PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 121, Number 4, August 1994

FULL SUBALGEBRAS OF JORDAN-BANACH ALGEBRAS AND ALGEBRA NORMS ON JB^*-ALGEBRAS

J. PÉREZ, L. RICO, AND A. RODRÍGUEZ

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. We introduce normed Jordan Q-algebras, namely, normed Jordan algebras in which the set of quasi-invertible elements is open, and we prove that a normed Jordan algebra is a Q-algebra if and only if it is a full subalgebra of its completion. Homomorphisms from normed Jordan Q-algebras onto semisimple Jordan-Banach algebras with minimality of norm topology are continuous. As a consequence, the topology of the norm of a JB^*-algebra is the smallest normable topology making the product continuous, and JB^*-algebras have minimality of the norm. Some applications to (associative) C^*-algebras are also given: (i) the associative normed algebras that are ranges of continuous (resp. contractive) Jordan homomorphisms from C^*-algebras are bicontinuously (resp. isometrically) isomorphic to C^*-algebras, and (ii) weakly compact Jordan homomorphisms from C^*-algebras are of finite rank.

INTRODUCTION

Associative normed algebras in which the set of quasi-invertible elements is open were considered first by Kaplansky [15], who called them "normed Q-algebras". Since then, normed Q-algebras were seldom studied (exceptions are Yood's relevant papers [32, 33]) until the Wilansky conjecture [30], which states that associative normed Q-algebras are nothing but full subalgebras of Banach algebras. In fact, Palmer [20] set the bases for a systematic study of associative normed Q-algebras, providing, in particular, an affirmative answer to Wilansky's conjecture (see also [3] for an independent proof of this result). It must also be mentioned that full subalgebras of Banach algebras have played a relevant role in connection with the nonassociative extension of Johnson's uniqueness-of-norm theorem [25] and with the nonassociative extension of the Civin-Yood decomposition theorem [10].

The general theory of Jordan-Banach algebras began with the paper by Balachandran and Rema [2]; since then it has been fully developed in a complete analogy with the case of (associative) Banach algebras (see, e.g., [29, 16, 1, 9, 25, 10, 5, 11]), although in most of the cases new methods have been needed for such Jordan extensions of associative results. Noncomplete normed Jordan algebras whose sets of quasi-invertible elements are open (called, of course, "normed Jordan Q-algebras") were only germinally considered in [29].

Received by the editors May 15, 1992 and, in revised form, November 9, 1992.
1991 *Mathematics Subject Classification*. Primary 46H70, 46L70, 46L05.

It is the aim of this paper to develop the theory of normed Jordan Q-algebras, providing also the complete analogy with the associative case. With no additional effort we shall even consider normed noncommutative Jordan Q-algebras so that the associative (or even alternative) case will remain contained in our approach. In the first part of this paper we give several characterizations of normed noncommutative Jordan Q-algebras (Theorem 4 and Proposition 6), including the one asserting that normed noncommutative Jordan Q-algebras are nothing but full subalgebras of noncommutative Jordan-Banach algebras (the affirmative answer to Wilansky's conjecture in the Jordan setting). It must also be emphasized that the normed complexification of a normed noncommutative Jordan real Q-algebra is also a Q-algebra (Proposition 3), whose proof needs an intrisical Jordan method as it is the Shirshov-Cohn theorem with inverses [18]. We end this section with a theorem on automatic continuity (Theorem 8) which is a Jordan extension of the main result in [27].

The second part of the paper is devoted to applying a part of the developed theory of normed Jordan Q-algebras in order to obtain new results on JB^*-algebras (hence on the Jordan structure of C^*-algebras). Thus in Theorem 10 we use the aforementioned result on automatic continuity to generalize Cleveland's theorem [8], which asserts that the topology of the norm of a C^*-algebra A is the smallest algebra-normable topology on A, to noncommutative JB^*-algebras. (As a consequence, every norm on the vector space of a C^*-algebra that makes the Jordan product continuous defines a topology which is stronger than the topology of the C^*-norm—a result that improves the original Cleveland theorem.) The JB^*-extension of Cleveland's result was obtained almost at the same time and with essentially identical techniques by Bensebah [4]. With the nonassociative Vidav-Palmer Theorem [24], it is also proved that noncommutative JB^*-algebras have minimality of the norm (Proposition 11); i.e., $|\cdot| = \|\cdot\|$ whenever $|\cdot|$ is any algebra norm satisfying $|\cdot| \leq \|\cdot\|$. Finally, with the main result in [26], we determine the associative normed algebras that are ranges of continuous Jordan homomorphisms from C^*-algebras (Corollary 12), and we show that ranges of weakly compact Jordan homomorphisms from C^*-algebras are finite dimensional (Corollary 13).

