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A NORM-PRESERVING H^∞ EXTENSION PROBLEM

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ABSTRACT. First we establish a Schwarz lemma for holomorphic mappings between bounded symmetric domains. Then, as an application, we solve a norm-preserving extension problem.

INTRODUCTION

The classical Schwarz lemma asserts that if f is a holomorphic function from the unit disk $\mathbf{U}$ into $\mathbf{U}$, then $|f'(0)| \leq 1$ and the equality holds if and only if $f(z) = cz$ with $|c| = 1$. A generalization to the high-dimensional case is given in [R]. Namely, if F is a holomorphic mapping from the unit ball in $\mathbf{C}^n$ into the unit ball of $\mathbf{C}^m$, then the complex Jacobian $F'(0)$ is a linear operator of norm at most one, and if $F'(0)$ is an isometry of $\mathbf{C}^n$ into $\mathbf{C}^m$, then $F(z) = F'(0)z$. (The norm of the linear operator $F'(0)$ equal to one is insufficient for F to be a linear mapping.) See [R, Chapter 8]. In this paper, we shall generalize this result to the case when F is a holomorphic mapping from one bounded symmetric domain to another one. As an application, we also consider the following norm-preserving H^∞ extension problem; that is, for two bounded symmetric domains D_1 and D_2, determine which holomorphic mappings $\Phi : D_1 \to D_2$ have the property that to every $f \in H^\infty(D_1)$ corresponds a $g \in H^\infty(D_2)$ such that

(a) $g \circ \Phi = f$ and
(b) $||g||_\infty = ||f||_\infty$.

The reason for calling this an extension problem, as pointed out in [R], is the following: (a) will force Φ to be one-to-one. Hence each $f \in H^\infty(D_1)$ corresponds to a function $\tilde{f}$ on $\Phi(D_1)$ such that $\tilde{f} \circ \Phi = f$ and any g in (b) is an extension of $\tilde{f}$. Our solution to this problem again is a generalization of a result in [R]; namely, such a mapping Φ must be a linear isometry up to an automorphism of D_2 (see Theorem 4).

1. SOME BACKGROUND ABOUT BOUNDED SYMMETRIC DOMAINS

A finite-dimensional complex vector space V is a Hermitian Jordan Triple System if V is endowed with a real trilinear map $\{\ \}: V \times V \times V \to V$,

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$(u, v, w) \to \{uv^*w\}$, which is $\mathbf{C}$-linear and symmetric in u and w and $\mathbf{C}$-antilinear in v and satisfies

$$\{\{uv^*w\}y^*x\} + \{\{uv^*x\}y^*w\} - \{uv^*\{wy^*x\}\} = \{w\{vu^*y\}^*x\}.$$

Given $u, v \in V$, define the operator $D(u, v^*)$ on V by

$$D(u, v^*)z = \{uv^*z\}.$$

A Hermitian Jordan Triple System is positive definite if the trace $\tau(u, v)$ of the $\mathbf{C}$-linear transformation $D(u, v^*)$ is a positive definite Hermitian form on V. A positive definite Hermitian Jordan Triple System will be abbreviated as PDHJTS. In the following, V will always denote a PDHJTS.

It is known that the Jordan unit ball

$$D = \{z \in V \mid \|D(z, z^*)\| < 1\}$$

is a bounded symmetric domain in V where $\| \ \|$ is the operator norm. Conversely, any bounded symmetric domain can be realized as a Jordan unit ball in some PDHJTS V (see [L]).

An element e in V is a tripotent if $\{ee^*e\} = e$. Two tripotents e, c are orthogonal if $D(e, c^*) = 0$. A tripotent is primitive if e is not a sum of two orthogonal tripotents. A frame of V is a maximal orthogonal system of primitive tripotents. Two frames of V always have the same number of elements, and the number is the rank of V which is also equal to the rank of the bounded symmetric domain D. Throughout this paper, $r(D)$ will denote the rank of a bounded symmetric domain D.

