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POSITIVE DEFINITE OPERATOR SEQUENCES

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ABSTRACT. An example is given of a linear mapping from $\mathbf{C}[x]$ to $\mathbf{M}_2(\mathbf{C})$ which is positive but not completely positive. It is shown that a positive linear mapping from $\mathbf{C}[x]$ to $\mathbf{B}(\mathscr{H})$ is completely positive if certain scalar moment sequences associated with it are determinate.

Suppose $\mathscr{A}$ is a complex algebra with involution, $\mathscr{B}$ is a C^*-algebra, and L is a linear mapping from $\mathscr{A}$ to $\mathscr{B}$. We say L is *positive* if $L(a^*a) \geq 0$ for each $a \in \mathscr{A}$. For each integer n the set $\mathbf{M}_n(\mathscr{A})$ of matrices of order n with entries in $\mathscr{A}$, with matrix multiplication and the involution $(a_{jk})^* = (a_{kj}^*)$, is again a complex algebra with involution and $\mathbf{M}_n(\mathscr{B})$ is a C^*-algebra. The mapping L is said to be *completely positive* if for each n the mapping $(a_{jk}) \to (L(a_{jk}))$ from $\mathbf{M}_n(\mathscr{A})$ to $\mathbf{M}_n(\mathscr{B})$ is positive. For studies of such mappings we refer to [2, 6–9, 11, 12]. See [5] for a fairly recent survey.

Evidently, every completely positive mapping is positive. The converse, however, is false, even if $\mathscr{A}$ is assumed to be a C^*-algebra, as evidenced by the example of Arveson [2, p. 169] that the transpose operation on $\mathbf{M}_2(\mathbf{C})$ is positive but not completely positive. It is known that if $\mathscr{A}$ is a commutative C^*-algebra or if $\mathscr{B}$ is commutative, then every positive linear mapping from $\mathscr{A}$ to $\mathscr{B}$ is completely positive (see [2, 11]). The main aim of the present note is to show that commutativity of $\mathscr{A}$ (without the condition that $\mathscr{A}$ be a C^*-algebra) does not imply that every positive linear mapping from $\mathscr{A}$ to $\mathscr{B}$ is completely positive. In fact, the conclusion fails even if $\mathscr{A} = \mathbf{C}[x]$, the polynomials in one indeterminate with complex coefficients, and $\mathscr{B} = \mathbf{M}_2(\mathbf{C})$. A counterexample is given in Theorem 1.

Theorem 2 provides a sufficient condition, in terms of determinacy of certain scalar moment sequences, for a positive linear mapping defined on $\mathbf{C}[x]$ to be completely positive. An application shows that a positive linear mapping $L : \mathbf{C}[x] \to \mathbf{B}(\mathscr{H})$ is completely positive if only $\|L(x^n)\|$ does not grow too rapidly (i.e., not much faster than n^n) as $n \to \infty$. Finally, Theorem 3 is a strengthening of Theorem 1, stating that a positive linear mapping from $\mathbf{C}[x]$

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to $\mathbf{B}(\mathscr{H})$ may fail to be completely positive even if many of the associated scalar moment sequences are determinate.

A sequence $(s_n)_{n\geq 0}$ of bounded operators on a Hilbert space $\mathscr{H}$ is said to be *positive definite* if $\sum_{j,k=0}^{n} \overline{c_j} c_k s_{j+k} \geq 0$ for all $n \geq 0$ and $c_0, \ldots, c_n \in \mathbf{C}$, and *of positive type* if $\sum_{j,k=0}^{n} \langle s_{j+k}\xi_k, \xi_j\rangle \geq 0$ for all $n \geq 0$ and $\xi_0, \ldots, \xi_n \in \mathscr{H}$ (cf. [4, 4.1.7] or [13]). Note that (s_n) is positive definite if and only if for each $\xi \in \mathscr{H}$ the scalar sequence $(\langle s_n\xi, \xi\rangle)$ is positive definite. Sequences (s_n) in $\mathbf{B}(\mathscr{H})$ are in a one-to-one correspondence with linear mappings $L : \mathbf{C}[x] \to \mathbf{B}(\mathscr{H})$, given by $L(x^n) = s_n$. Such a sequence (s_n) is positive definite if and only if the corresponding mapping L is positive and is of positive type if and only if L is completely positive. We identify $\mathbf{B}(\mathbf{C}^n)$ with $\mathbf{M}_n(\mathbf{C})$.

