FULL SUBALGEBRAS OF JORDAN-BANACH ALGEBRAS
AND ALGEBRA NORMS ON $JB^*$-ALGEBRAS

J. PÉREZ, L. RICO, AND A. RODRÍGUEZ

(Communicated by Palle E. T. Jorgensen)

Abstract. We introduce normed Jordan $Q$-algebras, namely, normed Jordan algebras in which the set of quasi-invertible elements is open, and we prove that a normed Jordan algebra is a $Q$-algebra if and only if it is a full subalgebra of its completion. Homomorphisms from normed Jordan $Q$-algebras onto semisimple Jordan-Banach algebras with minimality of norm topology are continuous. As a consequence, the topology of the norm of a $JB^*$-algebra is the smallest normable topology making the product continuous, and $JB^*$-algebras have minimality of the norm. Some applications to (associative) $C^*$-algebras are also given: (i) the associative normed algebras that are ranges of continuous (resp. contractive) Jordan homomorphisms from $C^*$-algebras are bicontinuously (resp. isometrically) isomorphic to $C^*$-algebras, and (ii) weakly compact Jordan homomorphisms from $C^*$-algebras are of finite rank.

Introduction

Associative normed algebras in which the set of quasi-invertible elements is open were considered first by Kaplansky [15], who called them “normed $Q$-algebras”. Since then, normed $Q$-algebras were seldom studied (exceptions are Yood’s relevant papers [32, 33]) until the Wilansky conjecture [30], which states that associative normed $Q$-algebras are nothing but full subalgebras of Banach algebras. In fact, Palmer [20] set the bases for a systematic study of associative normed $Q$-algebras, providing, in particular, an affirmative answer to Wilansky’s conjecture (see also [3] for an independent proof of this result). It must also be mentioned that full subalgebras of Banach algebras have played a relevant role in connection with the nonassociative extension of Johnson’s uniqueness-of-norm theorem [25] and with the nonassociative extension of the Civin-Yood decomposition theorem [10].

The general theory of Jordan-Banach algebras began with the paper by Balachandran and Rema [2]; since then it has been fully developed in a complete analogy with the case of (associative) Banach algebras (see, e.g., [29, 16, 1, 9, 25, 10, 5, 11]), although in most of the cases new methods have been needed for such Jordan extensions of associative results. Noncomplete normed Jordan algebras whose sets of quasi-invertible elements are open (called, of course, “normed Jordan $Q$-algebras”) were only germinally considered in [29].
It is the aim of this paper to develop the theory of normed Jordan Q-algebras, providing also the complete analogy with the associative case. With no additional effort we shall even consider normed noncommutative Jordan Q-algebras so that the associative (or even alternative) case will remain contained in our approach. In the first part of this paper we give several characterizations of normed noncommutative Jordan Q-algebras (Theorem 4 and Proposition 6), including the one asserting that normed noncommutative Jordan Q-algebras are nothing but full subalgebras of noncommutative Jordan-Banach algebras (the affirmative answer to Wilansky's conjecture in the Jordan setting). It must also be emphasized that the normed complexification of a normed noncommutative Jordan real Q-algebra is also a Q-algebra (Proposition 3), whose proof needs an intrinsic Jordan method as it is the Shirshov-Cohn theorem with inverses [18]. We end this section with a theorem on automatic continuity (Theorem 8) which is a Jordan extension of the main result in [27].

The second part of the paper is devoted to applying a part of the developed theory of normed Jordan Q-algebras in order to obtain new results on JB*-algebras (hence on the Jordan structure of C*-algebras). Thus in Theorem 10 we use the aforementioned result on automatic continuity to generalize Cleveland's theorem [8], which asserts that the topology of the norm of a C*-algebra A is the smallest algebra-normable topology on A, to noncommutative JB*-algebras. (As a consequence, every norm on the vector space of a C*-algebra that makes the Jordan product continuous defines a topology which is stronger than the topology of the C*-norm—a result that improves the original Cleveland theorem.) The JB*-extension of Cleveland's result was obtained almost at the same time and with essentially identical techniques by Bensebah [4]. With the nonassociative Vidav-Palmer Theorem [24], it is also proved that noncommutative JB*-algebras have minimality of the norm (Proposition 11); i.e., \(|\cdot| = \|\cdot\|\) whenever \(|\cdot|\) is any algebra norm satisfying \(|\cdot| \leq \|\cdot\|\). Finally, with the main result in [26], we determine the associative normed algebras that are ranges of continuous Jordan homomorphisms from C*-algebras (Corollary 12), and we show that ranges of weakly compact Jordan homomorphisms from C*-algebras are finite dimensional (Corollary 13).