For the Jordan unit ball D in V, let $G = \mathrm{Aut}_0(D)$ be the identity component of the automorphism group of D and K the isotropy subgroup of G at 0. Suppose that $\{e_1, \dots, e_r\}$ is a frame. Then every element $z \in V$ can be written as

$$z = k \cdot \sum_{i=1}^{r} \lambda_i e_i, \tag{1}$$

for some $k \in K$ with $\lambda_1 \geq \cdots \geq \lambda_r \geq 0$. Furthermore, for any $z \in V$, there exists a frame $\{c_1, \dots, c_r\}$ such that

$$z = \sum_{i=1}^{r} \lambda_i c_i. \tag{2}$$

Then $z \in D$ if and only if $|\lambda_i| < 1$, $i = 1, \dots, r$.

For a tripotent e, let $V_\alpha(e)$ be the α-eigenspace of the operator $D(e, e^*)$ $\{v \in V \mid (D(e, e^*)v = \alpha v\}$. The only possible nonzero eigenspaces are $V_1(e)$, $V_{1/2}(e)$, and $V_0(e)$. Then $V_0(e)$ and $V_1(e)$ are positive definite Hermitian Jordan subtriple of V. A tripotent e is of rank j if $V_1(e)$ is of rank j.

Suppose that $\{e_1, \dots, e_r\}$ is a frame. Then V has the following Peirce decomposition:

$$V = \sum_{0 \leq i \leq j \leq r} V_{i<j} \tag{3}$$

where $V_{ii} = V_1(e_i)$ $(i = 1, \dots, r)$; $V_{ij} = V_{1/2}(e_i) \cap V_{1/2}(e_j)$ $(1 \leq i < j \leq r)$; $V_{0i} = V_{1/2}(e_i) \cap \bigcap_{j \neq i} V_0(e_j)$ $(i = 1, \dots, r)$; and $V_{00} = 0$.

For z, $w \in V$, the Bergman operators $B(z, w)$ on V are defined by

$$B(z, w) = I - 2D(z, w^*) + Q_z Q_w \tag{4}$$

where I is the identity map on V and $Q_z u = \{zu^* z\}$ for all z, $u \in V$.

Fix a frame $\{e_1, \dots, e_r\}$ in the following.

If $z = \sum_{i=1} \lambda_i e_i$, $\lambda_i \in C$, and $\lambda_0 = 0$, then, for $y \in V_{ij}$ $(0 \le i \le j \le r)$,

$$B(z, z)y = (1 - |\lambda_i|^2)(1 - |\lambda_j|^2)y. \tag{5}$$

Let $\langle\ \rangle$ be the Hermitian inner product on V which is a scalar multiple of the trace τ such that $\langle e, e\rangle = 1$ for a primitive tripotent e. Throughout this paper, $|\ |$ will denote the Euclidean norm $\langle\ \rangle^{1/2}$ on V. Suppose the rank of V is r. Then the Jordan unit ball D of V is contained in the Euclidean ball $B_{\sqrt{r}} = \{z \in V \mid |z| < \sqrt{r}\}$, and the Shilov boundary S of D is the intersection of $\partial B_{\sqrt{r}}$ and $\overline{D}$. In particular, this implies that $\eta \in \overline{D}$ is on S if and only if $|\eta| = \sqrt{r}$.

A point z is on the topological boundary ∂D of D if and only if there exists a frame $\{c_1, \dots, c_r\}$ such that $z = \sum_{i=1}^r \lambda_i c_i$ with $\lambda_1 = \cdots = \lambda_s = 1 > \lambda_{s+1} \ge \cdots \ge 0$. In particular, z is on S if and only if z is a tripotent of rank r, and z is on ∂D with $|z| = 1$ if and only if z is a primitive tripotent.

For details about this section, see [L].

2. A Schwarz lemma

In this section, we want to establish a Schwarz lemma for holomorphic mappings from one bounded symmetric domain into another one. First, we need two known results.

An open set E is star-shaped circular if $\alpha z \in E$ whenever $z \in E$ and $\alpha \in \mathbf{C}$, $|\alpha| \le 1$.

