A NORM-PRESERVING $H^\infty$ EXTENSION PROBLEM

ZHIMIN YAN

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Abstract. First we establish a Schwarz lemma for holomorphic mappings between bounded symmetric domains. Then, as an application, we solve a norm-preserving extension problem.

Introduction

The classical Schwarz lemma asserts that if $f$ is a holomorphic function from the unit disk $U$ into $U$, then $|f'(0)| \leq 1$ and the equality holds if and only if $f(z) = cz$ with $|c| = 1$. A generalization to the high-dimensional case is given in [R]. Namely, if $F$ is a holomorphic mapping from the unit ball in $\mathbb{C}^n$ into the unit ball of $\mathbb{C}^m$, then the complex Jacobian $F'(0)$ is a linear operator of norm at most one, and if $F'(0)$ is an isometry of $\mathbb{C}^n$ into $\mathbb{C}^m$, then $F(z) = F'(0)z$. (The norm of the linear operator $F'(0)$ equal to one is insufficient for $F$ to be a linear mapping.) See [R, Chapter 8]. In this paper, we shall generalize this result to the case when $F$ is a holomorphic mapping from one bounded symmetric domain to another one. As an application, we also consider the following norm-preserving $H^\infty$ extension problem; that is, for two bounded symmetric domains $D_1$ and $D_2$, determine which holomorphic mappings $\Phi : D_1 \to D_2$ have the property that to every $f \in H^\infty(D_1)$ corresponds a $g \in H^\infty(D_2)$ such that

1. Some background about bounded symmetric domains

A finite-dimensional complex vector space $V$ is a Hermitian Jordan Triple System if $V$ is endowed with a real trilinear map $\{\cdot,\cdot,\cdot\} : V \times V \times V \to V$,
Given \( u, v \in V \), define the operator \( D(u, v^*) \) on \( V \) by
\[
D(u, v^*)z = \{uv^*z\}.
\]
A Hermitian Jordan Triple System is positive definite if the trace \( \tau(u, v) \) of the \( C \)-linear transformation \( D(u, v^*) \) is a positive definite Hermitian form on \( V \). A positive definite Hermitian Jordan Triple System will be abbreviated as PDHJTS. In the following, \( V \) will always denote a PDHJTS.

It is known that the Jordan unit ball
\[
D = \{ z \in V \mid \| D(z, z^*) \| < 1 \}
\]
is a bounded symmetric domain in \( V \) where \( \| \| \) is the operator norm. Conversely, any bounded symmetric domain can be realized as a Jordan unit ball in some PDHJTS \( V \) (see [L]).

An element \( e \) in \( V \) is a tripotent if \( \{ee^*e\} = e \). Two tripotents \( e, c \) are orthogonal if \( D(e, c^*) = 0 \). A tripotent is primitive if \( e \) is not a sum of two orthogonal tripotents. A frame of \( V \) is a maximal orthogonal system of primitive tripotents. Two frames of \( V \) always have the same number of elements, and the number is the rank of \( V \) which is also equal to the rank of the bounded symmetric domain \( D \). Throughout this paper, \( r(D) \) will denote the rank of a bounded symmetric domain \( D \).

For the Jordan unit ball \( D \) in \( V \), let \( G = \text{Aut}_0(D) \) be the identity component of the automorphism group of \( D \) and \( K \) the isotropy subgroup of \( G \) at 0. Suppose that \( \{e_1, \ldots, e_r\} \) is a frame. Then every element \( z \in V \) can be written as
\[
z = k \cdot \sum_{i=1}^{r} \lambda_i e_i,
\]
for some \( k \in K \) with \( \lambda_1 \geq \cdots \geq \lambda_r \geq 0 \). Furthermore, for any \( z \in V \), there exists a frame \( \{c_1, \ldots, c_r\} \) such that
\[
z = \sum_{i=1}^{r} \lambda_i c_i.
\]
Then \( z \in D \) if and only if \( |\lambda_i| < 1, \ i = 1, \ldots, r \).

For a tripotent \( e \), let \( V_\alpha(e) \) be the \( \alpha \)-eigenspace of the operator \( D(e, e^*) \)
\[
\{ v \in V \mid (D(e, e^*)v = \alpha v) \}.
\]
The only possible nonzero eigenspaces are \( V_1(e) \), \( V_{1/2}(e) \), and \( V_0(e) \). Then \( V_0(e) \) and \( V_1(e) \) are positive definite Hermitian Jordan subtriple of \( V \). A tripotent \( e \) is of rank \( j \) if \( V_1(e) \) is of rank \( j \).