1. Preliminaries and notation

All the algebras we consider here are real or complex. A nonassociative algebra \(A\) satisfying \(x(\cancel{yx}) = (\cancel{xy})x\) and \(x^2(\cancel{yx}) = (\cancel{x^2y})x\) for all \(x, y\) in \(A\) is called a noncommutative Jordan (in short, n.c.J.) algebra. As usual \(A^+\) denotes the symmetrized algebra of \(A\) with product \(x \cdot y = \frac{1}{2}(xy + yx)\). Recall that \(A^+\) is a Jordan algebra whenever \(A\) is a n.c.J. algebra. For any element \(a\) in a n.c.J. algebra \(A\), \(U_a\) denotes the linear operator on \(A\) defined by

\[
U_a(x) = a(ax + xa) - a^2x = (ax + xa)a - xa, \quad x \in A.
\]

Recall that \(U_a^+ = U_a\), where \(U_a^+\) is the usual \(U\)-operator on the Jordan algebra \(A^+\). An element \(a\) in a n.c.J. algebra \(A\) with unit \(1\) is invertible with inverse \(b\) if \(ab = ba = 1\) and \(a^2b = ba^2 = a\). This is equivalent to being invertible with inverse \(b\) in the Jordan algebra \(A^+\) [19], whence, if \(\text{Inv}(A)\) denotes the set of invertible elements in \(A\), then we have \(\text{Inv}(A) = \text{Inv}(A^+)\). We recall the
following basic results (see [14, Theorem 13, p. 52]). For elements \( x, y \) in \( A \)
- \( x \) is invertible if and only if \( U_x \) is an invertible operator, and in that case \( x^{-1} = U_x^{-1}(x) \) and \( U_x^{-1} = U_{x^{-1}}. \)
- \( x \) and \( y \) are invertible if and only if \( U_x(y) \) is invertible.
- \( x \) is invertible if and only if \( x^n \ (n \geq 1) \) is invertible.

An element \( a \) in a n.c.J. algebra \( A \) is quasi-invertible with quasi-inverse \( b \) if \( 1 - a \) has inverse \( 1 - b \) in the n.c.J. algebra \( A_1 \) (the unitization of \( A \)) obtained by adjoining a unit to \( A \) in the usual way. Let \( q - \text{Inv}(A) \) denote the set of quasi-invertible elements in \( A \). Any real n.c.J. algebra \( A \) can be regarded as a real subalgebra of a complex n.c.J. algebra \( A_C \) which satisfies \( A_C = A \oplus iA \) and is called the complexification of \( A \) (see [6, Definition 3.1]). The spectrum of an element \( x \) in a n.c.J. algebra \( A \), denoted by \( \text{sp}(x, A) \), is defined as in the associative case (see [6, Definitions 5.1 and 13.6]). The “algebraic” spectral radius of \( x \) is defined by

\[
\rho(x, A) := \sup\{|\lambda| : \lambda \in \text{sp}(x, A)\}.
\]

We write \( \text{sp}(x, A) \) or \( \rho(x, A) \) instead of \( \text{sp}(x, A) \) or \( \rho(x, A) \) when no confusion can occur. A subalgebra \( B \) of a n.c.J. algebra \( A \) is called a full subalgebra of \( A \) if \( B \) contains the quasi inverses of its elements that are quasi-invertible in \( A \), that is, the equality \( q - \text{Inv}(B) = B \cap q - \text{Inv}(A) \) holds. Easy examples of full subalgebras are left or right ideals and strict inner ideals (see definition later). It is clear that if \( B \) is a full subalgebra of a complex n.c.J. algebra \( A \), then

\[
\text{sp}(x, A) \cup \{0\} = \text{sp}(x, B) \cup \{0\} \quad (x \in B).
\]

A n.c.J. algebra \( A \) is said to be normed if an algebra norm (a norm \( \| \cdot \| \) on the vector space of \( A \) satisfying \( \|ab\| \leq \|a\| \|b\| \) for all \( a, b \) in \( A \)) is given on \( A \). In that case the “geometric” spectral radius of \( x \in A \) is the number \( r(\|\cdot\|)(x) := \lim \|x^n\|^{1/n} \). When no confusion can occur, we write \( r(x) \) to denote \( r(\|\cdot\|)(x) \). The unitization \( A_1 \) of n.c.J. normed algebra \( A \) becomes normed by defining \( \|x + \alpha\| := \|x\| + \|\alpha\| \) for \( x + \alpha \) in \( A_1 \). Also the complexification of a real n.c.J. normed algebra can be normed as in [6, Proposition 13.3]. Since every element \( x \) in a n.c.J. normed algebra \( A \) can be immersed in a closed associative full subalgebra of \( A \) [5, Théorème 1], it follows that the properties of the spectrum and the classical functional calculus for a single element in (associative) Banach algebras remain valid for n.c.J. complete normed algebras. In particular, the Gelfand-Beurling formula, \( r(x) = \rho(x) \), holds for any element \( x \) in a n.c.J. complete normed algebra.