Theorem 1. *There is a positive linear mapping from* $\mathbf{C}[x]$ *to* $\mathbf{M}_2(\mathbf{C})$ *which is not completely positive.*

Proof. We shall exhibit a sequence (s_n) in $\mathbf{M}_2(\mathbf{C})$ which is positive definite but not of positive type. The corresponding linear mapping then has the properties claimed. □

Define $a_n = 2^{(n+2)!}$ for $n \geq 0$ and

$$s_0 = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \qquad s_1 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \qquad s_2 = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix},$$

$$s_{2n-1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad s_{2n} = \begin{pmatrix} a_n & 0 \\ 0 & a_n \end{pmatrix}$$

for $n \geq 2$. To show that (s_n) is positive definite it suffices to show that the determinant

$$D_n(\xi) = \begin{vmatrix} \langle s_0\xi, \xi\rangle & \ldots & \langle s_n\xi, \xi\rangle \\ \vdots & & \vdots \\ \langle s_n\xi, \xi\rangle & \ldots & \langle s_{2n}\xi, \xi\rangle \end{vmatrix} \tag{1}$$

is positive for each $n \geq 0$ and each ξ in the unit sphere of $\mathbf{C}^2$. So suppose $\xi = (\xi_1, \xi_2)$ with $|\xi_1|^2 + |\xi_2|^2 = 1$. Then

$$D_0(\xi) = 4|\xi_1|^2 + |\xi_2|^2 \geq 1,$$

$$\begin{aligned} D_1(\xi) &= (4|\xi_1|^2 + |\xi_2|^2)(|\xi_1|^2 + 4|\xi_2|^2) - (4\operatorname{Re}(\overline{\xi_1}\xi_2))^2 \\ &\geq 4|\xi_1|^4 + 4|\xi_2|^4 \geq 2(|\xi_1|^2 + |\xi_2|^2)^2 \geq 1. \end{aligned}$$

Now assume $n \geq 2$ and $D_{n-1}(\xi) \geq 1$. Expand $D_n(\xi)$ completely, then collect the $n!$ terms containing $\langle s_{2n}\xi, \xi\rangle$ into one. For the remaining $(n+1)! - n!$ terms use

$$|\langle s_k\xi, \xi\rangle| \leq \| s_k \| \leq \max\{4, a_2, \ldots, a_{n-1}\} \leq a_{n-1}$$

($k = 0, \ldots, 2n-1$) to get a bound. The result is

$$\begin{aligned} D_n(\xi) &\geq \langle s_{2n}\xi, \xi\rangle D_{n-1}(\xi) - ((n+1)! - n!)a_{n-1}^{n+1} \\ &\geq a_n - (n+1)!a_{n-1}^{n+1} = 2^{(n+1)(n+1)!}(2^{(n+1)!} - (n+1)!) \geq 1. \end{aligned}$$

So (s_n) is positive definite. To see that (s_n) is not of positive type, take $\xi_0 = (0, 1)$ and $\xi_1 = (-1, 0)$ and verify $\sum_{j,k=0}^{1}\langle s_{j+k}\xi_k, \xi_j\rangle = -2$.

A sequence (s_n) in $\mathbf{B}(\mathscr{H})$ is a(n operator) *moment sequence* if there is a measure μ, defined on the Borel field $\mathscr{B}(\mathbf{R})$ and taking values in $\mathbf{B}(\mathscr{H})_+$, such that

$$\langle s_n\xi, \eta\rangle = \int_{\mathbf{R}} x^n \, d\langle \mu\xi, \eta\rangle(x) \tag{2}$$

for all $\xi, \eta \in \mathscr{H}$ and $n \geq 0$. Here $\langle \mu\xi, \eta\rangle$ is the complex measure defined by $\langle \mu\xi, \eta\rangle(E) = \langle \mu(E)\xi, \eta\rangle$, $E \in \mathscr{B}(\mathbf{R})$. (By definition, $\mu : \mathscr{B}(\mathbf{R}) \to \mathbf{B}(\mathscr{H})_+$ is a measure if and only if $\langle \mu\xi, \xi\rangle$ is a measure for each $\xi \in \mathscr{H}$.) By Hamburger's Theorem ([4, 6.2.2] or [1]), a sequence $(s_n)_{n\geq 0}$ in $\mathbf{R}$ is a moment sequence if and only if it is positive definite. General references for this classical moment problem are [1, 10]; see [4] for the generalization to semigroups other than the integers. For an arbitrary Hilbert space $\mathscr{H}$ moment sequences in $\mathbf{B}(\mathscr{H})$ are precisely the sequences of positive type; see, for example, [9].