Suppose that \( \{e_1, \ldots, e_r\} \) is a frame. Then \( V \) has the following Peirce decomposition:
\[
V = \sum_{0 \leq i \leq j \leq r} V_{i<j}
\]
where \( V_{ii} = V_1(e_i) \ (i = 1, \ldots, r) \); \( V_{ij} = V_{1/2}(e_i) \cap V_{1/2}(e_j) \ (1 \leq i < j \leq r) \);
\( V_{0i} = V_{1/2}(e_i) \cap \bigcap_{j \neq i} V_0(e_j) \ (i = 1, \ldots, r) \); and \( V_{00} = 0 \).
For $z, w \in V$, the Bergman operators $B(z, w)$ on $V$ are defined by

$$B(z, w) = I - 2D(z, w^*) + Q_z Q_w$$

where $I$ is the identity map on $V$ and $Q_z u = \{zu^* z\}$ for all $z, u \in V$.

Fix a frame $\{e_1, \ldots, e_r\}$ in the following.

If $z = \sum_{i=1}^r \lambda_i e_i$, $\lambda_i \in \mathbb{C}$, and $\lambda_0 = 0$, then, for $y \in V_{ij}$ ($0 \leq i \leq j \leq r$),

$$B(z, z)y = (1 - |\lambda_j|^2)(1 - |\lambda_i|^2)y.$$  

Let $\langle \cdot, \cdot \rangle$ be the Hermitian inner product on $V$ which is a scalar multiple of the trace $\tau$ such that $\langle e, e \rangle = 1$ for a primitive tripotent $e$. Throughout this paper, $\| \|$ will denote the Euclidean norm $\| \|_2$ on $V$. Suppose the rank of $V$ is $r$. Then the Jordan unit ball $D$ of $V$ is contained in the Euclidean ball $B_{\sqrt{r}} = \{z \in V \mid |z| < \sqrt{r} \}$, and the Shilov boundary $S$ of $D$ is the intersection of $\partial B_{\sqrt{r}}$ and $D$. In particular, this implies that $\eta \in \overline{D}$ is on $S$ if and only if $|\eta| = \sqrt{r}$.

A point $z$ is on the topological boundary $\partial D$ of $D$ if and only if there exists a frame $\{c_1, \ldots, c_r\}$ such that $z = \sum_{i=1}^r \lambda_i c_i$ with $\lambda_1 = \cdots = \lambda_r = 1 > \lambda_{s+1} \geq \cdots \geq 0$. In particular, $z$ is on $S$ if and only if $z$ is a tripotent of rank $r$, and $z$ is on $\partial D$ with $|z| = 1$ if and only if $z$ is a primitive tripotent.

For details about this section, see [L].

### 2. A Schwarz Lemma

In this section, we want to establish a Schwarz lemma for holomorphic mappings from one bounded symmetric domain into another one. First, we need two known results.

An open set $E$ is star-shaped circular if $az \in E$ whenever $z \in E$ and $a \in \mathbb{C}$, $|a| \leq 1$.

**Lemma 1** ([R, Theorem 8.1.2]). Suppose that

(i) $D_1$ and $D_2$ are two star-shaped circular regions in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively,

(ii) $D_2$ is convex and bounded,

(iii) $F : D_1 \rightarrow D_2$ is holomorphic.

Then

(a) $F'(0)$ maps $D_1$ into $D_2$, and

(b) $F(rD_1) \subset rD_2$ $(0 < r \leq 1)$ if $F(0) = 0$.

For $a \in D$, let $\varphi_a$ be the holomorphic automorphism of $D$ which interchanges 0 and $a$. Then one has (see [AY])

**Lemma 2.** If $D$ is the Jordan unit ball of $V$ and $a \in D$, then the complex Jacobian of $\varphi_a$ is given by

$$\varphi'_a(z) = -B(a, a)^{1/2}B(z, a)^{-1} \forall z \in D.$$  

**Remark.** In the proof of Theorem 3, we only need the fact that $\varphi'_a(a)$ can be written as a composition of a unitary operator and $B(a, a)^{-1/2}$. This fact follows easily from [L, Lemma 2.11].

From now on, whenever we say that $D_1$ and $D_2$ are two bounded symmetric domains, we always assume that they are realized as two Jordan unit balls of PDHJS's $V_1$ and $V_2$ respectively.