2. Noncommutative Jordan \( Q \)-algebras

A n.c.J. normed algebra \( A \) in which the set \( q - \text{Inv}(A) \) is open is called a n.c.J. \( Q \)-algebra. Taking into account that \( A^+ \) with the same norm as \( A \) is a Jordan normed algebra and \( q - \text{Inv}(A^+) = q - \text{Inv}(A) \), it is clear that \( A \) is a n.c.J. \( Q \)-algebra if and only if \( A^+ \) is a Jordan \( Q \)-algebra. This fact will be used without comment in what follows. Also note that when \( A \) has a unit, \( q - \text{Inv}(A) = \{1 - x : x \in \text{Inv}(A)\} \), so \( q - \text{Inv}(A) \) is open if and only if \( \text{Inv}(A) \) is open. If \( A \) is a n.c.J. complete normed algebra and \( \phi \) denotes the mapping \( x \to U_{1-x} \) from \( A \) into the Banach algebra \( BL(A_1) \) of bounded linear operators on \( A_1 \), then \( \phi \) is continuous and \( q - \text{Inv}(A) = \phi^{-1} \left( \text{Inv}(BL(A_1)) \right) \), so
$q - \text{Inv}(A)$ is open and $A$ is a n.c.J. $Q$-algebra. It is clear that full subalgebras of n.c.J. $Q$-algebras are also n.c.J. $Q$-algebras; in particular, full subalgebras of n.c.J. complete normed algebras are examples of n.c.J. $Q$-algebras. In fact, we shall prove in Theorem 4 that these examples exhaust the class of n.c.J. $Q$-algebras.

**Proposition 1.** If the set of quasi-invertible elements of a n.c.J. normed algebra $A$ has some interior point, then $A$ is a n.c.J. $Q$-algebra.

**Proof.** Suppose first that $A$ has a unit and the set $\text{Inv}(A)$ has some interior point, say $x_0$. Choose $y \in \text{Inv}(A)$. Then the linear operator $U_y$ is a homeomorphism on $A$, and it leaves invariant the set $\text{Inv}(A)$, so $U_y(x_0)$ is an interior point of $\text{Inv}(A)$. Since the mapping $z \rightarrow U_z(x_0)$, $z \in A$, is continuous, it follows that there is some number $\rho > 0$ such that $U_z(x_0) \in \text{Inv}(A)$, (hence, $z \in \text{Inv}(A)$) whenever $||z - y|| < \rho$. Hence, $\text{Inv}(A)$ is open.

Suppose now that the set $q - \text{Inv}(A)$ has some interior point, say $u_0$, and let $\rho > 0$ be such that $u \in q - \text{Inv}(A)$ whenever $||u_0 - u|| < \rho$. Put $\delta = \rho/(1 + \rho + ||u_0||)$ and $x_0 = 1 - u_0 \in \text{Inv}(A_1)$. Then for $z = \alpha + u$ in $A_1$ such that $||z - x_0|| < \delta$ we have $|1 - \alpha| < \delta$ and $||\alpha u + u|| < \delta + |1 + \alpha| ||u||$, which implies $\alpha \neq 0$ and $||u_0 - (-u/\alpha)|| < \delta(1 + ||u_0||)/(1 - \delta) < \rho$, so $-u/\alpha \in q - \text{Inv}(A)$; that is, $z = \alpha + u \in \text{Inv}(A_1)$, and, as we noted in the beginning, this implies that the set $\text{Inv}(A_1)$ is open. Since $A$ is an ideal of $A_1$, it is also a full subalgebra of $A_1$; hence, $q - \text{Inv}(A) = A \cap q - \text{Inv}(A_1)$, which shows that the set $q - \text{Inv}(A)$ is open. □

As a consequence of Proposition 1 and its proof we obtain

**Proposition 2.** Let $A$ be a n.c.J. normed algebra and $A_1$ its normed unitization. Then $A$ is a n.c.J. $Q$-algebra if and only if the same is true for $A_1$.

**Proposition 3.** Let $A$ be a n.c.J. real normed algebra and $A_C$ its normed complexification. Then $A$ is a n.c.J. $Q$-algebra if and only if the same is true for $A_C$.

**Proof.** Assume that $A$ is a real n.c.J. $Q$-algebra. We can suppose that $A$ actually is a Jordan algebra and, by Proposition 2, that $A$ has a unit. Let $p$ denote the algebra norm on $A_C$ defined as in [6, Proposition 13.3]. Choose $0 < \alpha < 1$ such that $x \in \text{Inv}(A)$ whenever $||1 - x|| < \alpha$. Put $\delta = \frac{\alpha}{3}$. For $a + ib \in A_C$ such that $p(1 - (a + ib)) < \delta$ we have max{$||1 - a||, ||b||$} $< \delta$, so $||1 - a|| < \delta < \alpha$; therefore, $a \in \text{Inv}(A)$. Now

$$||1 - (a + U_b(a^{-1}()))|| \leq ||1 - a|| + ||U_b(a^{-1}())|| \leq ||1 - a|| + 3||b||^2||a^{-1}|| \leq ||1 - a|| + \frac{3||b||^2}{1 - ||1 - a||} < \delta + \frac{3\delta^2}{1 - \delta} < \alpha,$$