Lemma 1 ([R, Theorem 8.1.2]). *Suppose that*

(i) *D_1 and D_2 are two star-shaped circular regions in $\mathbf{C}^n$ and $\mathbf{C}^m$ respectively,*

(ii) *D_2 is convex and bounded,*

(iii) *$F : D_1 \to D_2$ is holomorphic.*

Then

(a) *$F'(0)$ maps D_1 into D_2, and*

(b) *$F(rD_1) \subset rD_2$ $(0 < r \le 1)$ if $F(0) = 0$.*

For $a \in D$, let φ_a be the holomorphic automorphism of D which interchanges 0 and a. Then one has (see [AY])

Lemma 2. *If D is the Jordan unit ball of V and $a \in D$, then the complex Jacobian of φ_a is given by*

$$\varphi_a'(z) = -B(a, a)^{1/2} B(z, a)^{-1} \quad \forall z \in D. \tag{6}$$

Remark. In the proof of Theorem 3, we only need the fact that $\varphi_a'(a)$ can be written as a composition of a unitary operator and $B(a, a)^{-1/2}$. This fact follows easily from [L, Lemma 2.11].

From now on, whenever we say that D_1 and D_2 are two bounded symmetric domains, we always assume that they are realized as two Jordan unit balls of PDHJTS's V_1 and V_2 respectively.

Now we have the following Schwarz lemma.

Theorem 3. *Suppose that D_1 and D_2 are two bounded symmetric domains in V_1 and V_2 respectively, with $r(D_1) \geq r(D_2)$; $F: D_1 \to D_2$ is a holomorphic mapping; and $F'(0)$ is an isometry of V_1 into V_2. Then*

(i) $r(D_1) = r(D_2)$;
(ii) $||F(z)|| \leq ||z||$, $\forall z \in D_1$;
(iii) $F(z) = F'(0)z$, $\forall z \in D_1$.

Proof. Let r_i be the rank of D_i and S_i the Shilov boundary of D_i $(i = 1, 2)$.

First, we show that $r_1 = r_2$ and $F(0) = 0$. Then (ii) follows from Lemma 1(b).

Let $a = F(0)$ and $G(z) = \varphi_a \circ F : D_1 \to D_2$ where $\varphi_a \in \mathrm{Aut}(D_2)$, which interchanges 0 and a. Then $G(0) = 0$.

By Lemma 1(a), $G'(0)z \in \overline{D_2}$ $\forall z \in \overline{D_1}$. This yields that

$$|\varphi_a'(a)F'(0)z| = |G'(0)z| \leq \sqrt{r_2}. \tag{7}$$

On the other hand, by (6) we have that $\varphi_a'(a) = -B(a, a)^{-1/2}$ which, together with (5), implies

$$|\varphi_a'(a)F'(0)\eta| \geq |F'(0)\eta|. \tag{8}$$

Now since $F'(0)$ is an isometry, for $\eta \in S_1$,

$$|F'(0)\eta| = \sqrt{r_1}. \tag{9}$$

(7)–(9) give that $r_1 = r_2$ and

$$|\varphi_a'(a)G'(0)\eta| = |F'(0)\eta| = \sqrt{r_1}. \tag{10}$$

Let $\{c_1, \dots, c_r\}$ be a frame of V_2 such that $a = \sum_{i=1}^r \lambda_i c_i$ with $\lambda_1 \geq \cdots \geq \lambda_j > 0 = \lambda_{j+1} = \cdots = \lambda_r$. From (5) we conclude that, for $w \in V_2$, $|B(a, a)^{-1/2}w| = |w|$ can happen if and only if $w \in V_0(c_1 + \cdots + c_j)$. Now it follows from (10) that $F'(0)\eta \in V_0(c_1 + \cdots + c_j) \cap \overline{D_2}$. Hence $F'(0)\eta$ is in the closure of the Jordan unit ball of the positive definite Hermitian Jordan subtriple system $V_0(c_1 + \cdots + c_j)$ of V which is of rank $r_2 - j$. Consequently, if $j \geq 1$, $|F'(0)\eta| \leq \sqrt{r_2} - j < \sqrt{r_2} = \sqrt{r_1}$ which contradicts (9). This proves $a = 0$.

Second, we show that, for every $\eta \in S_1$ and every $\lambda \in \mathbf{U}$, $F(\lambda\eta) = \lambda F'(0)\eta$, which will imply (iii) since F is holomorphic.