Now we have the following Schwarz lemma.
Theorem 3. Suppose that $D_1$ and $D_2$ are two bounded symmetric domains in $V_1$ and $V_2$ respectively, with $r(D_1) \geq r(D_2)$; $F : D_1 \to D_2$ is a holomorphic mapping; and $F'(0)$ is an isometry of $V_1$ into $V_2$. Then

(i) $r(D_1) = r(D_2)$;
(ii) $\|F(z)\| \leq \|z\|$, $\forall z \in D_1$;
(iii) $F(z) = F'(0)z$, $\forall z \in D_1$.

Proof. Let $r_1$ be the rank of $D_i$ and $S_i$ the Shilov boundary of $D_i$ ($i = 1, 2$).

First, we show that $r_1 = r_2$ and $F(0) = 0$. Then (ii) follows from Lemma 1(b).

Let $a = F(0)$ and $G(z) = \varphi_a \circ F : D_1 \to D_2$ where $\varphi_a \in \text{Aut}(D_2)$, which interchanges 0 and $a$. Then $G(0) = 0$.

By Lemma 1(a), $G'(0)z \in D_2 \\forall z \in D_1$. This yields that

\begin{equation}
|\varphi_a'(a)F'(0)z| = |G'(0)z| \leq \sqrt{r_2}.
\end{equation}

On the other hand, by (6) we have that $\varphi_a'(a) = -B(a, a)^{-1/2}$ which, together with (5), implies

\begin{equation}
|\varphi_a'(a)F'(0)\eta| \geq |F'(0)\eta|.
\end{equation}

Now since $F'(0)$ is an isometry, for $\eta \in S_1$,

\begin{equation}
|F'(0)\eta| = \sqrt{r_1}.
\end{equation}

(7)-(9) give that $r_1 = r_2$ and

\begin{equation}
|\varphi_a'(a)G'(0)\eta| = |F'(0)\eta| = \sqrt{r_1}.
\end{equation}

Let $\{c_1, \ldots, c_r\}$ be a frame of $V_2$ such that $a = \sum_{i=1}^r \lambda_ic_i$ with $\lambda_1 \geq \cdots \geq \lambda_j > 0 = \lambda_{j+1} = \cdots = \lambda_r$. From (5) we conclude that, for $w \in V_2$, $|B(a, a)^{-1/2}w| = |w|$ can happen if and only if $w \in V_0(c_1 + \cdots + c_j)$. Now it follows from (10) that $F'(0)\eta \in V_0(c_1 + \cdots + c_j) \cap D_2$. Hence $F'(0)\eta$ is in the closure of the Jordan unit ball of the positive definite Hermitian Jordan subtriple system $V_0(c_1 + \cdots + c_j)$ of $V$ which is of rank $r_2 - j$. Consequently, if $j \geq 1$, $|F'(0)\eta| \leq \sqrt{r_2} - j < \sqrt{r_2} = \sqrt{r_1}$ which contradicts (9). This proves $a = 0$.

Second, we show that, for every $\eta \in S_1$ and every $\lambda \in U$, $F(\lambda \eta) = \lambda F'(0)\eta$, which will imply (iii) since $F$ is holomorphic.

Let $A = F'(0)$ and $r = r_1 = r_2$. Then $A\eta \in S_2$, for $\eta \in S_1$, since $|A\eta| = |\eta| = \sqrt{r}$. Now, for any $w \in D_2$, $\alpha \in S_2$, since $|w| < \sqrt{r}$, we have

\[|\langle w, \alpha \rangle| < r.\]

For $\eta \in S_1$, we define an analytic function $h$ from the unit disk $U$ in $C$ into $U$ by

\[h(\lambda) = \frac{1}{r}(F(\lambda \eta), A\eta).\]

Then $h(0) = 0$, and $h'(0) = \frac{1}{r}(F'(0)\eta, A\eta) = 1$. This leads, by the classical Schwarz lemma, to $h(\lambda) = \lambda$. Thus we obtain

\begin{equation}
\lambda = \langle \frac{1}{r}F(\lambda \eta), A\eta \rangle.
\end{equation}

(ii) shows

\[|F(\lambda \eta)| \leq |\lambda||\eta| \leq \sqrt{|\lambda|}.\]
Therefore, we have that \( \frac{1}{\lambda} F(\lambda \eta) \in D_2 \subset B_{\sqrt{r}} \) for \( \lambda \neq 0 \). Since \( A \eta \in S_2 \subset \partial B_{\sqrt{r}} \), one can easily see that (11) can hold if and only if \( \frac{1}{\lambda} F(\lambda \eta) = A \eta \).