which implies that $a + U_b(a^{-1}) \in \text{Inv}(A)$. Next we shall prove

$$U_{a+ib}(U_{a^{-1}}(a - ib)^2)) = (a + U_b(a^{-1}))^2.$$ To this end note that if $c = 1 + b$ then $||1 - c|| < \alpha$, so $c \in \text{Inv}(A)$. Also note that the above equality can be localized to the subalgebra $B$ of $A_C$ generated by $c, a, c^{-1},$ and $a^{-1}$. By the Shirshov-Cohn theorem with inverses [18], $B$ is a special Jordan algebra. Now in terms of the associative product of any associative envelop of $B$ our equality is

$$(a + ib)a^{-1}(a - ib)(a - ib)a^{-1}(a + ib) = (a + ba^{-1}b)^2,$$
which can be easily verified. The equality just proved together with the fact that 
\( a + U_b(a^{-1}) \in \text{Inv}(A) \subset \text{Inv}(A_C) \) gives that \( a + bi \in \text{Inv}(A_C) \). Hence the unit is 
an interior point of \( \text{Inv}(A_C) \) and, by Proposition 1, \( A_C \) is a n.c.J. \( Q \)-algebra. 
The converse is an easy consequence of the fact that \( A \) is a full real subalgebra 
of \( A_C \). □

**Theorem 4.** Let \( A \) be a n.c.J. normed algebra. The following are equivalent:

(i) \( A \) is a n.c.J. \( Q \)-algebra.

(ii) \( \rho(x) = r(x) \) for all \( x \) in \( A \).

(iii) \( \rho(x) \leq \|x\| \) for all \( x \) in \( A \).

(iv) \( A \) is a full subalgebra of its normed completion.

(v) \( A \) is a full subalgebra of some n.c.J. complete normed algebra.

(vi) Every element \( x \) in \( A \) with \( \|x\| < 1 \) is quasi-invertible in \( A \).

*Proof.* Suppose (i). Then there is some number \( \alpha > 0 \) such that \( x \in q - \text{Inv}(A) \) whenever \( \|x\| < \alpha \). By Propositions 2 and 3 we can assume that \( A \) is a complex Jordan \( Q \)-algebra with unit. Given \( x \) in \( A \) choose \( \lambda \in \mathbb{C} \) such that \( \|x\|/\alpha < |\lambda| \). Then \( |\lambda X| < \alpha \), so \( 1 - x/\lambda \in \text{Inv}(A) \); that is, \( \lambda \notin \text{sp}(x) \). This shows that \( \rho(x) \leq \|x\|/\alpha \). Repeating with \( x \) replaced by \( x^n \) \((n \geq 1)\), we obtain \( \rho(x^n) \leq \|x^n\|/\alpha \). Since \( \text{sp}(x^n) = \{\lambda^n : \lambda \in \text{sp}(x)\} \) [16, Theorem 1.1] it follows 
that \( \rho(x^n) = \rho(x)^n \). Now taking \( n \)th roots in the above inequality and letting 
\( n \to \infty \), we see that \( \rho(x) \leq r(x) \). Now if \( \hat{A} \) denotes the normed completion 
of \( A \), we have \( r(x) = \rho(x, \hat{A}) \). Since \( \rho(x, \hat{A}) \leq \rho(x) \), it follows that \( \rho(x) = r(x) \), so (ii) is obtained. Clearly (ii) implies (iii). Next suppose (iii). Since for 
z = \alpha + x in \( A_1 \) we have \( \rho(z, A_1) \leq \rho(x) + |\alpha| \), (iii) is valid for both \( A \) and 
\( A_1 \), so we can assume that \( A \) has a unit. Let \( \hat{A} \) denote the normed completion 
of \( A \) and choose \( a \in A \cap \text{Inv}(\hat{A}) \). Then \( U_a \) is a linear homeomorphism on 
\( \hat{A} \), and, in particular, \( U_a(A) \) is dense in \( A \). Therefore, there is \( b \in A \) such that 
\( \|1 - U_a(b)\| < 1 \); whence, \( \rho(1 - U_a(b)) < 1 \), so \( U_a(b) \in \text{Inv}(A) \), which 
implies that \( a \in \text{Inv}(A) \). We have proved that \( A \cap \text{Inv}(\hat{A}) \subset \text{Inv}(A) \). Since 
the opposite inclusion is always true, we have \( A \cap \text{Inv}(\hat{A}) = \text{Inv}(A) \) and (iv) 
follows. Clearly (iv) implies (v). Suppose now that \( A \) is a full subalgebra of a 
n.c.J. complete normed algebra \( J \). Then \( x \in q - \text{Inv}(J) \) whenever \( x \in J \) with 
\( \|x\| < 1 \), because \( J \) is complete. In particular, if \( x \in A \) and \( \|x\| < 1 \), then 
x \( \in A \cap q - \text{Inv}(J) = q - \text{Inv}(A) \) and (vi) follows. Finally, by Proposition 1, 
(vi) implies (i). □

As a clear consequence of (v) the spectrum of an element in a n.c.J. \( Q \)-algebra 
is a compact (nonempty) subset of \( \mathbb{C} \). For associative \( Q \)-algebras the equivalence 
of (i), (ii), and (iii) of Theorem 4 was proved by Yood [32, Lemma 2.1]. Also Palmer in [20, Theorem 3.1 and Proposition 5.10] states the associative 
version of Theorem 4. Next we are going to give a characterization of n.c.J. 
\( Q \)-algebras as those n.c.J. normed algebras in which the maximal modular inner 
ideals are closed.

