

**Theorem 8.** Let $A$ be a n.c.J. complex $Q$-algebra, and let $B$ be a semisimple complete normed complex n.c.J. algebra with minimality of norm topology. Then every homomorphism from $A$ onto $B$ is continuous.

### 3. Algebra norms on noncommutative $JB^*$-algebras

A *not necessarily commutative* (for short n.c.) $JB^*$-algebra $A$ is a complete normed n.c.J. complex algebra with (conjugate linear) algebra involution $\dagger$ such that $\|U_a(a^\dagger)\| = \|a\|^3$ for all $a$ in $A$. Thus $C^*$-algebras and (commutative) $JB^*$-algebras are particular types of n.c. $JB^*$-algebras. If $A$ is a n.c. $JB^*$-algebra, then $A^+$ is a $JB^*$-algebra with the same norm and involution as those of $A$. $JB^*$-algebras were introduced by Kaplansky in 1976, and they have been extensively studied since the paper by Wright [31].

**Lemma 9.** If $| \cdot |$ is any algebra norm on a n.c. $JB^*$-algebra $A$, then $(A, | \cdot |)$ is a n.c.J. $Q$-algebra.

**Proof.** Since n.c.J. algebras are power-associative, the closed subalgebra of $A$ generated by a symmetric element ($a = a^\dagger$) is a commutative $C^*$-algebra. Given $a$ in $A$, we can consider the commutative $C^*$-algebra generated by the
symmetric element \( a^* \cdot a = \frac{1}{2}(aa^* + a^*a) \) and make use of a well-known result due to Kaplansky, according to which any algebra norm on a commutative \( C^* \)-algebra is greater than the original norm, to get that \( \|a^* \cdot a\| \leq \|a^* \cdot a\| \). Also it is known that \( \|a\|^2 \leq 2\|a^* \cdot a\| \) [21, Proposition 2.2]. So we have that \( \|a\|^2 \leq 2|a^* \cdot a| \leq 2|a^*| |a| \) for all \( a \) in \( A \). Hence \( \|a^n\|^2 \leq 2|a^n| |a^n| \) for all \( n \) in \( \mathbb{N} \), which implies that \( (r_n||a||)^2 \leq r_n(a^*)r_n(a) \). Now, if \( (C, \cdot \cdot) \) denotes the completion of \( (A, \cdot \cdot) \), we have \( r_n(a) = \rho(a, C) \leq \rho(a, A) = r_n||a|| \) for all \( a \) in \( A \). Thus \( (r_n||a||)^2 \leq r_n(a^*)r_n(a) \) and consequently \( r_n||a|| \leq r_n(a) \). We deduce that \( r_n(a) = r_n||a|| = \rho(a, A) \) for all \( a \) in \( A \), and by Theorem 4 we conclude that \( (A, \cdot \cdot) \) is a n.c. Q-algebra. □

**Theorem 10.** The topology of the norm of a n.c. \( JB^* \)-algebra \( A \) is the smallest algebra normable topology on \( A \).

**Proof.** If \( \cdot \cdot \) is any algebra norm on \( A \), it has been shown in the proof of Lemma 9 that \( \|a\|^2 \leq 2|a^*| |a| \) for all \( a \) in \( A \). If we know additionally that \( \cdot \cdot \leq M\cdot\cdot \) for some nonnegative number \( M \), then \( \|a\|^2 \leq 2M\|a^*\| |a| = 2M|a||a| \), so \( \|a\| \leq 2M|a| \) for all \( a \) in \( A \). Hence the norm \( \cdot \cdot \) is equivalent to the norm of \( A \). Therefore, \( (A, \|\cdot\|) \) has minimality of norm topology. Now, for an arbitrary algebra norm \( \cdot \cdot \) on \( A \), we can use Lemma 9 and apply Theorem 8 to the identity mapping from \( (A, \cdot \cdot) \) into \( (A, \|\cdot\|) \) to obtain that this mapping is continuous, which concludes the proof. □

If \( A \) is a \( C^* \)-algebra, then the particularization of Theorem 10 to the \( JB^* \)-algebra \( A^+ \) gives that any algebra norm on \( A^+ \) defines a topology on \( A \) which is stronger than the original one. This is an improvement of the classical result by Cleveland [8] which states the same for algebra norms on \( A \).

Unlike the preceding results, which are of an algebraic-topologic nature, the following one is geometric.

Let \( A \) be a complete normed complex nonassociative algebra with unit \( 1 \) such that \( \|1\| = 1 \). Denote by \( A^* \) the dual Banach space of \( A \). For \( a \) in \( A \) the subset of \( \mathbb{C} \), \( V_{\|\cdot\|}(a) = \{f(a): f \in A^*, \|f\| = 1 = f(1)\} \) is called the numerical range of \( a \). The set of hermitian elements of \( A \), denoted by \( H(A) \), is defined as the set of those elements \( a \) in \( A \) such that \( V_{\|\cdot\|}(a) \subset \mathbb{R} \). If \( A = H(A) + iH(A) \), then \( A \) is called a \( V \)-algebra. The general nonassociative Vidav-Palmer theorem [24] says that the class of (nonassociative) \( V \)-algebras coincides with the one of unital n.c. \( JB^* \)-algebras.

**Proposition 11.** Every n.c. \( JB^* \)-algebra \( A \) has the property of minimality of the norm; that is, if \( \cdot \cdot \) is an algebra norm on \( A \) such that \( \cdot \cdot \leq \|\cdot\| \), then the equality \( \cdot \cdot = \|\cdot\| \) holds.

