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THE AUTOMORPHISM GROUP OF A FREE GROUP IS NOT A CAT(0) GROUP

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ABSTRACT. If F is a finitely generated free group, then the group $\mathrm{Aut}(F)$, if $\mathrm{rank}(F) \geq 3$, and $\mathrm{Out}(F)$, if $\mathrm{rank}(F) \geq 4$, are not isomorphic to a subgroup of a group which acts properly discontinuously and cocompactly on a 1-connected geodesic metric space satisfying Gromov's condition CAT(0).

1. INTRODUCTION

In his thesis, Bridson [Br1] established that, for $n \geq 3$, the Culler-Vogtmann space $X(F_n)$, which is a contractible finite-dimensional CW complex on which $\mathrm{Out}(F_n)$ acts properly, cellularly, and with compact quotient does not admit a piecewise Euclidean metric of nonpositive curvature, which is invariant under the group action; here F_n denotes a finitely generated free group of rank n. This left open the question whether $\mathrm{Out}(F_n)$, $n \geq 3$, can act properly by isometries on any simply connected geodesic metric space satisfying Gromov's condition CAT(0) [Gr] with a compact quotient. The question was originally raised in the attempt to prove that $\mathrm{Out}(F_n)$ is combable (or, more optimistically, automatic), since such an action would give a combing [ECHLPT]. In this note we shall establish the following result.

Theorem. *If F is a finitely generated free group, then the group* $\mathrm{Aut}(F)$, *if* $\mathrm{rank}(F) \geq 3$, *and* $\mathrm{Out}(F)$, *if* $\mathrm{rank}(F) \geq 4$, *are not isomorphic to a subgroup of a group which acts properly, discontinuously, and cocompactly by isometries on a 1-connected geodesic metric space satisfying the condition* CAT(0).

In this connection, we note that $\mathrm{Out}(F_2) = \mathrm{Gl}_2(\mathbb{Z})$ acts on a simplicial tree with finite stabilizers [Se]. In addition, $\mathrm{Aut}(F_2)$ is known to be commensurable with the quotient of the braid group B_4 by its $\mathbb{Z}_2$ center and hence is automatic [ECHLPT]. Otherwise it is open whether $\mathrm{Out}(F_n)$ and $\mathrm{Aut}(F_n)$ are either (bi-)automatic or (bi-)combable for $n \geq 3$.

Our result is an application of a geometric result due to Bridson [Br2], which is in turn a generalization of the flat subspace theorem of Gromoll and Wolf

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[GW] and Lawson and Yau [LY]. Bridson's result states the following (the terms will be explained in the next paragraph).

Theorem (Bridson). *If X is a simply connected geodesic metric space satisfying* $\mathrm{CAT}(0)$ *which is acted on properly discontinuously and isometrically by a group G with compact quotient $G\backslash X$ and if $A < G$ is a free Abelian subgroup of* rank r, *then there is a flat subspace Y isometric to $\mathbb{R}^r$ which is isometrically and totally geodesically embedded in X and such that Y is stabilized by A.*

We recall here that a metric space X is called *geodesic* if for any two points x, y of X there is an isometric embedding of an interval $f: [a, b] \to X$ with $f(a) = x$, $f(b) = y$, and $d(x, y) = b - a$. There are various equivalent ways of formulating the condition $\mathrm{CAT}(0)$ for a geodesic metric space [Br1, GH]. For definiteness, we adopt the following definition. Let $\Delta = [x, y, z]$ be a geodesic triangle in X, so the sides $[x, y]$, $[y, z]$, and $[z, x]$ are geodesic segments. Let $\Delta' = [x', y', z']$ be a comparison triangle in $\mathbb{R}^2$ so that corresponding sides of Δ and Δ' are of equal length. We require that if p is any point on $[y, z]$ and p' is the corresponding point on $[y', z']$ (so p' divides the segment $[y', z']$ in the same ratio of lengths as p divides $[y, z]$), then $d(x, p) \le d'(x', p')$, where d' denotes the Euclidean metric of $\mathbb{R}^2$.

To say that Y is totally geodesically embedded in X in the statement of the theorem means that the (unique) geodesic in X connecting any two given points of Y lies entirely in Y.

As an example, if M is a closed Riemannian manifold with all sectional curvatures nonpositive, then the universal cover $\widetilde{M}$ is a geodesic metric space satisfying $\mathrm{CAT}(0)$. For examples of $\mathrm{CAT}(0)$ spaces which are not manifolds, see [Br1].

2. Translation lengths

We assume in this section that X is a 1-connected geodesic metric space which is acted on properly discontinuously by a discrete group G of isometries so that the quotient $G\backslash X$ is compact. In addition we assume that X satisfies $\mathrm{CAT}(0)$.

If $g \in G$ is of infinite order, then by Bridson's theorem g stabilizes a flat $\mathbb{R}^1$ isometrically embedded in X. Thus g acts on the flat by translation by a positive real number $\tau_{\mathrm{geo}}(g)$, which is defined to be infimum of the displacement function $x \mapsto d(x, gx)$, $x \in X$. Thus the number $\tau_{\mathrm{geo}}(g)$ is independent of the flat $\mathbb{R}^1$ stabilized by g. This can be seen geometrically by using the fact, proved in the course of establishing Bridson's theorem, that two such flats stabilized by g cobound a flat strip bounded by parallel straight lines and stabilized by g. From the fact that g acts isometrically, it follows that g translates one boundary $\mathbb{R}^1$ the same amount as it translates the other.