A vector subspace \( M \) of a Jordan algebra \( A \) such that \( U_m(A) \subset M \) for all 
m \( \in M \) is called an inner ideal of \( A \). If, in addition, \( M \) is also a subalgebra 
of \( A \), then it is called a strict inner ideal of \( A \). Recall that for \( a, b \) in \( A \) the operator 
\( U_{a,b} \) is defined by \( U_{a,b} = (U_{a^2} - U_a - U_b)/2 \). The element 
\( U_{a,b}(x) \) is usually written as \( \{a, x, b\} \). A strict inner ideal \( M \) of \( A \) is called
**x-modular** for some \( x \in A \) when the following three conditions are satisfied:

1. \( U_{1-x}(A) \subseteq M \).
2. \( \{1-x, z, m\} \subseteq M \) for all \( z \in A \) and all \( m \in M \).
3. \( x^2 - x \in M \).

This concept of modularity in Jordan algebras is due to Hogben and McCrimmon [13]. The next result has been used in [9], giving the clue for its proof in the case of Jordan-Banach algebras, although it has not been explicitly stated.

**Proposition 5.** The closure \( \overline{M} \) of a proper \( x \)-modular strict inner ideal \( M \) of a Jordan \( Q \)-algebra \( A \) is a proper \( x \)-modular strict inner ideal of \( A \).

**Proof.** Using the continuity of the product of \( A \), it is easily obtained that \( \overline{M} \) is an \( x \)-modular strict inner ideal of \( A \). Let us show it is proper. Choose \( m \in M \), and let \( z = x - m \). If \( ||z|| < 1 \), then by Theorem 4 we know that \( z \in q - \text{Inv}(A) \). If \( w \) is the quasi inverse of \( z \), then \( 1 - z = U_{1-z}(1 - w) = U_{1-z}(1 - w)^2 - U_{1-z}(w^2 - w) = 1 - U_{1-z}(w^2 - w) \), so \( z = U_{1-z}(t) \), where \( t = w^2 - w \in A \). Now

\[
z = U_{1-z}(t) = U_{1-x-m}(t) = U_{1-x}(t) + U_m(t) + 2U_{1-x,m}(t),
\]

and it follows that \( z \in M \), but then \( x \in M \), and this implies that \( M = A \) [13, Proposition 3.1], which contradicts the assumption that \( M \) is proper. Hence it must be \( ||x - m|| \geq 1 \) for every \( m \in M \), so \( x \notin \overline{M} \). Thus \( \overline{M} \) is proper. \( \square \)

A **maximal modular** inner ideal of a Jordan algebra \( A \) is a strict inner ideal which is \( x \)-modular for some \( x \in A \) and maximal among all proper \( x \)-modular strict inner ideals of \( A \) (for \( x \) fixed). The maximal modular inner ideals of a n.c.J. algebra \( A \) are, by definition, the maximal modular inner ideals of the Jordan algebra \( A^+ \).

**Proposition 6.** Let \( A \) be a n.c.J. normed algebra. The following are equivalent:

1. \( A \) is a n.c.J. \( Q \)-algebra.
2. The maximal modular inner ideals of \( A \) are closed.

**Proof.** As a consequence of Proposition 5 we have that (i) implies (ii). To prove the converse we can suppose that \( A \) is a Jordan algebra. Let \( \hat{A} \) denote the normed completion of \( A \). Choose \( x \in A \cap q - \text{Inv}(A) \). Then \( 1-x \) is invertible in \( \hat{A}_1 \), so \( U_{1-x} \) is a homeomorphism on \( \hat{A} \); in particular, \( U_{1-x}(A_1) \) is dense in \( A_1 \). Therefore, if \( z \in A \), there is a sequence \( \{\alpha_n + z_n\} \) in \( A_1 \) such that \( \lim \{U_{1-x}(\alpha_n + z_n)\} = z \). Since \( U_{1-x}(\alpha_n + z_n) \) can be written in the form \( \alpha_n + w_n \) with \( w_n \in A \), it follows that \( \lim \{\alpha_n\} = 0 \), and we deduce that \( \lim \{U_{1-x}(z_n)\} = z \). Hence \( U_{1-x}(A) \) is dense in \( A \). Note that \( U_{1-x}(A) \subseteq A \), since \( A \) is an ideal of \( A_1 \). If \( U_{1-x}(A) \neq A \), then it follows from [13, Remark 2.8] that there is a maximal modular inner ideal \( M \) of \( A \) such that \( U_{1-x}(A) \subseteq M \). Since, by assumption, \( M \) is closed, we have a contradiction with the density of \( U_{1-x}(A) \) in \( A \). Hence \( U_{1-x}(A) = A \). It has been seen in the proof of Proposition 5 that the quasi inverse \( y \) of \( x \) is given by \( y = U_{1-y}(x^2 - x) = U_{1-x}^{-1}(x^2 - x) \), so it follows that \( y \) lies in \( A \). We have proved that \( A \) is a full subalgebra of \( \hat{A} \), and therefore \( A \) is a Jordan \( Q \)-algebra. \( \square \)

