POSITIVE DEFINITE OPERATOR SEQUENCES

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Abstract. An example is given of a linear mapping from \( C[x] \) to \( M_2(C) \) which is positive but not completely positive. It is shown that a positive linear mapping from \( C[x] \) to \( B(\mathcal{H}) \) is completely positive if certain scalar moment sequences associated with it are determinate.

Suppose \( \mathcal{A} \) is a complex algebra with involution, \( \mathcal{B} \) is a \( C^* \)-algebra, and \( L \) is a linear mapping from \( \mathcal{A} \) to \( \mathcal{B} \). We say \( L \) is positive if \( L(a^*a) \geq 0 \) for each \( a \in \mathcal{A} \). For each integer \( n \) the set \( M_n(\mathcal{A}) \) of matrices of order \( n \) with entries in \( \mathcal{A} \), with matrix multiplication and the involution \( (a_{jk})^* = (a_{kj})^* \), is again a complex algebra with involution and \( M_n(\mathcal{B}) \) is a \( C^* \)-algebra. The mapping \( L \) is said to be completely positive if for each \( n \) the mapping \( (a_{jk}) \mapsto (L(a_{jk})) \) from \( M_n(\mathcal{A}) \) to \( M_n(\mathcal{B}) \) is positive. For studies of such mappings we refer to [2, 6–9, 11, 12]. See [5] for a fairly recent survey.

Evidently, every completely positive mapping is positive. The converse, however, is false, even if \( \mathcal{A} \) is assumed to be a \( C^* \)-algebra, as evidenced by the example of Arveson [2, p. 169] that the transpose operation on \( M_2(C) \) is positive but not completely positive. It is known that if \( \mathcal{A} \) is a commutative \( C^* \)-algebra or if \( \mathcal{B} \) is commutative, then every positive linear mapping from \( \mathcal{A} \) to \( \mathcal{B} \) is completely positive (see [2, 11]). The main aim of the present note is to show that commutativity of \( \mathcal{A} \) (without the condition that \( \mathcal{A} \) be a \( C^* \)-algebra) does not imply that every positive linear mapping from \( \mathcal{A} \) to \( \mathcal{B} \) is completely positive. In fact, the conclusion fails even if \( \mathcal{A} = C[x] \), the polynomials in one indeterminate with complex coefficients, and \( \mathcal{B} = M_2(C) \). A counterexample is given in Theorem 1.

Theorem 2 provides a sufficient condition, in terms of determinacy of certain scalar moment sequences, for a positive linear mapping defined on \( C[x] \) to be completely positive. An application shows that a positive linear mapping \( L : C[x] \to B(\mathcal{H}) \) is completely positive if only \( \|L(x^n)\| \) does not grow too rapidly (i.e., not much faster than \( n^n \)) as \( n \to \infty \). Finally, Theorem 3 is a strengthening of Theorem 1, stating that a positive linear mapping from \( C[x] \)
to $B(\mathcal{H})$ may fail to be completely positive even if many of the associated scalar moment sequences are determinate.

A sequence $(s_n)_{n \geq 0}$ of bounded operators on a Hilbert space $\mathcal{H}$ is said to be positive definite if $\sum_{j, k=0}^{n} c_j c_k s_{j+k} \geq 0$ for all $n \geq 0$ and $c_0, \ldots, c_n \in \mathbb{C}$, and of positive type if $\sum_{j, k=0}^{n} (s_{j+k} \xi_k, \xi_j) \geq 0$ for all $n \geq 0$ and $\xi_0, \ldots, \xi_n \in \mathcal{H}$ (cf. [4, 4.1.7] or [13]). Note that $(s_n)$ is positive definite if and only if for each $\xi \in \mathcal{H}$ the scalar sequence $(\langle s_n \xi, \xi \rangle)$ is positive definite. Sequences $(s_n)$ in $B(\mathcal{H})$ are in a one-to-one correspondence with linear mappings $L : \mathbb{C}[x] \to B(\mathcal{H})$, given by $L(x^n) = s_n$. Such a sequence $(s_n)$ is positive definite if and only if the corresponding mapping $L$ is positive and is of positive type if and only if $L$ is completely positive. We identify $B(\mathbb{C})$ with $M_n(\mathbb{C})$.

**Theorem 1.** There is a positive linear mapping from $\mathbb{C}[x]$ to $M_2(\mathbb{C})$ which is not completely positive.