The function τ_{geo} satisfies the following properties, which are established easily from the definition:

(1) $\tau_{\mathrm{geo}}(g) = \tau_{\mathrm{geo}}(hgh^{-1})$ for all $g, h \in G$.
(2) $\tau_{\mathrm{geo}}(g^n) = |n|\tau_{\mathrm{geo}}(g)$ for $n \in \mathbb{Z}$.

If $A < G$ is a free abelian subgroup of rank r, let $Y \subset X$ be a flat $\mathbb{R}^r$ stabilized by A. Since Y is isometrically and totally geodesically embedded in X, it follows that the translation numbers of elements of A calculated in

$\mathbb{R}^r$ and Y are the same. If we choose a point $y \in Y \cong \mathbb{R}^r$ as the origin, then $a \in A$ can be identified with its displacement vector v_a at the origin, since the group A, considered as a group of isometries $\mathbb{R}^r$, consists only of translations. One has the equality $\tau_{\text{geo}}(a) = \|v_a\|$, where $\|v\|$ denotes the Euclidean norm of the vector $v \in \mathbb{R}^r$.

Let $F = F(a, b, c)$ be the free group freely generated by a, b, and c, and let $\phi\colon F \to F$ be the automorphism given by $a \mapsto a$, $b \mapsto ba$, $c \mapsto ca^2$. Thus the split extension $H = F \rtimes_\phi \mathbb{Z}$ has presentation

$$\mathscr{P} = \langle a, b, c, t | tat^{-1} = a,\ tbt^{-1} = ba,\ tct^{-1} = ca^2 \rangle.$$

Proposition 2.1. *The group H above is not isomorphic to a subgroup of any group G of isometries acting properly discontinuously on a 1-connected geodesic metric space X with compact quotient $G\backslash X$ where X satisfies* CAT(0).

Proof. Observe that the second and third relations of $\mathscr{P}$ can be rewritten as $b^{-1}tb = at$ and $c^{-1}tc = a^2t$. By the first relation of $\mathscr{P}$, $A := \langle a, t \rangle$ is free abelian of rank 2.

If $H < G$, where G acts properly discontinuously on the 1-connected geodesic metric space X, $G\backslash X$ is compact, and X satisfies CAT(0), then the function τ_{geo} associated to G and X satisfies $\tau_{\text{geo}}(t) = \tau_{\text{geo}}(at) = \tau_{\text{geo}}(a^2t)$, where we have used property (1) of translation numbers. By Bridson's theorem, A stabilizes a flat $\mathbb{R}^2$. If we pick a base point in this flat, then t, a can be identified with translations by independent vectors v_t, v_a in this flat. Hence we have $\|v_t\| = \|v_t + v_a\| = \|v_t + 2v_a\|$. But these equalities cannot be satisfied for two independent vectors $v_t, v_a \in \mathbb{R}^2$, since they say that a line intersects a circle in 3 points. This contradiction shows that H cannot be isomorphic to a subgroup of G.

Question. We would like to know whether or not the group H is (bi-)combable (resp. (bi-)automatic). Bestvina informed us that it follows from his joint work with Feighn [BF] that H satisfies the quadratic isoperimetric inequality, so this can be taken as positive evidence.

Theorem 2.2. *Each of the following groups cannot be isomorphic to a subgroup of a group G which acts properly discontinuously by isometries with compact quotient on a 1-connected geodesic metric space satisfying* CAT(0):

(1) $\mathrm{Aut}(F_n)$ *if* $n \geq 3$, *and*
(2) $\mathrm{Out}(F_n)$ *if* $n \geq 4$.

Proof. We let $H = F(a, b, c) \rtimes_\phi \mathbb{Z}$, where $\phi\colon a \mapsto a,\ b \mapsto ba,\ c \mapsto ca^2$. We embed H in $\mathrm{Aut}(F(a, b, c))$ as follows. Let $F = F(a, b, c)$, and let $\iota\colon F \to \mathrm{Aut}(F)$ send $f \in F$ to the inner automorphism ι_f, so $\iota_f(x) = fxf^{-1}$, for $x \in F$. With ϕ as above, one checks that $\phi\iota_f\phi^{-1} = \iota_{\phi(f)}$, which produces a homomorphism $H \to \mathrm{Aut}(F)$. This homomorphism is easily seen to be injective. Hence H embeds in $\mathrm{Aut}(F_3)$. Since $\mathrm{Aut}(F_3)$ embeds in $\mathrm{Aut}(F_n)$ for all $n \geq 3$, we have embeddings of H in $\mathrm{Aut}(F_n)$ for all $n \geq 3$.

Next one observes that $\mathrm{Aut}(F_n)$ embeds in $\mathrm{Out}(F_{n+1})$, by stabilizing the last element of a free basis. Thus H embeds in $\mathrm{Out}(F_n)$ for all $n \geq 4$.

The result follows from these observations by applying Proposition 2.1.

Remark. It is interesting to compare our result with that of [FP], which gives the same values of n for which $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$ are not linear.

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