The maximal modular left or right ideals in associative algebras are also maximal modular inner ideals [13, Example 3.3]. In this respect the above
proof can be easily modified to show that, if $A$ is a normed associative algebra and the maximal modular left ideals of $A$ are closed, then $A$ is an associative $Q$-algebra (see also [33, Theorem 2.9]).

Since for any element $x$ in a n.c.J. $Q$-algebra we have $p(x) = r(x)$, it follows that homomorphisms of n.c.J. $Q$-algebras decrease the (geometric) spectral radius. Moreover, if $r(x) = 0$ then $sp(x) = \{0\}$, so $x$ is quasi-invertible. Taking this into account, it is easily seen that the proof given by Aupetit [1] and the recent and more simple proof given by Ransford [23] of Johnson's uniqueness of norm theorem yield immediately to the following result (see also [25, Proposition 3.1]). If $X$ and $Y$ are normed spaces and $F$ is a linear mapping from $X$ into $Y$, we denote by $S(F)$ (the separating subspace of $F$) the set of those $y$ in $Y$ for which there is a sequence $\{x_n\}$ in $X$ such that $\lim{x_n} = 0$ and $\lim{F(x_n)} = y$. If $A$ is a n.c.J. algebra, $Rad(A)$ means the Jacobson radical of $A$ [19]; namely, $Rad(A)$ is the largest quasi-invertible ideal of $A$. If $Rad(A) = \{0\}$, $A$ is called semisimple.

**Proposition 7** [1, 23]. Let $A$ and $B$ be n.c.J. complex $Q$-algebras, and let $F$ be a homomorphism from $A$ into $B$. Then $r(b) = 0$ for every $b$ in $S(F) \cap F(A)$. Moreover, if $F$ is a surjective homomorphism, then $S(F) \subseteq Rad(B)$.

Suppose $A$ is a n.c.J. $Q$-algebra, and let $M$ be a closed ideal of $A$. Then the algebra $A/M$ is a n.c.J. $Q$-algebra. (Indeed, if $\pi$ denotes the canonical projection of $A$ onto $A/M$, then $\pi$ is open and $\pi(q - Inv(A)) \subseteq q - Inv(A/M)$. Hence we may apply Proposition 1 to $A/M$.) Moreover, if $B$ is a semisimple n.c.J. algebra and $\varphi$ is a homomorphism from $A$ onto $B$, then $Ker(\varphi)$ is closed (just use Theorem 4(vi) to obtain in the usual way that $\varphi(Ker(\varphi))$ is a quasi-invertible ideal of $B$). With Proposition 7 and these considerations the proof of the main result in [27] yields directly to the following result. Recall that a normed algebra $(A, \| \cdot \|)$ is said to have minimality of norm topology if any algebra norm on $A$, $\| \cdot \|$, minorizing $\| \cdot \|$, i.e., $\| \cdot \| \leq \alpha \| \cdot \|$ for some $\alpha > 0$, is actually equivalent to $\| \cdot \|$.

**Theorem 8**. Let $A$ be a n.c.J. complex $Q$-algebra, and let $B$ be a semisimple complete normed complex n.c.J. algebra with minimality of norm topology. Then every homomorphism from $A$ onto $B$ is continuous.

### 3. Algebra norms on noncommutative JB$^*$-algebras

A not necessarily commutative (for short n.c.) JB$^*$-algebra $A$ is a complete normed n.c.J. complex algebra with (conjugate linear) algebra involution $^*$ such that $\|U_a(a^*)\| = \|a\|^3$ for all $a$ in $A$. Thus $C^*$-algebras and (commutative) JB$^*$-algebras are particular types of n.c. JB$^*$-algebras. If $A$ is a n.c. JB$^*$-algebra, then $A^+$ is a JB$^*$-algebra with the same norm and involution as those of $A$. JB$^*$-algebras were introduced by Kaplansky in 1976, and they have been extensively studied since the paper by Wright [31].

**Lemma 9**. If $\| \cdot \|$ is any algebra norm on a n.c. JB$^*$-algebra $A$, then $(A, \| \cdot \|)$ is a n.c.J. $Q$-algebra.