1. Preliminaries and notation

All the algebras we consider here are real or complex. A nonassociative algebra A satisfying $x(yx) = (xy)x$ and $x^2(yx) = (x^2y)x$ for all x, y in A is called a *noncommutative Jordan* (in short, n.c.J.) *algebra.* As usual A^+ denotes the symmetrized algebra of A with product $x \cdot y = \frac{1}{2}(xy + yx)$. Recall that A^+ is a Jordan algebra whenever A is a n.c.J. algebra. For any element a in a n.c.J. algebra A, U_a denotes the linear operator on A defined by

$$U_a(x) = a(ax + xa) - a^2x = (ax + xa)a - xa^2, \qquad x \in A.$$

Recall that $U_a = U_a^+$, where U_a^+ is the usual U-operator on the Jordan algebra A^+. An element a in a n.c.J. algebra A with unit $\mathbf{1}$ is *invertible* with inverse b if $ab = ba = \mathbf{1}$ and $a^2b = ba^2 = a$. This is equivalent to a being invertible with inverse b in the Jordan algebra A^+ [19], whence, if $\mathrm{Inv}(A)$ denotes the set of invertible elements in A, then we have $\mathrm{Inv}(A) = \mathrm{Inv}(A^+)$. We recall the following basic results (see [14, Theorem 13, p. 52]). For elements x, y in A

- x is invertible if and only if U_x is an invertible operator, and in that case $x^{-1} = U_x^{-1}(x)$ and $U_x^{-1} = U_{x^{-1}}$.
- x and y are invertible if and only if $U_x(y)$ is invertible.
- x is invertible if and only if x^n $(n \geq 1)$ is invertible

An element a in a n.c.J. algebra A is *quasi-invertible* with quasi-inverse b if $\mathbf{1} - a$ has inverse $\mathbf{1} - b$ in the n.c.J. algebra A_1 (the *unitization* of A) obtained by adjoining a unit to A in the usual way. Let $q - \mathrm{Inv}(A)$ denote the set of quasi-invertible elements in A. Any real n.c.J. algebra A can be regarded as a real subalgebra of a complex n.c.J. algebra $A_{\mathbb{C}}$ which satisfies $A_{\mathbb{C}} = A \oplus iA$ and is called the *complexification* of A (see [6, Definition 3.1]). The *spectrum* of an element x in a n.c.J. algebra A, denoted by $\mathrm{sp}(x, A)$, is defined as in the associative case (see [6, Definitions 5.1 and 13.6]). The "*algebraic*" *spectral radius* of x is defined by

$$\rho(x, A) := \sup\{|\lambda| : \lambda \in \mathrm{sp}(x, A)\}.$$

We write $\mathrm{sp}(x)$ or $\rho(x)$ instead of $\mathrm{sp}(x, A)$ or $\rho(x, A)$ when no confusion can occur. A subalgebra B of a n.c.J. algebra A is called a *full* subalgebra of A if B contains the quasi inverses of its elements that are quasi-invertible in A, that is, the equality $q - \mathrm{Inv}(B) = B \cap q - \mathrm{Inv}(A)$ holds. Easy examples of full subalgebras are left or right ideals and strict inner ideals (see definition later). It is clear that if B is a full subalgebra of a complex n.c.J. algebra A, then

$$\mathrm{sp}(x, A) \cup \{0\} = \mathrm{sp}(x, B) \cup \{0\} \qquad (x \in B).$$

A n.c.J. algebra A is said to be *normed* if an *algebra norm* (a norm $\|\cdot\|$ on the vector space of A satisfying $\|ab\| \leq \|a\|\,\|b\|$ for all a, b in A) is given on A. In that case the "*geometric*" *spectral radius* of $x \in A$ is the number $r_{\|\cdot\|}(x) := \lim \|x^n\|^{1/n}$. When no confusion can occur, we write $r(x)$ to denote $r_{\|\cdot\|}(x)$. The unitization A_1 of n.c.J. normed algebra A becomes normed by defining $\|x + \alpha\| := \|x\| + \|\alpha\|$ for $x + \alpha$ in A_1. Also the complexification of a real n.c.J. normed algebra can be normed as in [6, Proposition 13.3]. Since every element x in a n.c.J. normed algebra A can be immersed in a closed associative full subalgebra of A [5, Théorème 1], it follows that the properties of the spectrum and the classical functional calculus for a single element in (associative) Banach algebras remain valid for n.c.J. complete normed algebras. In particular, the Gelfand-Beurling formula, $r(x) = \rho(x)$, holds for any element x in a n.c.J. complete normed algebra.

2. Noncommutative Jordan Q-algebras

A n.c.J. normed algebra A in which the set $q - \mathrm{Inv}(A)$ is open is called a n.c.J. *Q-algebra*. Taking into account that A^+ with the same norm as A is a Jordan normed algebra and $q - \mathrm{Inv}(A^+) = q - \mathrm{Inv}(A)$, it is clear that A is a n.c.J. Q-algebra if and only if A^+ is a Jordan Q-algebra. This fact will be used without comment in what follows. Also note that when A has a unit, $q - \mathrm{Inv}(A) = \{\mathbf{1} - x : x \in \mathrm{Inv}(A)\}$, so $q - \mathrm{Inv}(A)$ is open if and only if $\mathrm{Inv}(A)$ is open. If A is a n.c.J. complete normed algebra and ϕ denotes the mapping $x \to U_{1-x}$ from A into the Banach algebra $\mathrm{BL}(A_1)$ of bounded linear operators on A_1, then ϕ is continuous and $q - \mathrm{Inv}(A) = \phi^{-1}$ $(\mathrm{Inv}(\mathrm{BL}(A_1)))$, so

$q - \mathrm{Inv}(A)$ is open and A is a n.c.J. Q-algebra. It is clear that full subalgebras of n.c.J. Q-algebras are also n.c.J. Q-algebras; in particular, full subalgebras of n.c.J. complete normed algebras are examples of n.c.J. Q-algebras. In fact, we shall prove in Theorem 4 that these examples exhaust the class of n.c.J. Q-algebras.