Suppose (s_n) is a positive definite sequence in $\mathbf{B}(\mathscr{H})$. For each $\xi \in \mathscr{H}$ the sequence $(\langle s_n\xi, \xi\rangle)$ is positive definite, so one may choose a measure λ_ξ on $\mathbf{R}$ such that

$$\langle s_n\xi, \xi\rangle = \int_{\mathbf{R}} x^n \, d\lambda_\xi(x) \tag{3}$$

for all $n \geq 0$. If a $\mathbf{B}(\mathscr{H})_+$-valued measure μ exists which satisfies

$$\langle \mu\xi, \xi\rangle = \lambda_\xi, \qquad \xi \in \mathscr{H}, \tag{4}$$

then μ satisfies equation (2) (by polarization), so (s_n) is a moment sequence. The existence of such a measure μ, however, cannot be taken for granted. Indeed, if (4) holds then

$$\begin{aligned}\lambda_{\xi+\eta} + \lambda_{\xi-\eta} &= \langle \mu(\xi+\eta), \xi+\eta\rangle + \langle \mu(\xi-\eta), \xi-\eta\rangle \\ &= 2(\langle \mu\xi, \xi\rangle + \langle \mu\eta, \eta\rangle) = 2(\lambda_\xi + \lambda_\eta)\end{aligned}$$

and

$$\lambda_{c\xi} = \langle \mu(c\xi), c\xi\rangle = |c|^2\langle \mu\xi, \xi\rangle = |c|^2\lambda_\xi$$

for all $\xi, \eta \in \mathscr{H}$ and $c \in \mathbf{C}$. In short, the family $(\lambda_\xi)_{\xi\in\mathscr{H}}$ has to satisfy the parallelogram law

$$\lambda_{\xi+\eta} + \lambda_{\xi-\eta} = 2(\lambda_\xi + \lambda_\eta) \tag{5}$$

($\xi, \eta \in \mathscr{H}$) and the rule of homogeneity

$$\lambda_{c\xi} = |c|^2\lambda_\xi \tag{6}$$

($\xi \in \mathscr{H}$ and $c \in \mathbf{C}$). *These conditions are also sufficient.* To see this, suppose they hold. For each Borel set E in $\mathbf{R}$ the family $(\lambda_\xi(E))_{\xi\in\mathscr{H}}$ satisfies the parallelogram law and the rule of homogeneity (same as above, only for scalars instead of measures), and it is a matter of algebra that there is a unique sesquilinear form f_E on $\mathscr{H}$ such that $\lambda_\xi(E) = f_E(\xi, \xi)$ for all $\xi \in \mathscr{H}$. Now

f_E is positive since the λ_ξ are positive measures and is bounded since (by (3))

$$f_E(\xi, \xi) = \lambda_\xi(E) \leq \lambda_\xi(\mathbf{R}) = \langle s_0\xi, \xi\rangle \leq \|s\| \, \|\xi\|^2 .$$

Hence there is a unique $\mu(E) \in \mathbf{B}(\mathscr{H})_+$ such that $f_E(\xi, \eta) = \langle \mu(E)\xi, \eta\rangle$ for all $\xi, \eta \in \mathscr{H}$. In particular $\lambda_\xi(E) = \langle \mu(E)\xi, \xi\rangle$ for all $\xi \in \mathscr{H}$. This being so for all $E \in \mathscr{B}(\mathbf{R})$, it follows that μ is a measure and that (4) holds.

A moment sequence (s_n) in $\mathbf{B}(\mathscr{H})$ is *determinate* if only one measure μ satisfies (2).

Theorem 2. *Let (s_n) be a positive definite sequence in $\mathbf{B}(\mathscr{H})$, and assume that for all $\xi, \eta \in \mathscr{H}$ the scalar moment sequence $(\langle s_n\xi, \xi\rangle + \langle s_n\eta, \eta\rangle)_{n\geq 0}$ is determinate. Then (s_n) is a determinate moment sequence.*

Proof. For each $\xi \in \mathscr{H}$ the moment sequence $(\langle s_n\xi, \xi\rangle)$ is determinate (take $\eta = 0$ in the statement of the theorem); let λ_ξ be the unique measure for which (3) holds. By the uniqueness of λ_ξ, any measure μ satisfying (2) must satisfy (4). By polarization it follows that (s_n), if a moment sequence, is determinate.