**Proof.** Since n.c.J. algebras are power-associative, the closed subalgebra of $A$ generated by a symmetric element $(a = a^*)$ is a commutative $C^*$-algebra. Given $a$ in $A$, we can consider the commutative $C^*$-algebra generated by the
symmetric element \( a^* \cdot a = \frac{1}{2}(aa^* + a^*a) \) and make use of a well-known result due to Kaplansky, according to which any algebra norm on a commutative \( C^* \)-algebra is greater than the original norm, to get that \( \|a^* \cdot a\| \leq \|a^* \cdot a\| \). Also it is known that \( \|a\|^2 \leq 2\|a^* \cdot a\| \) [21, Proposition 2.2]. So we have that \( \|a\|^2 \leq 2|a^* \cdot a| \leq 2|a^*||a| \) for all \( a \) in \( A \). Hence \( \|a^n\|^2 \leq 2|a^*|^n|a^n| \) for all \( n \) in \( \mathbb{N} \), which implies that \( (r_{||.||}(a))^2 \leq r_{||.||}(a^*)r_{||.||}(a) \). Now, if \( (C, |\cdot|) \) denotes the completion of \( (A, |\cdot|) \), we have \( r_{||.||}(a) = \rho(a, C) \leq \rho(a, A) = r_{||.||}(a) \) for all \( a \) in \( A \). Thus \( (r_{||.||}(a))^2 \leq r_{||.||}(a^*)r_{||.||}(a) \) and consequently \( r_{||.||}(a) \leq r_{||.||}(a) \). We deduce that \( r_{||.||}(a) = r_{||.||}(a) = \rho(a, A) \) for all \( a \) in \( A \), and by Theorem 4 we conclude that \( (A, |\cdot|) \) is a n.c. \( Q \)-algebra. □

**Theorem 10.** The topology of the norm of a n.c. \( JB^* \)-algebra \( A \) is the smallest algebra normable topology on \( A \).

*Proof.* If \( |\cdot| \) is any algebra norm on \( A \), it has been shown in the proof of Lemma 9 that \( \|a\|^2 \leq 2|a^*||a| \) for all \( a \) in \( A \). If we know additionally that \( |\cdot| \leq M||\cdot|| \) for some nonnegative number \( M \), then \( \|a\|^2 \leq 2M||a^*|||a| = 2M||a|||a| \), so \( \|a\| \leq 2M|a| \) for all \( a \) in \( A \). Hence the norm \( |\cdot| \) is equivalent to the norm of \( A \). Therefore, \( (A, ||\cdot||) \) has minimality of norm topology. Now, for an arbitrary algebra norm \( |\cdot| \) on \( A \), we can use Lemma 9 and apply Theorem 8 to the identity mapping from \( (A, |\cdot|) \) into \( (A, ||\cdot||) \) to obtain that this mapping is continuous, which concludes the proof. □

If \( A \) is a \( C^* \)-algebra, then the particularization of Theorem 10 to the \( JB^* \)-algebra \( A^+ \) gives that any algebra norm on \( A^+ \) defines a topology on \( A \) which is stronger than the original one. This is an improvement of the classical result by Cleveland [8] which states the same for algebra norms on \( A \).

Unlike the preceding results, which are of an algebraic-topologic nature, the following one is geometric.

Let \( A \) be a complete normed complex nonassociative algebra with unit \( 1 \) such that \( ||1|| = 1 \). Denote by \( A^* \) the dual Banach space of \( A \). For \( a \) in \( A \) the subset of \( \mathbb{C} \), \( V_{||.||}(a) = \{f(a) : f \in A^*, ||f|| = 1 = f(1)\} \) is called the numerical range of \( a \). The set of hermitian elements of \( A \), denoted by \( H(A) \), is defined as the set of those elements \( a \) in \( A \) such that \( V_{||.||}(a) \subset \mathbb{R} \). If \( A = H(A) + iH(A) \), then \( A \) is called a \( V \)-algebra. The general nonassociative Vidav-Palmer theorem [24] says that the class of (nonassociative) \( V \)-algebras coincides with the one of unital n.c. \( JB^* \)-algebras.

**Proposition 11.** Every n.c. \( JB^* \)-algebra \( A \) has the property of minimality of the norm; that is, if \( |\cdot| \) is an algebra norm on \( A \) such that \( |\cdot| \leq ||\cdot|| \), then the equality \( |\cdot| = ||\cdot|| \) holds.