Let $A = F'(0)$ and $r = r_1 = r_2$. Then $A\eta \in S_2$, for $\eta \in S_1$, since $|A\eta| = |\eta| = \sqrt{r}$. Now, for any $w \in D_2$, $\alpha \in S_2$, since $|w| < \sqrt{r}$, we have

$$|\langle w, \alpha\rangle| < r.$$

For $\eta \in S_1$, we define an analytic function h from the unit disk $\mathbf{U}$ in $\mathbf{C}$ into $\mathbf{U}$ by

$$h(\lambda) = \tfrac{1}{r}\langle F(\lambda\eta), A\eta\rangle.$$

Then $h(0) = 0$, and $h'(0) = \frac{1}{r}\langle F'(0)\eta, A\eta\rangle = 1$. This leads, by the classical Schwarz lemma, to $h(\lambda) = \lambda$. Thus we obtain

$$\lambda = \langle \tfrac{1}{r}F(\lambda\eta), A\eta\rangle. \tag{11}$$

(ii) shows

$$||F(\lambda\eta)|| \leq ||\lambda\eta|| = |\lambda|||\eta|| \leq \sqrt{r}|\lambda|.$$

Therefore, we have that $\frac{1}{\lambda}F(\lambda\eta) \in \overline{D_2} \subset \overline{B_{\sqrt{r}}}$ for $\lambda \neq 0$. Since $A\eta \in S_2 \subset \partial B_{\sqrt{r}}$, one can easily see that (11) can hold if and only if $\frac{1}{\lambda}F(\lambda\eta) = A\eta$. Therefore, we have obtained that

$$F(\lambda\eta) = \lambda A\eta = \lambda F'(0)\eta \quad \forall \eta \in S_1, \lambda \in \mathbf{U},$$

proving the theorem.

Remark. We can easily construct a holomorphic mapping F from the unit ball of $\mathbf{C}^2$ into an irreducible bounded symmetric domain of rank two in $\mathbf{C}^4$ such that $F'(0)$ is an isometry of $\mathbf{C}^2$ into $\mathbf{C}^4$, but F fails to be a linear mapping.

For another version of the Schwarz lemma for a bounded symmetric domain, see [K].

3. An extension problem

Let D_1 and D_2 be two bounded symmetric domains. A holomorphic mapping $\Phi : D_1 \to D_2$ is said to have the norm-preserving H^∞ extension property

(*) if, for every $f \in H^\infty(D_1)$, there exists a $g \in H^\infty(D_2)$ such that (a) $g \circ \Phi = f$ and (b) $\|g\|_\infty = \|f\|_\infty$.

Remark. To study such Φ's, it is enough to study those Φ's with $\Phi(0) = 0$, since it is obvious that if Φ has the norm-preserving H^∞ extension property, so is $\varphi \circ \Phi$ where $\varphi \in \mathrm{Aut}(D_2)$.

Now we state our main result.

Theorem 4. *Suppose that D_1 and D_2 are two bounded symmetric domains in V_1 and V_2 respectively. For a holomorphic mapping $\Phi: D_1 \to D_2$ with $r(D_1) \geq r(D_2)$, the following are equivalent:*

(i) *Φ has the property (*);*

(ii) *$\Phi = \varphi \circ \mathscr{L}$ where φ is an automorphism of D_2 and $\mathscr{L}: D_1 \to D_2$ is a linear isometry of V_1 into V_2 mapping all primitive tripotents in V_1 into primitive tripotents in V_2;*

(iii) *there is a multiplicative linear operator $E: H^\infty(D_1) \to H^\infty(D_2)$ such that $(Ef) \circ \Phi = f$, $\forall f \in H^\infty(D_1)$.*

Proof. We divide the proof into several steps. By the remark before the theorem, it can be assumed that $\Phi(0) = 0$. In this case Φ becomes linear.

Step 1. We show (iii) implies (i). It remains to show $\|Ef\|_\infty = \|f\|_\infty$. The identity $(Ef) \circ \Phi = f$ implies $f \equiv 0$ if $Ef \equiv 0$. Now $Ef = E(f \cdot 1) = (Ef) \cdot (E1)$ gives that $E1 = 1$. Suppose that $\|Ef\|_\infty > \|f\|_\infty$. Then there exists a sequence z_n in D_2 such that $Ef(z_n) \to c$ with $|c| = \|Ef\|_\infty$. Since

$$(f-c)^{-1} \in H^\infty(D_1), \; E\left(\frac{1}{f-c}\right) \in H^\infty(D_2)$$

and we also have

$$1 = E\left(\frac{1}{f-c} \cdot (f-c)\right) = E\left(\frac{1}{f-c}\right) E(f-c).$$

However, this implies

$$\left| E\left(\frac{1}{f-c}\right)(z_n) \right| \to \infty$$

since $E(f-c)(z_n) \to 0$; we get a contradiction.