For $\xi, \eta \in \mathscr{H}$ we have

$$\begin{aligned}\int x^n \, d\lambda_{\xi+\eta}(x) + \int x^n \, d\lambda_{\xi-\eta}(x) &= \langle s_n(\xi+\eta), \xi+\eta\rangle + \langle s_n(\xi-\eta), \xi-\eta\rangle \\ &= 2\langle s_n\xi, \xi\rangle + 2\langle s_n\eta, \eta\rangle = 2\int x^n \, d\lambda_\xi(x) + 2\int x^n \, d\lambda_\eta(x)\end{aligned}$$

$(n \geq 0)$. By the determinacy hypothesis it follows that (5) holds; (6) is gotten by a similar argument. This proves the existence of a $\mathbf{B}(\mathscr{H})_+$-valued measure μ satisfying (4) and hence (by (3) and polarization) (2), thus completing the proof of the theorem. □

Recall the following two criteria of Carleman for determinacy in the classical moment problem: First, suppose (s_n) is a sequence of reals such that the determinants

$$D_n = \begin{vmatrix} s_0 & \cdots & s_n \\ \vdots & & \vdots \\ s_n & \cdots & s_{2n} \end{vmatrix}$$

$(n \geq 0)$ are positive. (In particular, (s_n) is positive definite, hence a moment sequence). If

$$\sum_{n=2}^{\infty} \frac{D_{n-1}}{\sqrt{D_{n-2}D_n}} = \infty ,$$

then (s_n) is determinate.

Second, every scalar moment sequence (s_n) satisfying

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{s_{2n}}} = \infty$$

is determinate (see [1]).

Corollary 1. *Let* (s_n) *be a positive definite sequence in* $\mathbf{B}(\mathscr{H})$, *and suppose*

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{\| s_{2n} \|}} = \infty.$$

Then (s_n) *is a determinate moment sequence.*

Proof. Given $\xi, \eta \in \mathscr{H}$, apply the second Carleman criterion to the sequence $(\langle s_n\xi, \xi\rangle + \langle s_n\eta, \eta\rangle)$. □

Though conceivably $(\langle s_n\xi, \xi\rangle + \langle s_n\eta, \eta\rangle)$ might be replaced by just $(\langle s_n\xi, \xi\rangle)$ in Theorem 2, a proof of this can hardly be simple. Certainly the naïve approach is doomed to failure; the sum of two determinate moment sequences need not be determinate. (In fact, *every* (scalar) moment sequence is the sum of two determinate ones. To see this, consider any moment sequence (s_n). If (s_n) is determinate, it is the sum of the two determinate moment sequences (0) and (s_n). Otherwise, choose a Nevanlinna extremal measure μ of which (s_n) is the moment sequence (see [1]). Such a measure has the form $\mu = \sum_{n=1}^{\infty} a_n\varepsilon_{x_n}$ where the x_n are distinct reals, $a_n > 0$, and ε_{x_n} denotes the Dirac measure at x_n. Moreover, $\mu - a_n\varepsilon_{x_n}$ is determinate for each n [3]. So (s_n) is the sum of the moment sequences of $a_1\varepsilon_{x_1}$ and $\mu - a_1\varepsilon_{x_1}$; both of which are determinate.) Determinacy of a great many of the scalar moment sequences $(\langle s_n\xi, \xi\rangle)$ is *not* sufficient.

Theorem 3. *There is a positive definite sequence* $(s_n)_{n\geq 0}$ *in* $\mathbf{B}(\mathbf{C}^2)$ *such that the scalar moment sequence* $(\langle s_n\xi, \xi\rangle)$ *is determinate for each* ξ *in a dense* G_δ *in* $\mathbf{C}^2$; *yet* (s_n) *is not a moment sequence.*

Proof. Let Σ be the unit sphere in $\mathbf{C}^2$, Φ a countable dense set in Σ. Choose a sequence $(\varphi_n)_{n\geq 2}$ in Φ in which each element of Φ appears infinitely often.

We shall construct the sequence (s_n) by induction, keeping the determinants (1) positive for all n and all $\xi \in \Sigma$ (hence for all $\xi \in \mathbf{C}^2 \setminus \{0\}$), thus making sure that (s_n) will be positive definite. As in the proof of Theorem 1, beginning the construction with

$$s_0 = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \qquad s_1 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \qquad s_2 = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$

ensures that the full sequence will not be of positive type (i.e., not a moment sequence).