Proposition 1. *If the set of quasi-invertible elements of a n.c.J. normed algebra A has some interior point, then A is a n.c.J. Q-algebra.*

Proof. Suppose first that A has a unit and the set $\mathrm{Inv}(A)$ has some interior point, say x_0. Choose $y \in \mathrm{Inv}(A)$. Then the linear operator U_y is a homeomorphism on A, and it leaves invariant the set $\mathrm{Inv}(A)$, so $U_y(x_0)$ is an interior point of $\mathrm{Inv}(A)$. Since the mapping $z \to U_z(x_0)$, $z \in A$, is continuous, it follows that there is some number $\rho > 0$ such that $U_z(x_0) \in \mathrm{Inv}(A)$, (hence, $z \in \mathrm{Inv}(A)$) whenever $\|z - y\| < \rho$. Hence, $\mathrm{Inv}(A)$ is open.

Suppose now that the set $q - \mathrm{Inv}(A)$ has some interior point, say u_0, and let $\rho > 0$ be such that $u \in q - \mathrm{Inv}(A)$ whenever $\|u_0 - u\| < \rho$. Put $\delta = \rho/(1+\rho+\|u_0\|)$ and $x_0 = \mathbf{1} - u_0 \in \mathrm{Inv}(A_1)$. Then for $z = \alpha + u$ in A_1 such that $\|z - x_0\| < \delta$ we have $|1-\alpha| < \delta$ and $\|\alpha u + u\| < \delta + |1+\alpha|\,\|u\|$, which implies $\alpha \neq 0$ and $\|u_0 - (-u/\alpha)\| < \delta(1+\|u_0\|)/(1-\delta) < \rho$, so $-u/\alpha \in q - \mathrm{Inv}(A)$; that is, $z = \alpha + u \in \mathrm{Inv}(A_1)$, and, as we noted in the beginning, this implies that the set $\mathrm{Inv}(A_1)$ is open. Since A is an ideal of A_1, it is also a full subalgebra of A_1; hence, $q - \mathrm{Inv}(A) = A \cap q - \mathrm{Inv}(A_1)$, which shows that the set $q - \mathrm{Inv}(A)$ is open. $\square$

As a consequence of Proposition 1 and its proof we obtain

Proposition 2. *Let A be a n.c.J. normed algebra and A_1 its normed unitization. Then A is a n.c.J. Q-algebra if and only if the same is true for A_1.*

Proposition 3. *Let A be a n.c.J. real normed algebra and $A_{\mathbb{C}}$ its normed complexification. Then A is a n.c.J. Q-algebra if and only if the same is true for $A_{\mathbb{C}}$.*

Proof. Assume that A is a real n.c.J. Q-algebra. We can suppose that A actually is a Jordan algebra and, by Proposition 2, that A has a unit. Let p denote the algebra norm on $A_{\mathbb{C}}$ defined as in [6, Proposition 13.3]. Choose $0 < \alpha < 1$ such that $x \in \mathrm{Inv}(A)$ whenever $\|\mathbf{1} - x\| < \alpha$. Put $\delta = \frac{\alpha}{3}$. For $a + ib \in A_{\mathbb{C}}$ such that $p(\mathbf{1} - (a+ib)) < \delta$ we have $\max\{\|\mathbf{1} - a\|, \|b\|\} < \delta$, so $\|\mathbf{1} - a\| < \delta < \alpha$; therefore, $a \in \mathrm{Inv}(A)$. Now

$$\begin{aligned}\|\mathbf{1} - (a + U_b(a^{-1}))\| &\leq \|\mathbf{1} - a\| + \|U_b(a^{-1})\| \leq \|\mathbf{1} - a\| + 3\|b\|^2\|a^{-1}\| \\ &\leq \|\mathbf{1} - a\| + \frac{3\|b\|^2}{1 - \|\mathbf{1} - a\|} < \delta + \frac{3\delta^2}{1-\delta} < \alpha,\end{aligned}$$

which implies that $a + U_b(a^{-1}) \in \mathrm{Inv}(A)$. Next we shall prove

$$U_{a+ib}(U_{a^{-1}}(a - ib)^2)) = (a + U_b(a^{-1}))^2.$$

To this end note that if $c = \mathbf{1} + b$ then $\|\mathbf{1} - c\| < \alpha$, so $c \in \mathrm{Inv}(A)$. Also note that the above equality can be localized to the subalgebra B of $A_{\mathbb{C}}$ generated by c, a, c^{-1}, and a^{-1}. By the Shirshov-Cohn theorem with inverses [18], B is a special Jordan algebra. Now in terms of the associative product of any associative envelop of B our equality is

$$(a + ib)a^{-1}(a - ib)(a - ib)a^{-1}(a + ib) = (a + ba^{-1}b)^2,$$

which can be easily verified. The equality just proved together with the fact that $a + U_b(a^{-1}) \in \mathrm{Inv}(A) \subset \mathrm{Inv}(A_{\mathbb{C}})$ gives that $a + bi \in \mathrm{Inv}(A_{\mathbb{C}})$. Hence the unit is an interior point of $\mathrm{Inv}(A_{\mathbb{C}})$ and, by Proposition 1, $A_{\mathbb{C}}$ is a n.c.J. Q-algebra. The converse is an easy consequence of the fact that A is a full real subalgebra of $A_{\mathbb{C}}$. □