*Proof.* By Theorem 10 and the assumptions made, \( |\cdot| \) and \( ||\cdot|| \) are equivalent norms an \( A \), so \( |\cdot| \) is a complete norm on \( A \). Suppose first that \( A \) has a unit element \( 1 \). \( |\cdot| \) being an algebra norm, we have \( 1 \leq |1| \leq ||1|| = 1 \), so \( |1| = 1 \). Let \( ||\cdot|| \) and \( |\cdot| \) also denote the corresponding dual norms of \( ||\cdot|| \) and \( |\cdot| \). Then for \( f \) in \( A^* \) we have \( ||f|| \leq |f| \), and we deduce easily that \( V_{||.||}(a) \subset V_{|.|}(a) \) for all \( a \) in \( A \). Since \( (A, ||\cdot||) \) is a \( V \)-algebra, it follows that \( (A, |\cdot|) \) is also a \( V \)-algebra, and, consequently, by the nonassociative Vidav-Palmer theorem, \( (A, |\cdot|) \) is a n.c. \( JB^* \)-algebra. Since the norm of a n.c. \( JB^* \)-algebra is unique [31], we conclude that \( |\cdot| = ||\cdot|| \). If \( A \) has no unit element, then it is known that \( (A^{**}, ||\cdot||) \), with the Aren's product and a convenient involution which
extends that of $A$, is a unital n.c. $JB^*$-algebra [21]. Since the bidual $A^{**}$ of $A$ is the same for both norms and $|\cdot|$ is an algebra norm on $A^{**}$ satisfying $|\cdot| \leq \|\cdot\|$ on $A^{**}$, it follows from what was previously seen that $|\cdot| = \|\cdot\|$ on $A^{**}$ and, in particular, $|\cdot| = \|\cdot\|$ on $A$. $\square$

Now we apply Theorem 10 and Proposition 11 to the study of the ranges of Jordan homomorphisms from $C^*$-algebras.

**Corollary 12.** Assume that a normed associative complex algebra $B$ is the range of a continuous (resp. contractive) Jordan homomorphism from a $C^*$-algebra. Then $B$ is bicontinuously (resp. isometrically) isomorphic to a $C^*$-algebra.

**Proof.** Let $A$ be a $C^*$-algebra and $\varphi$ a Jordan homomorphism from $A$ onto $B$ under the assumptions in the statement. Since closed Jordan ideals of a $C^*$-algebra are associative ideals (see [7, Theorem 5.3.] or [21, Theorem 4.3]), $A/\text{Ker}(\varphi)$ is a $C^*$-algebra and we may assume that $\varphi$ is a one-to-one mapping. Then, by Theorem 10 (resp. Proposition 11) applied to the $JB^*$-algebra $A^+$, it follows that $\varphi$ is a bicontinuous (resp. isometric) Jordan isomorphism from $A$ onto $B$. Let $C$ denote the associative complex algebra with vector space that of $A$ and product $\square$ defined by $x \square y := \varphi^{-1}(\varphi(x)\varphi(y))$. Then $C^+ (= A^+)$ is a $JB^*$-algebra under the norm and involution of $A$, so, with the same norm and involution, $C$ becomes a $C^*$-algebra [26, Theorem 2] and, clearly, $\varphi$ becomes a bicontinuous (resp. isometric) associative isomorphism from $C$ onto $B$. $\square$

**Corollary 13.** The range of any weakly compact Jordan homomorphism from a $C^*$-algebra into a normed algebra is finite dimensional.

**Proof.** If $A$ is a $C^*$-algebra, $B$ a normed algebra, and $\varphi$ a weakly compact Jordan homomorphism from $A$ into $B$, then, as above, $A/\text{Ker}(\varphi)$ is a $C^*$-algebra and, easily, the induced Jordan homomorphism $A/\text{Ker}(\varphi) \rightarrow B$ is weakly compact, so again we may assume that $\varphi$ is a one-to-one mapping. Now, by Theorem 10 applied to $A^+$, $\varphi$ is a weakly compact topological embedding, so $A$ is a $C^*$-algebra with reflexive Banach space, and so $A$ (and hence the range of $\varphi$) is finite dimensional [28]. $\square$

**Remark 14.** The fact that weakly compact (associative) homomorphisms from $C^*$-algebras have finite-dimensional ranges was proved first in [12] as a consequence of a more general result, and later a very simple proof (that we imitate above) was obtained by Mathieu [17]. If $A$ is a n.c. $JB^*$-algebra and $\varphi$ is any weakly compact homomorphism from $A$ into a normed algebra $B$, since $A/\text{Ker}(\varphi)$ is a n.c. $JB^*$-algebra [21, Corollary 1.11], to obtain some information about the range of $\varphi$ we may assume that $\varphi$ is a one-to-one mapping, and then, as above, the range of $\varphi$ is bicontinuously isomorphic to a n.c. $JB^*$-algebra with reflexive Banach space, namely, a finite product of simple n.c. $JB^*$-algebras which are either finite dimensional or quadratic [22, Theorem 3.5] (note that infinite-dimensional quadratic $JB^*$-algebras do exist and the identity mapping on such a $JB^*$-algebra is weakly compact). This result on the range of a weakly compact homomorphism from a n.c. $JB^*$-algebra was proved first in [11] by using Theorem 10 and a nonassociative extension of the above-mentioned general result in [12]. The proof given above (also suggested in [11]) is analogous to Mathieu’s proof for the particular case of $C^*$-algebras.
References


Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071-Granada, Spain

Current address, L. Rico: Departamento de Didáctica de la Matemática, Facultad de Ciencias de la Educación, Universidad de Granada, 18077-Granada, Spain

E-mail address, A. Rodríguez: apalacios@ugr.es