Suppose that $n \geq 2$ and that $s_0, \dots, s_{2n-2}$ have been chosen properly. Put $s_{2n-1} = 0$ and define a function F_n on $\mathbf{C}^2$ by the condition that

$$\begin{vmatrix} \langle s_0\xi, \xi\rangle & \dots & \langle s_{n-1}\xi, \xi\rangle & \langle s_n\xi, \xi\rangle \\ \vdots & & \vdots & \vdots \\ \langle s_{n-1}\xi, \xi\rangle & \dots & \langle s_{2n-2}\xi, \xi\rangle & \langle s_{2n-1}\xi, \xi\rangle \\ \langle s_n\xi, \xi\rangle & \dots & \langle s_{2n-1}\xi, \xi\rangle & t \end{vmatrix} = D_{n-1}(\xi)t + F_n(\xi) \tag{7}$$

for all $\xi \in \mathbf{C}^2$ and $t \in \mathbf{R}$. Let $p : \mathbf{C}^2 \to \mathbf{C}\varphi_n$ be the orthogonal projection. Since $D_{n-1}(\varphi_n) > 0$ by assumption, we may choose $a \in \mathbf{R}$ such that

$$0 < D_{n-1}(\varphi_n)a + F_n(\varphi_n) < \frac{D_{n-1}(\varphi_n)^2}{D_{n-2}(\varphi_n)}.$$

Note that $a > 0$ since the matrix in (7) becomes strictly positive definite upon substituting a for t and φ_n for ξ. The obvious homogeneity properties of D_{n-1} and F_n imply that

$$D_{n-1}(\xi)a + F_n(\xi) = D_{n-1}(\varphi_n)a + F_n(\varphi_n)$$

for each $\xi \in \mathbf{T}\varphi_n$ where $\mathbf{T} = \{c \in \mathbf{C} \mid |c| = 1\}$. Since D_{n-1} and F_n are continuous, it follows that

$$D_{n-1}(\xi) \parallel p\xi \parallel^2 a + F_n(\xi) > 0$$

for all ξ in some neighbourhood of $\mathbf{T}\varphi_n$. On the remainder of Σ, the function $\xi \to \|(1-p)\xi\|$ is bounded below away from zero. Hence, if $b > 0$ is chosen large enough, then

$$D_{n-1}(\xi)(a \parallel p\xi \parallel^2 + b \parallel (1-p)\xi \parallel^2) + F_n(\xi) > 0$$

for all $\xi \in \Sigma$. Defining $s_{2n} = ap + b(1-p)$ we have

$$\begin{aligned} D_n(\xi) &= D_{n-1}(\xi)\langle s_{2n}\xi, \xi\rangle + F_n(\xi) \\ &= D_{n-1}(\xi)(a \parallel p\xi \parallel^2 + b \parallel (1-p)\xi \parallel^2) + F_n(\xi) > 0 \end{aligned}$$

for all $\xi \in \Sigma$. Note that

$$\begin{aligned} D_n(\varphi_n) &= D_{n-1}(\varphi_n)\langle s_{2n}\varphi_n, \varphi_n\rangle + F_n(\varphi_n) \\ &= D_{n-1}(\varphi_n)a + F_n(\varphi_n) < \frac{D_{n-1}(\varphi_n)^2}{D_{n-2}(\varphi_n)}. \end{aligned} \tag{8}$$

For each $n \geq 2$ the set

$$G_n = \{\xi \in \mathbf{C}^2 \setminus \{0\} \mid D_{n-2}(\xi)D_n(\xi) < D_{n-1}(\xi)^2\}$$

is open, contains φ_n according to (8), and is stable under multiplication by nonzero scalars (by the homogeneity properties of the functions D_k). The G_δ-set

$$A = \bigcap_{n=2}^{\infty} \bigcup_{m=n}^{\infty} G_m$$

contains Φ and is stable under multiplication by nonzero scalars; so A is dense in $\mathbf{C}^2$. If $\xi \in A$, then $D_{n-2}(\xi)D_n(\xi) < D_{n-1}(\xi)^2$ for infinitely many n, hence

$$\sum_{n=2}^{\infty} \frac{D_{n-1}(\xi)}{\sqrt{D_{n-2}(\xi)D_n(\xi)}} = \infty.$$

By the Carleman criterion it follows that the moment sequence $(\langle s_n\xi, \xi\rangle)_{n\geq 0}$ is determinate. □

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