Step 2. We show (ii) implies (iii). Let P be the orthogonal projection of V_2 onto $\Phi(V_1)$. We claim that

$$P(D_2) \subset \Phi(D_1).$$

In fact, for $w \in D_2$, $P(w)$ can be written as

$$P(w) = \sum_{i=1}^{r} \lambda_i \Phi(c_i)$$

for some frame of V_1 with $\lambda_1 \geq \cdots \geq \lambda_r \geq 0$. Since $\Phi(c_1)$ is a primitive tripotent in V_2, by Lemma 5, $\lambda_1 = \langle w, \Phi(c_1)\rangle < 1$, which shows that $P(w) \in \Phi(D_1)$.

Now define

$$(Ef)(w) = f(A^{-1}P(w)), \qquad w \in D_2,$$

for all $F \in H^\infty(D_1)$ where $A = \Phi'(0)$. It is clear that E satisfies (iii).

Step 3. We show that (i) implies (ii) by establishing some lemmas in which we assume that Φ satisfies (i).

Lemma 5. *Let e be any primitive tripotent of V and $z \in D$. Then $|\langle z, e\rangle| < 1$.*

Proof. First, $|\langle z, e\rangle|^2 \leq \langle z, \{ee^*z\}\rangle = \langle z, \{ze^*e\}\rangle$.

Next, by [L], Lemma 2.6 (4),

$$\langle z, \{ze^*e\}\rangle = \langle \{ez^*z\}, e\rangle = \langle \{zz^*e\}, e\rangle,$$

which is not greater than $||D(z, z)|| < 1$.

Lemma 6. *Let ζ be a primitive tripotent in V_1. Then $|\Phi'(0)\zeta| \geq 1$.*

Proof. Define a holomorphic function $f(z)$ on D_1 by $f(z) = \langle z, \zeta\rangle$. Then Lemma 5 yields $||f||_\infty = 1$. Since Φ satisfies (i), there exists a $g \in H^\infty(D_2)$ such that $g(\Phi(z)) = \langle z, \zeta\rangle$ with $||g||_\infty = 1$. Then Lemma 1(a) implies that

$$|g'(0)| \leq 1. \tag{12}$$

Since $\langle \zeta, \zeta\rangle = 1$, $\forall \lambda \in \mathbf{U}$, $g(\Phi(\lambda\zeta)) = \lambda\langle \zeta, \zeta\rangle = \lambda$ which leads to $g'(0)\Phi'(0)\zeta = 1$. Now combining inequality (12) and $1 = |g'(0)\Phi'(0)\zeta| \leq |g'(0)|\,|\Phi'(0)\zeta|$ gives

$$|\Phi'(0)\zeta| \geq \frac{1}{|g'(0)|} \geq 1.$$

Lemma 7. *Let ζ be a tripotent of rank $r_1 = r(D_1)$. Then $\Phi'(0) \leq \sqrt{r_1}$.*

Proof. For $\lambda \in U$, define $G(\lambda) = \Phi(\lambda\zeta)$. Then $G(\lambda) \in D_2 \subset B_{\sqrt{r_2}}$, and $G(0) = 0$. It follows from Lemma 1(b) that $|\lambda|\,|\Phi'(0)| = |G'(0)\lambda| \leq |\lambda|\sqrt{r_2} \leq |\lambda|\sqrt{r_1}$, proving the lemma.

Let $\{c_1, \ldots, c_r\}$ be a frame of V_1. Define

$$h_{kl} = \langle \Phi'(0)c_k, \Phi'(0)c_l\rangle, \qquad k, l = 1, \ldots, r.$$

Lemma 8. *For any frame $\{c_1, \ldots, c_r\}$ of V_1, $h_{kl} = \delta_{kl}$, $k, l = 1, \ldots, r$. Consequently, Φ is an isometry of V_1 into V_2.*

Proof. We use induction on $r = r(D_1)$.