Theorem 4. *Let A be a n.c.J. normed algebra. The following are equivalent:*

(i) *A is a n.c.J. Q-algebra.*
(ii) *$\rho(x) = r(x)$ for all x in A.*
(iii) *$\rho(x) \leq \|x\|$ for all x in A.*
(iv) *A is a full subalgebra of its normed completion.*
(v) *A is a full subalgebra of some n.c.J. complete normed algebra.*
(vi) *Every element x in A with $\|x\| < 1$ is quasi-invertible in A.*

Proof. Suppose (i). Then there is some number $\alpha > 0$ such that $x \in q-\mathrm{Inv}(A)$ whenever $\|x\| < \alpha$. By Propositions 2 and 3 we can assume that A is a complex Jordan Q-algebra with unit. Given x in A choose $\lambda \in \mathbb{C}$ such that $\|x\|/\alpha < |\lambda|$. Then $\|x/\lambda\| < \alpha$, so $\mathbf{1} - x/\lambda \in \mathrm{Inv}(A)$; that is, $\lambda \notin \mathrm{sp}(x)$. This shows that $\rho(x) \leq \|x\|/\alpha$. Repeating with x replaced by x^n $(n \geq 1)$, we obtain $\rho(x^n) \leq \|x^n\|/\alpha$. Since $\mathrm{sp}(x^n) = \{\lambda^n : \lambda \in \mathrm{sp}(x)\}$ [16, Theorem 1.1] it follows that $\rho(x^n) = \rho(x)^n$. Now taking nth roots in the above inequality and letting $n \to \infty$, we see that $\rho(x) \leq r(x)$. Now if $\widehat{A}$ denotes the normed completion of A, we have $r(x) = \rho(x, \widehat{A})$. Since $\rho(x, \widehat{A}) \leq \rho(x)$, it follows that $\rho(x) = r(x)$, so (ii) is obtained. Clearly (ii) implies (iii). Next suppose (iii). Since for $z = \alpha + x$ in A_1 we have $\rho(z, A_1) \leq \rho(x) + |a|$, (iii) is valid for both A and A_1, so we can assume that A has a unit. Let $\widehat{A}$ denote the normed completion of A and choose $a \in A \cap \mathrm{Inv}(\widehat{A})$. Then U_a is a linear homeomorphism on $\widehat{A}$, and, in particular, $U_a(A)$ is dense in A. Therefore, there is $b \in A$ such that $\|\mathbf{1} - U_a(b)\| < 1$; whence, $\rho(\mathbf{1} - U_a(b)) < 1$, so $U_a(b) \in \mathrm{Inv}(A)$, which implies that $a \in \mathrm{Inv}(A)$. We have proved that $A \cap \mathrm{Inv}(\widehat{A}) \subset \mathrm{Inv}(A)$. Since the opposite inclusion is always true, we have $A \cap \mathrm{Inv}(\widehat{A}) = \mathrm{Inv}(A)$ and (iv) follows. Clearly (iv) implies (v). Suppose now that A is a full subalgebra of a n.c.J. complete normed algebra J. Then $x \in q-\mathrm{Inv}(J)$ whenever $x \in J$ with $\|x\| < 1$, because J is complete. In particular, if $x \in A$ and $\|x\| < 1$, then $x \in A \cap q - \mathrm{Inv}(J) = q - \mathrm{Inv}(A)$ and (vi) follows. Finally, by Proposition 1, (vi) implies (i). □

As a clear consequence of (v) the spectrum of an element in a n.c.J. Q-algebra is a compact (nonempty) subset of $\mathbb{C}$. For associative Q-algebras the equivalence of (i), (ii), and (iii) of Theorem 4 was proved by Yood [32, Lemma 2.1]. Also Palmer in [20, Theorem 3.1 and Proposition 5.10] states the associative version of Theorem 4. Next we are going to give a characterization of n.c.J. Q-algebras as those n.c.J. normed algebras in which the maximal modular inner ideals are closed.

A vector subspace M of a Jordan algebra A such that $U_m(A) \subset M$ for all $m \in M$ is called an *inner ideal* of A. If, in addition, M is also a subalgebra of A, then it is called a *strict* inner ideal of A. Recall that for a, b in A the operator $U_{a,b}$ is defined by $U_{a,b} = (U_{a+b} - U_a - U_b)/2$. The element $U_{a,b}(x)$ is usually written as $\{a, x, b\}$. A strict inner ideal M of A is called *x-modular* for some $x \in A$ when the following three conditions are satisfied:

(i) $U_{\mathbf{1}-x}(A) \subset M$.
(ii) $\{\mathbf{1}-x, z, m\} \in M$ for all $z \in A_1$ and all $m \in M$.
(iii) $x^2 - x \in M$.

This concept of modularity in Jordan algebras is due to Hogben and McCrimmon [13]. The next result has been used in [9], giving the clue for its proof in the case of Jordan-Banach algebras, although it has not been explicitly stated.