Therefore, we have obtained that

\[
F(\lambda \eta) = \lambda A \eta = \lambda F'(0) \eta \quad \forall \eta \in S_1, \lambda \in U,
\]

proving the theorem.

**Remark.** We can easily construct a holomorphic mapping \( F \) from the unit ball of \( C^2 \) into an irreducible bounded symmetric domain of rank two in \( C^4 \) such that \( F'(0) \) is an isometry of \( C^2 \) into \( C^4 \), but \( F \) fails to be a linear mapping.

For another version of the Schwarz lemma for a bounded symmetric domain, see [K].

### 3. AN EXTENSION PROBLEM

Let \( D_1 \) and \( D_2 \) be two bounded symmetric domains. A holomorphic mapping \( \Phi : D_1 \to D_2 \) is said to have the norm-preserving \( H^\infty \) extension property if, for every \( f \in H^\infty(D_1) \), there exists \( g \in H^\infty(D_2) \)

(*)

such that (a) \( g \circ \Phi = f \) and (b) \( ||g||_\infty = ||f||_\infty \).

**Remark.** To study such \( \Phi \)'s, it is enough to study those \( \Phi \)'s with \( \Phi(0) = 0 \), since it is obvious that if \( \Phi \) has the norm-preserving \( H^\infty \) extension property, so is \( \varphi \circ \Phi \) where \( \varphi \in \text{Aut}(D_2) \).

Now we state our main result.

**Theorem 4.** Suppose that \( D_1 \) and \( D_2 \) are two bounded symmetric domains in \( V_1 \) and \( V_2 \) respectively. For a holomorphic mapping \( \Phi : D_1 \to D_2 \) with \( r(D_1) \geq r(D_2) \), the following are equivalent:

(i) \( \Phi \) has the property (*)

(ii) \( \Phi = \varphi \circ \mathcal{L} \) where \( \varphi \) is an automorphism of \( D_2 \) and \( \mathcal{L} : D_1 \to D_2 \) is a linear isometry of \( V_1 \) into \( V_2 \) mapping all primitive tripotents in \( V_1 \) into primitive tripotents in \( V_2 \);

(iii) there is a multiplicative linear operator \( E : H^\infty(D_1) \to H^\infty(D_2) \) such that \( (Ef) \circ \Phi = f, \forall f \in H^\infty(D_1) \).

**Proof.** We divide the proof into several steps. By the remark before the theorem, it can be assumed that \( \Phi(0) = 0 \). In this case \( \Phi \) becomes linear.

**Step 1.** We show (iii) implies (i). It remains to show \( ||Ef||_\infty = ||f||_\infty \). The identity \( (Ef) \circ \Phi = f \) implies \( f \equiv 0 \) if \( Ef \equiv 0 \). Now \( Ef = E(f \cdot 1) = (Ef) \cdot (E1) \) gives that \( E1 = 1 \). Suppose that \( ||Ef||_\infty > ||f||_\infty \). Then there exists a sequence \( z_n \) in \( D_2 \) such that \( Ef(z_n) \to c \) with \( |c| = ||Ef||_\infty \). Since

\[
(f - c)^{-1} \in H^\infty(D_1), \quad E \left( \frac{1}{f - c} \right) \in H^\infty(D_2)
\]

and we also have

\[
1 = E \left( \frac{1}{f - c} \cdot (f - c) \right) = E \left( \frac{1}{f - c} \right) E(f - c).
\]

However, this implies

\[
|E \left( \frac{1}{f - c} \right) (z_n)| \to \infty
\]

since \( E(f - c)(z_n) \to 0 \); we get a contradiction.
Step 2. We show (ii) implies (iii). Let \( P \) be the orthogonal projection of \( V_2 \) onto \( \Phi(V_1) \). We claim that
\[
P(D_2) \subset \Phi(D_1).
\]
In fact, for \( w \in D_2, P(w) \) can be written as
\[
P(w) = \sum_{i=1}^{r} \lambda_i \Phi(c_i)
\]
for some frame of \( V_1 \) with \( \lambda_1 \geq \cdots \geq \lambda_r \geq 0 \). Since \( \Phi(c_1) \) is a primitive tripotent in \( V_2 \), by Lemma 5, \( \lambda_1 = \langle w, \Phi(c_1) \rangle < 1 \), which shows that \( P(w) \in \Phi(D_1) \).