**Proof.** We shall exhibit a sequence $(s_n)$ in $M_2(\mathbb{C})$ which is positive definite but not of positive type. The corresponding linear mapping then has the properties claimed. \[ \Box \]

Define $a_n = 2^{(n+2)!}$ for $n \geq 0$ and

$$ s_0 = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \quad s_1 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}, $$

$$ s_{2n-1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad s_{2n} = \begin{pmatrix} a_n & 0 \\ 0 & a_n \end{pmatrix} $$

for $n \geq 2$. To show that $(s_n)$ is positive definite it suffices to show that the determinant

$$ D_n(\xi) = \begin{vmatrix} \langle s_0 \xi, \xi \rangle & \ldots & \langle s_n \xi, \xi \rangle \\ \vdots & \ddots & \vdots \\ \langle s_n \xi, \xi \rangle & \ldots & \langle s_{2n} \xi, \xi \rangle \end{vmatrix} $$

is positive for each $n \geq 0$ and each $\xi$ in the unit sphere of $\mathbb{C}^2$. So suppose $\xi = (\xi_1, \xi_2)$ with $|\xi_1|^2 + |\xi_2|^2 = 1$. Then

$$ D_0(\xi) = 4|\xi_1|^2 + |\xi_2|^2 \geq 1, $$

$$ D_1(\xi) = (4|\xi_1|^2 + |\xi_2|^2)(|\xi_1|^2 + 4|\xi_2|^2) - (4 \text{Re}(\overline{\xi_1} \xi_2))^2 $$

$$ \geq 4|\xi_1|^4 + 4|\xi_2|^4 \geq 2(|\xi_1|^2 + |\xi_2|^2)^2 \geq 1. $$

Now assume $n \geq 2$ and $D_{n-1}(\xi) \geq 1$. Expand $D_n(\xi)$ completely, then collect the $n!$ terms containing $\langle s_{2n} \xi, \xi \rangle$ into one. For the remaining $(n + 1)! - n!$ terms use

$$ |\langle s_k \xi, \xi \rangle| \leq \|s_k\| \leq \max\{4, a_2, \ldots, a_{n-1}\} \leq a_{n-1} $$

($k = 0, \ldots, 2n - 1$) to get a bound. The result is

$$ D_n(\xi) \geq \langle s_{2n} \xi, \xi \rangle D_{n-1}(\xi) - ((n + 1)! - n!) a_{n-1}^{n+1} $$

$$ \geq a_n - (n + 1)! a_{n-1}^{n+1} = 2^{(n+1)(n+1)!} - (n + 1)! \geq 1. $$
So \( (s_n) \) is positive definite. To see that \( (s_n) \) is not of positive type, take \( \xi_0 = (0, 1) \) and \( \xi_1 = (-1, 0) \) and verify \( \sum_{j=0}^{1} s_{j+1} \xi_{j+1}, \xi_j = -2 \).

A sequence \( (s_n) \) in \( B(\mathcal{H}) \) is a(n operator) moment sequence if there is a measure \( \mu \), defined on the Borel field \( \mathcal{B}(\mathbb{R}) \) and taking values in \( B(\mathcal{H})^+ \), such that
\[
(2) \quad \langle s_{n+1}, \xi, \eta \rangle = \int_{\mathbb{R}} x^n d\langle \mu \xi, \eta \rangle(x)
\]
for all \( \xi, \eta \in \mathcal{H} \) and \( n \geq 0 \). Here \( \langle \mu \xi, \eta \rangle \) is the complex measure defined by \( \langle \mu \xi, \eta \rangle(E) = \langle \mu(E) \xi, \eta \rangle \), \( E \in \mathcal{B}(\mathbb{R}) \). (By definition, \( \mu : \mathcal{B}(\mathbb{R}) \to B(\mathcal{H})^+ \) is a measure if and only if \( \langle \mu \xi, \zeta \rangle \) is a measure for each \( \zeta \in \mathcal{H} \).) By Hamburger's Theorem ([4, 6.2.2] or [1]), a sequence \( (s_n)_{n \geq 0} \) in \( \mathbb{R} \) is a moment sequence if and only if it is positive definite. General references for this classical moment problem are [1, 10]; see [4] for the generalization to semigroups other than the integers. For an arbitrary Hilbert space \( \mathcal{H} \) moment sequences in \( B(\mathcal{H}) \) are precisely the sequences of positive type; see, for example, [9].