Proposition 5. *The closure $\overline{M}$ of a proper x-modular strict inner ideal M of a Jordan Q-algebra A is a proper x-modular strict inner ideal of A.*

Proof. Using the continuity of the product of A, it is easily obtained that $\overline{M}$ is an x-modular strict inner ideal of A. Let us show it is proper. Choose $m \in M$, and let $z = x - m$. If $\|z\| < 1$, then by Theorem 4 we know that $z \in q-\operatorname{Inv}(A)$. If w is the quasi inverse of z, then $\mathbf{1} - z = U_{1-z}(\mathbf{1} - w) = U_{1-z}((\mathbf{1}-w)^2) - U_{1-z}(w^2 - w) = \mathbf{1} - U_{1-z}(w^2 - w)$, so $z = U_{1-z}(t)$, where $t = w^2 - w \in A$. Now

$$z = U_{\mathbf{1}-z}(t) = U_{\mathbf{1}-x-m}(t) = U_{\mathbf{1}-x}(t) + U_m(t) + 2U_{\mathbf{1}-x,m}(t),$$

and it follows that $z \in M$, but then $x \in M$, and this implies that $M = A$ [13, Proposition 3.1], which contradicts the assumption that M is proper. Hence it must be $\|x - m\| \geq 1$ for every $m \in M$, so $x \notin \overline{M}$. Thus $\overline{M}$ is proper. □

A *maximal modular* inner ideal of a Jordan algebra A is a strict inner ideal which is x-modular for some $x \in A$ and maximal among all proper x-modular strict inner ideals of A (for x fixed). The maximal modular inner ideals of a n.c.J. algebra A are, by definition, the maximal modular inner ideals of the Jordan algebra A^+.

Proposition 6. *Let A be a n.c.J. normed algebra. The following are equivalent:*

(i) *A is a n.c.J. Q-algebra.*
(ii) *The maximal modular inner ideals of A are closed.*

Proof. As a consequence of Proposition 5 we have that (i) implies (ii). To prove the converse we can suppose that A is a Jordan algebra. Let $\widehat{A}$ denote the normed completion of A. Choose $x \in A \cap q-\operatorname{Inv}(\widehat{A})$. Then $\mathbf{1}-x$ is invertible in $\widehat{A}_1$, so $U_{\mathbf{1}-x}$ is a homeomorphism on $\widehat{A}_1$; in particular, $U_{\mathbf{1}-x}(A_1)$ is dense in A_1. Therefore, if $z \in A$, there is a sequence $\{\alpha_n + z_n\}$ in A_1 such that $\lim\{U_{\mathbf{1}-x}(\alpha_n+z_n)\} = z$. Since $U_{\mathbf{1}-x}(\alpha_n+z_n)$ can be written in the form α_n+w_n with $w_n \in A$, it follows that $\lim\{\alpha_n\} = 0$, and we deduce that $\lim\{U_{\mathbf{1}-x}(z_n)\} = z$. Hence $U_{\mathbf{1}-x}(A)$ is dense in A. Note that $U_{\mathbf{1}-x}(A) \subset A$, since A is an ideal of A_1. If $U_{\mathbf{1}-x}(A) \neq A$, then it follows from [13, Remark 2.8] that there is a maximal modular inner ideal M of A such that $U_{\mathbf{1}-x}(A) \subset M$. Since, by assumption, M is closed, we have a contradiction with the density of $U_{\mathbf{1}-x}(A)$ in A. Hence $U_{\mathbf{1}-x}(A) = A$. It has been seen in the proof of Proposition 5 that the quasi inverse y of x is given by $y = U_{\mathbf{1}-y}(x^2 - x) = U^{-1}_{\mathbf{1}-x}(x^2 - x)$, so it follows that y lies in A. We have proved that A is a full subalgebra of $\widehat{A}$, and therefore A is a Jordan Q-algebra. □

The maximal modular left or right ideals in associative algebras are also maximal modular inner ideals [13, Example 3.3]. In this respect the above proof can be easily modified to show that, if A is a normed associative algebra and the maximal modular left ideals of A are closed, then A is an associative Q-algebra (see also [33, Theorem 2.9]).

Since for any element x in a n.c.J. Q-algebra we have $\rho(x) = r(x)$, it follows that homomorphisms of n.c.J. Q-algebras decrease the (geometric) spectral radius. Moreover, if $r(x) = 0$ then $\mathrm{sp}(x) = \{0\}$, so x is quasi-invertible. Taking this into account, it is easily seen that the proof given by Aupetit [1] and the recent and more simple proof given by Ransford [23] of Johnson's uniqueness of norm theorem yield immediately to the following result (see also [25, Proposition 3.1]). If X and Y are normed spaces and F is a linear mapping from X into Y, we denote by $S(F)$ (the separating subspace of F) the set of those y in Y for which there is a sequence $\{x_n\}$ in X such that $\lim\{x_n\} = 0$ and $\lim\{F(x_n)\} = y$. If A is a n.c.J. algebra, $\mathrm{Rad}(A)$ means the *Jacobson radical* of A [19]; namely, $\mathrm{Rad}(A)$ is the largest quasi-invertible ideal of A. If $\mathrm{Rad}(A) = \{0\}$, A is called *semisimple.*

Proposition 7 [1, 23]. *Let A and B be n.c.J. complex Q-algebras, and let F be a homomorphism from A into B. Then $r(b) = 0$ for every b in $S(F) \cap F(A)$. Moreover, if F is a surjective homomorphism, then $S(F) \subset \mathrm{Rad}(B)$.*