Now define
\[
(Ef)(w) = f(A^{-1} P(w)), \quad w \in D_2,
\]
for all \( F \in H^\infty(D_1) \) where \( A = \Phi'(0) \). It is clear that \( E \) satisfies (iii).

Step 3. We show that (i) implies (ii) by establishing some lemmas in which we assume that \( \Phi \) satisfies (i).

Lemma 5. Let \( e \) be any primitive tripotent of \( V \) and \( z \in D \). Then \( |\langle z, e \rangle| < 1 \).

Proof. First, \( |\langle z, e \rangle|^2 \leq \langle z, \{ ee^* z \} \rangle = \langle z, \{ ze^* e \} \rangle \).

Next, by [L], Lemma 2.6 (4),
\[
\langle z, \{ ze^* e \} \rangle = \langle \{ ez^* z \}, e \rangle = \langle \{ zz^* e \}, e \rangle,
\]
which is not greater than \( ||D(z, z)|| < 1 \).

Lemma 6. Let \( \zeta \) be a primitive tripotent in \( V_1 \). Then \( |\Phi'(0)\zeta| \geq 1 \).

Proof. Define a holomorphic function \( f(z) \) on \( D_1 \) by \( f(z) = \langle z, \zeta \rangle \). Then Lemma 5 yields \( ||f||_\infty = 1 \). Since \( \Phi \) satisfies (i), there exists a \( g \in H^\infty(D_2) \) such that \( g(\Phi(z)) = \langle z, \zeta \rangle \) with \( ||g||_\infty = 1 \). Then Lemma 1(a) implies that
\[
|g'(0)| \leq 1.
\]
Since \( \langle \zeta, \zeta \rangle = 1 \), for all \( \lambda \in U \), \( g(\Phi(\lambda \zeta)) = \lambda \langle \zeta, \zeta \rangle = \lambda \) which leads to \( g'(0) \Phi'(0) \zeta = 1 \). Now combining inequality (12) and \( 1 = |g'(0)\Phi'(0)\zeta| = |g'(0)||\Phi'(0)\zeta| \) gives
\[
|\Phi'(0)\zeta| \geq 1
\]
which is not greater than \( ||D(z, z)|| < 1 \).

Lemma 7. Let \( \zeta \) be a tripotent of rank \( r_1 = r(D_1) \). Then \( \Phi'(0) \leq \sqrt{r_1} \).

Proof. For \( \lambda \in U \), define \( G(\lambda) = \Phi(\lambda \zeta) \). Then \( G(\lambda) \in D_2 \subset B_{\sqrt{r_2}} \), and \( G(0) = 0 \). It follows from Lemma 1(b) that \( |\lambda| |\Phi'(0)| = |G'(0)\lambda| \leq |\lambda| \sqrt{r_2} \leq |\lambda| \sqrt{r_1} \), proving the lemma.

Let \( \{c_1, \ldots, c_r\} \) be a frame of \( V_1 \). Define
\[
h_{kl} = \langle \Phi'(0)c_k, \Phi'(0)c_l \rangle, \quad k, l = 1, \ldots, r.
\]

Lemma 8. For any frame \( \{c_1, \ldots, c_r\} \) of \( V_1 \), \( h_{kl} = \delta_{kl} \), \( k, l = 1, \ldots, r \). Consequently, \( \Phi \) is an isometry of \( V_1 \) into \( V_2 \).

Proof. We use induction on \( r = r(D_1) \).

(i) If \( r = 1 \), the lemma follows from Lemmas 6 and 7.

(ii) Assume the lemma holds for \( r - 1 \).
For \( s = 1, \ldots, r \), we define a function \( H_s \) of \( \theta_1, \ldots, \theta_s \in \mathbb{R} \)
\[
H_s(\theta_1, \ldots, \theta_s) = \sum_{k \neq l}^s e^{i\theta_k} \cdot h_{kl} \cdot e^{-i\theta_l}.
\]

For any \( \theta_1, \ldots, \theta_r \in \mathbb{R} \), \( \sum_{k=1}^r h_{kk} + H_r(\theta_1, \ldots, \theta_r) \leq r \). Lemma 7 gives
\[
\sum_{k=1}^r h_{kk} + H_r(\theta_1, \ldots, \theta_r) \leq r. \tag{13}
\]