**Proof.** By Theorem 10 and the assumptions made, \( \cdot \cdot \) and \( \|\cdot\| \) are equivalent norms on \( A \), so \( \cdot \cdot \) is a complete norm on \( A \). Suppose first that \( A \) has a unit element \( 1 \). \( \cdot \cdot \) being an algebra norm, we have \( 1 \leq |1| \leq \|1\| = 1 \), so \( |1| = 1 \). Let \( \|\cdot\| \) and \( \cdot \cdot \) also denote the corresponding dual norms of \( \|\cdot\| \) and \( \cdot \cdot \). Then for \( f \) in \( A^* \) we have \( \|f\| \leq |f| \), and we deduce easily that \( V_{\|\cdot\|}(a) \subset V_{\|\cdot\|}(a) \) for all \( a \) in \( A \). Since \( (A, \|\cdot\|) \) is a \( V \)-algebra, it follows that \( (A, \cdot \cdot) \) is also a \( V \)-algebra, and, consequently, by the nonassociative Vidav-Palmer theorem, \( (A, \cdot \cdot) \) is a n.c. \( JB^* \)-algebra. Since the norm of a n.c. \( JB^* \)-algebra is unique [31], we conclude that \( \cdot \cdot = \|\cdot\| \). If \( A \) has no unit element, then it is known that \( (A^{**}, \|\cdot\|) \), with the Aren's product and a convenient involution which

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extends that of \( A \), is a unital n.c. \( JB^* \)-algebra [21]. Since the bidual \( A^{**} \) of \( A \) is the same for both norms and \( | \cdot | \) is an algebra norm on \( A^{**} \) satisfying \( | \cdot | \leq \| \cdot \| \) on \( A^{**} \), it follows from what was previously seen that \( | \cdot | = \| \cdot \| \) on \( A^{**} \) and, in particular, \( | \cdot | = \| \cdot \| \) on \( A \). \( \square \)

Now we apply Theorem 10 and Proposition 11 to the study of the ranges of Jordan homomorphisms from \( C^* \)-algebras.

**Corollary 12.** Assume that a normed associative complex algebra \( B \) is the range of a continuous (resp. contractive) Jordan homomorphism from a \( C^* \)-algebra. Then \( B \) is bicontinuously (resp. isometrically) isomorphic to a \( C^* \)-algebra.

**Proof.** Let \( A \) be a \( C^* \)-algebra and \( \phi \) a Jordan homomorphism from \( A \) onto \( B \) under the assumptions in the statement. Since closed Jordan ideals of a \( C^* \)-algebra are associative ideals (see [7, Theorem 5.3.] or [21, Theorem 4.3]), \( A/\text{Ker}(\phi) \) is a \( C^* \)-algebra and we may assume that \( \phi \) is a one-to-one mapping. Then, by Theorem 10 (resp. Proposition 11) applied to the \( JB^* \)-algebra \( A^+ \), it follows that \( \phi \) is a bicontinuous (resp. isometric) Jordan isomorphism from \( A \) onto \( B \). Let \( C \) denote the associative complex algebra with vector space that of \( A \) and product \( \Box \) defined by \( x \Box y := \phi^{-1}(\phi(x)\phi(y)) \). Then \( C^+ (= A^+) \) is a \( JB^* \)-algebra under the norm and involution of \( A \), so, with the same norm and involution, \( C \) becomes a \( C^* \)-algebra [26, Theorem 2] and, clearly, \( \phi \) becomes a bicontinuous (resp. isometric) associative isomorphism from \( C \) onto \( B \). \( \square \)

**Corollary 13.** The range of any weakly compact Jordan homomorphism from a \( C^* \)-algebra into a normed algebra is finite dimensional.

**Proof.** If \( A \) is a \( C^* \)-algebra, \( B \) a normed algebra, and \( \phi \) a weakly compact Jordan homomorphism from \( A \) into \( B \), then, as above, \( A/\text{Ker}(\phi) \) is a \( C^* \)-algebra and, easily, the induced Jordan homomorphism \( A/\text{Ker}(\phi) \rightarrow B \) is weakly compact, so again we may assume that \( \phi \) is a one-to-one mapping. Now, by Theorem 10 applied to \( A^+ \), \( \phi \) is a weakly compact topological embedding, so \( A \) is a \( C^* \)-algebra with reflexive Banach space, and so \( A \) (and hence the range of \( \phi \)) is finite dimensional [28]. \( \square \)

**Remark 14.** The fact that weakly compact (associative) homomorphisms from \( C^* \)-algebras have finite-dimensional ranges was proved first in [12] as a consequence of a more general result, and later a very simple proof (that we imitate above) was obtained by Mathieu [17]. If \( A \) is a n.c. \( JB^* \)-algebra and \( \phi \) is any weakly compact homomorphism from \( A \) into a normed algebra \( B \), since \( A/\text{Ker}(\phi) \) is a n.c. \( JB^* \)-algebra [21, Corollary 1.11], to obtain some information about the range of \( \phi \) we may assume that \( \phi \) is a one-to-one mapping, and then, as above, the range of \( \phi \) is bicontinuously isomorphic to a n.c. \( JB^* \)-algebra with reflexive Banach space, namely, a finite product of simple n.c. \( JB^* \)-algebras which are either finite dimensional or quadratic [22, Theorem 3.5] (note that infinite-dimensional quadratic \( JB^* \)-algebras do exist and the identity mapping on such a \( JB^* \)-algebra is weakly compact). This result on the range of a weakly compact homomorphism from a n.c. \( JB^* \)-algebra was proved first in [11] by using Theorem 10 and a nonassociative extension of the above-mentioned general result in [12]. The proof given above (also suggested in [11]) is analogous to Mathieu’s proof for the particular case of \( C^* \)-algebras.
References


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