Suppose A is a n.c.J. Q-algebra, and let M be a closed ideal of A. Then the algebra A/M is a n.c.J. Q-algebra. (Indeed, if π denotes the canonical projection of A onto A/M, then π is open and $\pi(q-\mathrm{Inv}(A)) \subset q-\mathrm{Inv}(A/M)$. Hence we may apply Proposition 1 to A/M.) Moreover, if B is a semisimple n.c.J. algebra and φ is a homomorphism from A onto B, then $\mathrm{Ker}(\varphi)$ is closed (just use Theorem 4(vi) to obtain in the usual way that $\varphi(\overline{\mathrm{Ker}(\varphi)})$ is a quasi-invertible ideal of B). With Proposition 7 and these considerations the proof of the main result in [27] yields directly to the following result. Recall that a normed algebra $(A, \|\cdot\|)$ is said to have *minimality of norm topology* if any algebra norm on A, $|\cdot|$, minorizing $\|\cdot\|$, i.e., $|\cdot| \leq \alpha\|\cdot\|$ for some $\alpha > 0$, is actually equivalent to $\|\cdot\|$.

Theorem 8. *Let A be a n.c.J. complex Q-algebra, and let B be a semisimple complete normed complex n.c.J. algebra with minimality of norm topology. Then every homomorphism from A onto B is continuous.*

3. Algebra norms on noncommutative JB^*-algebras

A *not necessarily commutative* (for short n.c.) *JB^*-algebra* A is a complete normed n.c.J. complex algebra with (conjugate linear) algebra involution $*$ such that $\|U_a(a^*)\| = \|a\|^3$ for all a in A. Thus C^*-algebras and (commutative) JB^*-algebras are particular types of n.c. JB^*-algebras. If A is a n.c. JB^*-algebra, then A^+ is a JB^*-algebra with the same norm and involution as those of A. JB^*-algebras were introduced by Kaplansky in 1976, and they have been extensively studied since the paper by Wright [31].

Lemma 9. *If $|\cdot|$ is any algebra norm on a n.c. JB^*-algebra A, then $(A, |\cdot|)$ is a n.c.J. Q-algebra.*

Proof. Since n.c.J. algebras are power-associative, the closed subalgebra of A generated by a symmetric element $(a = a^*)$ is a commutative C^*-algebra. Given a in A, we can consider the commutative C^*-algebra generated by the symmetric element $a^* \cdot a = \frac{1}{2}(aa^* + a^*a)$ and make use of a well-known result due to Kaplansky, according to which any algebra norm on a commutative C^*-algebra is greater than the original norm, to get that $\|a^* \cdot a\| \leq |a^* \cdot a|$. Also it is known that $\|a\|^2 \leq 2\|a^* \cdot a\|$ [21, Proposition 2.2]. So we have that $\|a\|^2 \leq 2|a^* \cdot a| \leq 2|a^*|\,|a|$ for all a in A. Hence $\|a^n\|^2 \leq 2|(a^*)^n|\,|a^n|$ for all n in $\mathbb{N}$, which implies that $(r_{\|\cdot\|}(a))^2 \leq r_{|\cdot|}(a^*)r_{|\cdot|}(a)$. Now, if $(C, |\cdot|)$ denotes the completion of $(A, |\cdot|)$, we have $r_{|\cdot|}(a) = \rho(a, C) \leq \rho(a, A) = r_{\|\cdot\|}(a)$ for all a in A. Thus $(r_{\|\cdot\|}(a))^2 \leq r_{\|\cdot\|}(a^*)r_{|\cdot|}(a)$ and consequently $r_{\|\cdot\|}(a) \leq r_{|\cdot|}(a)$. We deduce that $r_{|\cdot|}(a) = r_{\|\cdot\|}(a) = \rho(a, A)$ for all a in A, and by Theorem 4 we conclude that $(A, |\cdot|)$ is a n.c.J. Q-algebra. □

Theorem 10. *The topology of the norm of a n.c. JB^*-algebra A is the smallest algebra normable topology on A.*

Proof. If $|\cdot|$ is any algebra norm on A, it has been shown in the proof of Lemma 9 that $\|a\|^2 \leq 2|a^*|\,|a|$ for all a in A. If we know additionally that $|\cdot| \leq M\|\cdot\|$ for some nonnegative number M, then $\|a\|^2 \leq 2M\|a^*\|\,|a| = 2M\|a\|\,|a|$, so $\|a\| \leq 2M|a|$ for all a in A. Hence the norm $|\cdot|$ is equivalent to the norm of A. Therefore, $(A, \|\cdot\|)$ has minimality of norm topology. Now, for an arbitrary algebra norm $|\cdot|$ on A, we can use Lemma 9 and apply Theorem 8 to the identity mapping from $(A, |\cdot|)$ into $(A, \|\cdot\|)$ to obtain that this mapping is continuous, which concludes the proof. □

If A is a C^*-algebra, then the particularization of Theorem 10 to the JB^*-algebra A^+ gives that any algebra norm on A^+ defines a topology on A which is stronger than the original one. This is an improvement of the classical result by Cleveland [8] which states the same for algebra norms on A.

Unlike the preceeding results, which are of an algebraic-topologic nature, the following one is geometric.

Let A be a complete normed complex nonassociative algebra with unit $\mathbf{1}$ such that $\|\mathbf{1}\| = 1$. Denote by A^* the dual Banach space of A. For a in A the subset of $\mathbb{C}$, $V_{\|\cdot\|}(a) = \{f(a): f \in A^*,\ \|f\| = 1 = f(\mathbf{1})\}$ is called the *numerical range* of a. The set of *hermitian* elements of A, denoted by $\mathrm{H}(A)$, is defined as the set of those elements a in A such that $V_{\|\cdot\|}(a) \subset \mathbb{R}$. If $A = \mathrm{H}(A) + i\mathrm{H}(A)$, then A is called a *V-algebra.* The general nonassociative Vidav-Palmer theorem [24] says that the class of (nonassociative) V-algebras coincides with the one of unital n.c. JB^*-algebras.