On the other hand, Lemma 6 yields \( \sum_{k=1}^r h_{kk} = r \). Thus we have
\[
H_r(\theta_1, \ldots, \theta_r) = H_{r-1}(\theta_1, \ldots, \theta_{r-1}) \tag{14}
\]
\[
+ \sum_{k=1}^r e^{i\theta_k} \cdot h_{kr} \cdot e^{-i\theta_r} + \sum_{k=1}^r e^{i\theta_r} \cdot h_{rl} \cdot e^{-i\theta_l} \leq 0.
\]

Then the induction assumption implies \( H_{r-1}(\theta_1, \ldots, \theta_{r-1}) = 0 \), which, together with (14), leads to
\[
\sum_{k=1}^r e^{i\theta_k} \cdot h_{kr} \cdot e^{-i\theta_r} + \sum_{l=1}^r e^{i\theta_r} \cdot h_{rl} \cdot e^{-i\theta_l} \leq 0 \quad \forall \theta_1, \ldots, \theta_r \in \mathbb{R}. \tag{15}
\]

Now replacing \( \theta_r \) in the above by \( \theta_r + \pi \) reverses the inequality. Hence
\[
H_r(\theta_1, \ldots, \theta_r) = 0 \quad \forall \theta_1, \ldots, \theta_r \in \mathbb{R}. \tag{16}
\]

Now (13), (15), and Lemma 6 show that \( h_{kk} = 1, k = 1, \ldots, r \). Finally,
\[
h_{kl} = \frac{1}{2\pi} \int_0^{2\pi} H_r(\theta_1, \ldots, \theta_r) e^{i\theta_l} e^{-i\theta_k} d\theta_1 d\theta_2 = 0, \quad k \neq l,
\]
completing the proof.

**Lemma 9.** (a) \( \Phi(D_1) = D_2 \cap \Phi(V_1) \); (b) \( \Phi(\partial D_1) \subset \partial D_2 \).

**Proof.** (a) It is enough to show that \( D_2 \cap \Phi(V_1) \subset \Phi(D_1) \). Suppose that there exists a \( v \in D_2 \cap \Phi(V_1) \) but \( v \notin \Phi(D_1) \). We note that \( \Phi(D_1) \) is convex and open in \( \Phi(V_1) \) and \( \partial(\Phi(D_1)) = \Phi(\partial D_1) \). Therefore, we can find a \( \zeta \in \partial D_1 \) and a \( t > 1 \) with \( v = t\Phi(\zeta) \). Then \( \zeta \) can be written as \( \zeta = c_1 + \sum_{i=2} c_i \) where \( \{c_1, \ldots, c_r\} \) is a frame of \( V_1 \). Define a holomorphic function \( f(z) \) on \( D_1 \) by \( f(z) = \langle z, c_1 \rangle \). Lemma 5 asserts \( \|f\|_\infty = 1 \). To \( f(z) \) corresponds a \( g \in H^\infty(D_2) \) with \( \|g\|_\infty = 1 \) such that
\[
g(\Phi(z)) = \langle z, c_1 \rangle, \quad \forall z \in D_1.
\]
In particular,
\[
g(\Phi(\lambda \zeta)) = \lambda \langle \zeta, c_1 \rangle = \lambda, \quad \forall \lambda \in U.
\]
The linearity of \( \Phi \) implies that
\[
g(\lambda v) = g(\Phi(\lambda \zeta)) = \lambda, \quad \forall \lambda \in U. \tag{16}
\]
Since \( D_2 \) is star-shaped circular, \( \lambda v \in D_2 \cap \Phi(V_1) \), \( \forall \lambda \in U \). Actually, (16) holds for all \( \lambda \) with \( |\lambda| \leq t \) since \( g \) is holomorphic on \( D_2 \). This yields \( g(v) = t > 1 \), which contradicts \( \|g\|_\infty = 1 \).

(a) implies (b) since \( \Phi \) is one-to-one.
Lemma 10. If $\zeta$ is a primitive tripotent in $V_1$, then $\Phi(\zeta)$ is a primitive tripotent in $V_2$.

Proof. By Lemma 8, $\Phi$ is an isometry of $V_1$ into $V_2$. Hence, $|\Phi(\zeta)| = |\zeta| = 1$. By Lemma 9(b), $\Phi(\zeta) \in \partial D_2$. Now we see that $\Phi(\zeta)$ is a primitive tripotent.

Finally, Lemmas 8 and 10 show that (i) implies (ii).

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT BERKELEY, BERKELEY, CALIFORNIA 94720