Suppose \( (s_n) \) is a positive definite sequence in \( B(\mathcal{H}) \). For each \( \zeta \in \mathcal{H} \) the sequence \( (s_{n+1}, \zeta, \zeta) \) is positive definite, so one may choose a measure \( \lambda_\zeta \) on \( \mathbb{R} \) such that
\[
(3) \quad \langle s_{n+1}, \xi \rangle = \int_{\mathbb{R}} x^n d\lambda_\zeta(x)
\]
for all \( n \geq 0 \). If a \( B(\mathcal{H})^+ \)-valued measure \( \mu \) exists which satisfies
\[
(4) \quad \langle \mu \xi, \zeta \rangle = \lambda_\zeta, \quad \zeta \in \mathcal{H},
\]
then \( \mu \) satisfies equation (2) (by polarization), so \( (s_n) \) is a moment sequence. The existence of such a measure \( \mu \), however, cannot be taken for granted. Indeed, if (4) holds then
\[
\lambda_{\zeta+n} + \lambda_{\zeta-n} = \langle \mu(\zeta + \eta), \zeta + \eta \rangle + \langle \mu(\zeta - \eta), \zeta - \eta \rangle = 2(\langle \mu \xi, \zeta \rangle + \langle \mu \eta, \eta \rangle) = 2(\lambda_\zeta + \lambda_\eta)
\]
and
\[
\lambda_{c\zeta} = \langle \mu(c \xi), c \xi \rangle = |c|^2 \langle \mu \xi, \xi \rangle = |c|^2 \lambda_\xi
\]
for all \( \zeta, \eta \in \mathcal{H} \) and \( c \in \mathbb{C} \). In short, the family \( (\lambda_\zeta)_{\zeta \in \mathcal{H}} \) has to satisfy the parallelogram law
\[
(5) \quad \lambda_{\zeta+n} + \lambda_{\zeta-n} = 2(\lambda_\zeta + \lambda_\eta)
\]
(\( \zeta, \eta \in \mathcal{H} \)) and the rule of homogeneity
\[
(6) \quad \lambda_{c\zeta} = |c|^2 \lambda_\xi
\]
(\( \zeta \in \mathcal{H} \) and \( c \in \mathbb{C} \)). These conditions are also sufficient. To see this, suppose they hold. For each Borel set \( E \) in \( \mathbb{R} \) the family \( (\lambda_\zeta(E))_{\zeta \in \mathcal{H}} \) satisfies the parallelogram law and the rule of homogeneity (same as above, only for scalars instead of measures), and it is a matter of algebra that there is a unique sesquilinear form \( f_E \) on \( \mathcal{H} \) such that \( \lambda_\zeta(E) = f_E(\zeta, \xi) \) for all \( \zeta \in \mathcal{H} \). Now
$f_E$ is positive since the $\lambda_\xi$ are positive measures and is bounded since (by (3))

$$f_E(\xi, \eta) = \lambda_\xi(E) \leq \lambda_\xi(R) = \langle s_0, \xi \rangle \leq \|s\| \|\xi\|^2.$$ 

Hence there is a unique $\mu(E) \in B(B\mathcal{H})_+$ such that $f_E(\xi, \eta) = \langle \mu(E)\xi, \eta \rangle$ for all $\xi, \eta \in \mathcal{H}$. In particular $\lambda_\xi(E) = \langle \mu(E)\xi, \xi \rangle$ for all $\xi \in \mathcal{H}$. This being so for all $E \in B(R)$, it follows that $\mu$ is a measure and that (4) holds.

A moment sequence $(s_n)$ in $B(B\mathcal{H})$ is determinate if only one measure $\mu$ satisfies (2).

**Theorem 2.** Let $(s_n)$ be a positive definite sequence in $B(B\mathcal{H})$, and assume that for all $\xi, \eta \in \mathcal{H}$ the scalar moment sequence $(\langle s_n\xi, \xi \rangle + \langle s_n\eta, \eta \rangle)_{n \geq 0}$ is determinate. Then $(s_n)$ is a determinate moment sequence.

**Proof.** For each $\xi, \eta \in \mathcal{H}$ the moment sequence $(\langle s_n\xi, \xi \rangle)$ is determinate (take $\eta = 0$ in the statement of the theorem); let $\lambda_\xi$ be the unique measure for which (3) holds. By the uniqueness of $\lambda_\xi$, any measure $\mu$ satisfying (2) must satisfy (4). By polarization it follows that $(s_n)$, if a moment sequence, is determinate.