Proposition 11. *Every n.c. JB^*-algebra A has the property of minimality of the norm*; *that is, if $|\cdot|$ is an algebra norm on A such that $|\cdot| \leq \|\cdot\|$, then the equality $|\cdot| = \|\cdot\|$ holds.*

Proof. By Theorem 10 and the assumptions made, $|\cdot|$ and $\|\cdot\|$ are equivalent norms an A, so $|\cdot|$ is a complete norm on A. Suppose first that A has a unit element $\mathbf{1}$. $|\cdot|$ being an algebra norm, we have $1 \leq |\mathbf{1}| \leq \|\mathbf{1}\| = 1$, so $|\mathbf{1}| = 1$. Let $\|\cdot\|$ and $|\cdot|$ also denote the corresponding dual norms of $\|\cdot\|$ and $|\cdot|$. Then for f in A^* we have $\|f\| \leq |f|$, and we deduce easily that $V_{|\cdot|}(a) \subset V_{\|\cdot\|}(a)$ for all a in A. Since $(A, \|\cdot\|)$ is a V-algebra, it follows that $(A, |\cdot|)$ is also a V-algebra, and, consequently, by the nonassociative Vidav-Palmer theorem, $(A, |\cdot|)$ is a n.c. JB^*-algebra. Since the norm of a n.c. JB^*-algebra is unique [31], we conclude that $|\cdot| = \|\cdot\|$. If A has no unit element, then it is known that $(A^{**}, \|\cdot\|)$, with the Aren's product and a convenient involution which extends that of A, is a unital n.c. JB^*-algebra [21]. Since the bidual A^{**} of A is the same for both norms and $|\cdot|$ is an algebra norm on A^{**} satisfying $|\cdot| \leq \|\cdot\|$ on A^{**}, it follows from what was previously seen that $|\cdot| = \|\cdot\|$ on A^{**} and, in particular, $|\cdot| = \|\cdot\|$ on A. $\square$

Now we apply Theorem 10 and Proposition 11 to the study of the ranges of Jordan homomorphisms from C^*-algebras.

Corollary 12. *Assume that a normed associative complex algebra B is the range of a continuous (resp. contractive) Jordan homomorphism from a C^*-algebra. Then B is bicontinuously (resp. isometrically) isomorphic to a C^*-algebra.*

Proof. Let A be a C^*-algebra and φ a Jordan homomorphism from A onto B under the assumptions in the statement. Since closed Jordan ideals of a C^*-algebra are associative ideals (see [7, Theorem 5.3.] or [21, Theorem 4.3]), $A/\operatorname{Ker}(\varphi)$ is a C^*-algebra and we may assume that φ is a one-to-one mapping. Then, by Theorem 10 (resp. Proposition 11) applied to the JB^*-algebra A^+, it follows that φ is a bicontinuous (resp. isometric) Jordan isomorphism from A onto B. Let C denote the associative complex algebra with vector space that of A and product $\square$ defined by $x\square y := \varphi^{-1}(\varphi(x)\varphi(y))$. Then $C^+(= A^+)$ is a JB^*-algebra under the norm and involution of A, so, with the same norm and involution, C becomes a C^*-algebra [26, Theorem 2] and, clearly, φ becomes a bicontinuous (resp. isometric) associative isomorphism from C onto B. $\square$

Corollary 13. *The range of any weakly compact Jordan homomorphism from a C^*-algebra into a normed algebra is finite dimensional.*

Proof. If A is a C^*-algebra, B a normed algebra, and φ a weakly compact Jordan homomorphism from A into B, then, as above, $A/\operatorname{Ker}(\varphi)$ is a C^*-algebra and, easily, the induced Jordan homomorphism $A/\operatorname{Ker}(\varphi) \to B$ is weakly compact, so again we may assume that φ is a one-to-one mapping. Now, by Theorem 10 applied to A^+, φ is a weakly compact topological embedding, so A is a C^*-algebra with reflexive Banach space, and so A (and hence the range of φ) is finite dimensional [28]. $\square$

Remark 14. The fact that weakly compact (associative) homomorphisms from C^*-algebras have finite-dimensional ranges was proved first in [12] as a consequence of a more general result, and later a very simple proof (that we imitate above) was obtained by Mathieu [17]. If A is a n.c. JB^*-algebra and φ is any weakly compact homomorphism from A into a normed algebra B, since $A/\operatorname{Ker}(\varphi)$ is a n.c. JB^*-algebra [21, Corollary 1.11], to obtain some information about the range of φ we may assume that φ is a one-to-one mapping, and then, as above, the range of φ is bicontinuously isomorphic to a n.c. JB^*-algebra with reflexive Banach space, namely, a finite product of simple n.c. JB^*-algebras which are either finite dimensional or quadratic [22, Theorem 3.5] (note that infinite-dimensional quadratic JB^*-algebras do exist and the identity mapping on such a JB^*-algebra is weakly compact). This result on the range of a weakly compact homomorphism from a n.c. JB^*-algebra was proved first in [11] by using Theorem 10 and a nonassociative extension of the above-mentioned general result in [12]. The proof given above (also suggested in [11]) is analogous to Mathieu's proof for the particular case of C^*-algebras.