For $\xi, \eta \in \mathcal{H}$ we have

$$\int x^n d\lambda_{\xi+\eta}(x) + \int x^n d\lambda_{\xi-\eta}(x) = \langle s_n(\xi + \eta), \xi + \eta \rangle + \langle s_n(\xi - \eta), \xi - \eta \rangle = 2\langle s_n\xi, \xi \rangle + 2\langle s_n\eta, \eta \rangle = 2 \int x^n d\lambda_\xi(x) + 2 \int x^n d\lambda_\eta(x)$$

($n \geq 0$). By the determinacy hypothesis it follows that (5) holds; (6) is gotten by a similar argument. This proves the existence of a $B(B\mathcal{H})_+$-valued measure $\mu$ satisfying (4) and hence (by (3) and polarization) (2), thus completing the proof of the theorem. □

Recall the following two criteria of Carleman for determinacy in the classical moment problem: First, suppose $(s_n)$ is a sequence of reals such that the determinants

$$D_n = \begin{vmatrix} s_0 & \cdots & s_n \\ \vdots & \ddots & \vdots \\ s_n & \cdots & s_{2n} \end{vmatrix}$$

($n \geq 0$) are positive. (In particular, $(s_n)$ is positive definite, hence a moment sequence). If

$$\sum_{n=2}^{\infty} \frac{D_{n-1}}{\sqrt{D_{n-2}D_n}} = \infty,$$

then $(s_n)$ is determinate.

Second, every scalar moment sequence $(s_n)$ satisfying

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{s_{2n}}} = \infty$$

is determinate (see [1]).
Corollary 1. Let \((s_n)\) be a positive definite sequence in \(B(\mathcal{H})\), and suppose
\[
\sum_{n=1}^{\infty} \frac{1}{\sqrt{\|s_{2n}\|}} = \infty.
\]
Then \((s_n)\) is a determinate moment sequence.

Proof. Given \(\xi, \eta \in \mathcal{H}\), apply the second Carleman criterion to the sequence
\[
((s_n \xi, \xi) + (s_n \eta, \eta)).
\]
Though conceivably \(\{(s_n \xi, \xi) + (s_n \eta, \eta)\}\) might be replaced by just \(\{(s_n \xi, \xi)\}\)
in Theorem 2, a proof of this can hardly be simple. Certainly the naïve approach is doomed to failure; the sum of two determinate moment sequences need not be determinate. (In fact, every (scalar) moment sequence is the sum of two determinate ones. To see this, consider any moment sequence \((s_n)\). If \((s_n)\) is determinate, it is the sum of the two determinate moment sequences \((0)\) and \((s_n)\). Otherwise, choose a Nevanlinna extremal measure \(\mu\) of which \((s_n)\) is the moment sequence (see [1]). Such a measure has the form
\[
\mu = \sum_{n=1}^{\infty} a_n \delta_{x_n}
\]
where the \(x_n\) are distinct reals, \(a_n > 0\), and \(\delta_{x_n}\) denotes the Dirac measure at \(x_n\). Moreover, \(\mu - a_n \delta_{x_n}\) is determinate for each \(n\) [3]. So \((s_n)\) is the sum of the moment sequences of \(a_1 \delta_{x_1}\) and \(\mu - a_1 \delta_{x_1}\); both of which are determinate.) Determinacy of a great many of the scalar moment sequences \((s_n \xi, \xi)\) is not sufficient.

Theorem 3. There is a positive definite sequence \((s_n)_{n \geq 0}\) in \(B(\mathbb{C}^2)\) such that the scalar moment sequence \((s_n \xi, \xi)\) is determinate for each \(\xi\) in a dense \(G_\delta\) in \(\mathbb{C}^2\), yet \((s_n)\) is not a moment sequence.

Proof. Let \(\Sigma\) be the unit sphere in \(\mathbb{C}^2\), \(\Phi\) a countable dense set in \(\Sigma\). Choose a sequence \((\varphi_n)_{n \geq 2}\) in \(\Phi\) in which each element of \(\Phi\) appears infinitely often.

We shall construct the sequence \((s_n)\) by induction, keeping the determinants (1) positive for all \(n\) and all \(\xi \in \Sigma\) (hence for all \(\xi \in \mathbb{C}^2 \setminus \{0\}\)), thus making sure that \((s_n)\) will be positive definite. As in the proof of Theorem 1, beginning the construction with
\[
s_0 = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \quad s_1 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}
\]
e nsures that the full sequence will not be of positive type (i.e., not a moment sequence).