REFERENCES

1. B. Aupetit, *The uniqueness of the complete norm topology in Banach algebras and Banach-Jordan algebras*, J. Funct. Anal. **47** (1982), 1–6.
2. V. K. Balachandran and P. S. Rema, *Uniqueness of the norm topology in certain Banach Jordan algebras*, Publ. Ramanujan Inst. **1** (1968/69), 283–289.
3. A. Beddaa and M. Oudadess, *On a question of A. Wilansky in normed algebras*, Studia Math. **95** (1989), 175–177.
4. A. Bensebah, *Weakness of the topology of a JB^*-algebra*, Canad. Math. Bull. **35** (1992), 449–454.
5. M. Benslimane and A. Rodríguez, *Caracterization spectrale des algèbres de Jordan Banach non commutatives complexes modulaires annihilatrices*, J. Algebra **140** (1991), 344–354.
6. F. F. Bonsall and J. Duncan, *Complete normed algebras*, Ergeb. Math. Grenzgeb., vol. 80, Springer-Verlag, New York, 1973.
7. P. Civin and B. Yood, *Lie and Jordan structures in Banach algebras*, Pacific J. Math. **15** (1965), 775–797.
8. S. B. Cleveland, *Homomorphisms of non-commutative $*$-algebras*, Pacific J. Math. **13** (1963), 1097–1109.
9. A. Fernández, *Modular annihilator Jordan algebras*, Comm. Algebra **13** (1985), 2597–2613.
10. A. Fernández and A. Rodríguez, *A Wedderburn theorem for non-associative complete normed algebras*, J. London Math. Soc. (2) **33** (1986), 328–338.
11. J. E. Galé, *Weakly compact homomorphisms in nonassociative algebras*, Workshop on Nonassociative Algebraic Models, Nova Science Publishers, New York, 1992, pp. 167–171.
12. J. E. Galé, T. J. Ransford, and M. C. White, *Weakly compact homomorphisms*, Trans. Amer. Math. Soc. **331** (1992), 815–824.
13. L. Hogben and McCrimmon, *Maximal modular inner ideals and the Jacobson radical of a Jordan algebra*, J. Algebra **68** (1981), 155–169.
14. N. Jacobson, *Structure and representations of Jordan algebras*, Amer. Math. Soc. Colloq. Publ., vol. 39, Amer. Math. Soc., Providence, RI, 1968.
15. I. Kaplansky, *Normed algebras*, Duke Math. J. **16** (1949), 399–418.
16. J. Martínez, *JV-algebras*, Math. Proc. Cambridge Philos. Soc. **87** (1980), 47–50.
17. M. Mathieu, *Weakly compact homomorphisms from C^*-algebras are of finite rank*, Proc. Amer. Math. Soc. **107** (1989), 761–762.
18. K. McCrimmon, *McDonald's theorem with inverses*, Pacific J. Math. **21** (1967), 315–321.
19. ——, *Noncommutative Jordan rings*, Trans. Amer. Math. Soc. **158** (1971), 1–33.
20. T. W. Palmer, *Spectral algebras*, Rocky Mountain J. Math. **22** (1992), 293–328.
21. R. Payá, J. Pérez, and A. Rodríguez, *Non-Commutative Jordan C^*-algebras*, Manuscripta Math. **37** (1982), 87–120.
22. ——, *Type* I *factor representations of noncommutative JB^*-algebras*, Proc. London Math. Soc. (3) **48** (1984), 428–444.
23. T. J. Ransford, *A short proof of Johnson's uniqueness-of-norm theorem*, Bull. London Math. Soc. **21** (1989), 487–488.
24. A. Rodríguez, *Non-associative normed algebras spanned by hermitian elements*, Proc. London Math. Soc. (3) **47** (1983), 258–274.
25. ——, *The uniqueness of the complete norm topology in complete normed nonassociative algebras*, J. Funct. Anal. **60** (1985), 1–15.
26. ——, *Jordan axioms for C^*-algebras*, Manuscripta Math. **61** (1988), 297–314.
27. ——, *Automatic continuity with applications to C^*-algebras*, Math. Proc. Cambridge Philos. Soc. **107** (1990), 345–347.
28. S. Sakai, *Weakly compact operators on operator algebras*, Pacific J. Math. **14** (1964), 659–664.

29. C. Viola Devapakkiam, *Jordan algebras with continuous inverse*, Math. Japon. **16** (1971), 115–125.

30. A. Wilansky, *Letter to the editor*, Amer. Math. Monthly **91** (1984), 531.

31. J. D. M. Wright, *Jordan C^*-algebras*, Michigan Math. J. **24** (1977), 291–302.

32. B. Yood, *Homomorphisms on normed algebras*, Pacific J. Math. **8** (1958), 373–381.

33. ______, *Ideals in topological rings*, Canad. J. Math. **16** (1964), 28–45.

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, FACULTAD DE CIENCIAS, UNIVERSIDAD DE GRANADA, 18071-GRANDA, SPAIN

Current address, L. Rico: Departamento de Didáctica de la Matemática, Facultad de Ciencias de la Educación, Universidad de Granada, 18077-Granada, Spain

E-mail address, A. Rodríguez: apalacios@ugn.es