Suppose that \(n \geq 2\) and that \(s_0, \ldots, s_{2n-2}\) have been chosen properly. Put \(s_{2n-1} = 0\) and define a function \(F_n\) on \(\mathbb{C}^2\) by the condition that
\[
\begin{vmatrix}
(s_0 \xi, \xi) & \cdots & (s_{n-1} \xi, \xi) & (s_n \xi, \xi) \\
\vdots & \ddots & \vdots & \vdots \\
(s_{n-1} \xi, \xi) & \cdots & (s_{2n-2} \xi, \xi) & (s_{2n-1} \xi, \xi) \\
(s_n \xi, \xi) & \cdots & (s_{2n-1} \xi, \xi) & t
\end{vmatrix} = D_{n-1}(\xi)t + F_n(\xi)
\]
for all \(\xi \in \mathbb{C}^2\) and \(t \in \mathbb{R}\). Let \(p : \mathbb{C}^2 \to \mathbb{C} \varphi_n\) be the orthogonal projection. Since \(D_{n-1}(\varphi_n) > 0\) by assumption, we may choose \(a \in \mathbb{R}\) such that
\[
0 < D_{n-1}(\varphi_n)a + F_n(\varphi_n) < \frac{D_{n-1}(\varphi_n)^2}{D_{n-2}(\varphi_n)}.
\]
Note that $a > 0$ since the matrix in (7) becomes strictly positive definite upon substituting $a$ for $t$ and $\varphi_n$ for $\xi$. The obvious homogeneity properties of $D_{n-1}$ and $F_n$ imply that

$$D_{n-1}(\xi)a + F_n(\xi) = D_{n-1}(\varphi_n)a + F_n(\varphi_n)$$

for each $\xi \in T\varphi_n$ where $T = \{ c \in \mathbb{C} \mid |c| = 1 \}$. Since $D_{n-1}$ and $F_n$ are continuous, it follows that

$$D_{n-1}(\xi) \| p\xi \|^2 a + F_n(\xi) > 0$$

for all $\xi$ in some neighbourhood of $T\varphi_n$. On the remainder of $\Sigma$, the function $\xi \to \|(1 - p)\xi\|$ is bounded below away from zero. Hence, if $b > 0$ is chosen large enough, then

$$D_{n-1}(\xi)(a \| p\xi \|^2 + b \| (1 - p)\xi \|^2) + F_n(\xi) > 0$$

for all $\xi \in \Sigma$. Defining $s_{2n} = ap + b(1 - p)$ we have

$$D_n(\xi) = D_{n-1}(\xi)(s_{2n}\xi, \xi) + F_n(\xi) = D_{n-1}(\xi)(a \| p\xi \|^2 + b \| (1 - p)\xi \|^2) + F_n(\xi) > 0$$

for all $\xi \in \Sigma$. Note that

$$D_n(\varphi_n) = D_{n-1}(\varphi_n)(s_{2n}\varphi_n, \varphi_n) + F_n(\varphi_n)$$

(8)

$$= D_{n-1}(\varphi_n)a + F_n(\varphi_n) < \frac{D_{n-1}(\varphi_n)^2}{D_{n-2}(\varphi_n)}.$$

For each $n \geq 2$ the set

$$G_n = \{ \xi \in \mathbb{C}^2 \setminus \{0\} \mid D_{n-2}(\xi)D_n(\xi) < D_{n-1}(\xi)^2 \}$$

is open, contains $\varphi_n$ according to (8), and is stable under multiplication by nonzero scalars (by the homogeneity properties of the functions $D_k$). The $G_3$-set

$$A = \bigcap_{n=2}^{\infty} \bigcup_{m=n}^{\infty} G_m$$

contains $\Phi$ and is stable under multiplication by nonzero scalars; so $A$ is dense in $\mathbb{C}^2$. If $\xi \in A$, then $D_{n-2}(\xi)D_n(\xi) < D_{n-1}(\xi)^2$ for infinitely many $n$, hence

$$\sum_{n=2}^{\infty} \frac{D_{n-1}(\xi)}{\sqrt{D_{n-2}(\xi)D_n(\xi)}} = \infty.$$

By the Carleman criterion it follows that the moment sequence $(\langle s_n\xi, \xi \rangle)_{n \geq 0}$ is determinate.  

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