(i) If $r = 1$, the lemma follows from Lemmas 6 and 7.

(ii) Assume the lemma holds for $r - 1$.

For $s = 1, \dots, r$, we define a function H_s of $\theta_1, \dots, \theta_s \in R$

$$H_s(\theta_1, \dots, \theta_s) = \sum_{k \neq l}^{s} e^{i\theta_k} \cdot h_{kl} \cdot e^{-i\theta_l}.$$

For any $\theta_1, \dots, \theta_r \in R$, $\sum_{l=1}^{r} e^{i\theta_l} c_l$ is a tripotent of rank r. Lemma 7 gives

$$\sum_{k=1}^{r} h_{kk} + H_r(\theta_1, \dots, \theta_r) \leq r. \tag{13}$$

On the other hand, Lemma 6 yields $\sum_{k=1}^{r} h_{kk} \geq r$. Thus we have

$$\begin{aligned} H_r(\theta_1, \dots, \theta_r) &= H_{r-1}(\theta_1, \dots, \theta_{r-1}) \\ &\quad + \sum_{k=1}^{r} e^{i\theta_k} \cdot h_{kr} \cdot e^{-i\theta_r} + \sum_{k=1}^{r} e^{i\theta_r} \cdot h_{rl} \cdot e^{-i\theta_l} \leq 0. \end{aligned} \tag{14}$$

Then the induction assumption implies $H_{r-1}(\theta_1, \dots, \theta_{r-1}) = 0$, which, together with (14), leads to

$$\sum_{k=1}^{r} e^{i\theta_k} \cdot h_{kr} \cdot e^{-i\theta_r} + \sum_{l=1}^{r} e^{i\theta_r} \cdot h_{rl} \cdot e^{-i\theta_l} \leq 0 \quad \forall \theta_1, \dots, \theta_r \in \mathbf{R}.$$

Now replacing θ_r in the above by $\theta_r + \pi$ reverses the inequality. Hence

$$H_r(\theta_1, \dots, \theta_r) = 0 \quad \forall \theta_1, \dots, \theta_r \in \mathbf{R}. \tag{15}$$

Now (13), (15), and Lemma 6 show that $h_{kk} = 1$, $k = 1, \dots, r$. Finally,

$$h_{kl} = \frac{1}{2\pi} \int_0^{2\pi} H_r(\theta_1, \dots, \theta_r) e^{i\theta_l} e^{-i\theta_k} d\theta_l d\theta_k = 0, \qquad k \neq l,$$

completing the proof.

Lemma 9. (a) $\Phi(D_1) = D_2 \cap \Phi(V_1)$; (b) $\Phi(\partial D_1) \subset \partial D_2$.

Proof. (a) It is enough to show that $D_2 \cap \Phi(V_1) \subset \Phi(D_1)$. Suppose that there exists a $v \in D_2 \cap \Phi(V_1)$ but $v \notin \Phi(D_1)$. We note that $\Phi(D_1)$ is convex and open in $\Phi(V_1)$ and $\partial(\Phi(D_1)) = \Phi(\partial D_1)$. Therefore, we can find a $\zeta \in \partial D_1$ and a $t > 1$ with $v = t\Phi(\zeta)$. Then ζ can be written as $\zeta = c_1 + \sum_{i=2}^{r} \lambda_i c_i$ where $\{c_1, \dots, c_r\}$ is a frame of V_1. Define a holomorphic function $f(z)$ on D_1 by $f(z) = \langle z, c_1 \rangle$. Lemma 5 asserts $\|f\|_\infty = 1$. To $f(z)$ corresponds a $g \in H^\infty(D_2)$ with $\|g\|_\infty = 1$ such that

$$g(\Phi(z)) = \langle z, c_1 \rangle, \quad \forall z \in D_1.$$

In particular,

$$g(\Phi(\lambda\zeta)) = \lambda \langle \zeta, c_1 \rangle = \lambda, \quad \forall \lambda \in \mathbf{U}.$$

The linearity of Φ implies that

$$g(\lambda \tfrac{v}{t}) = g(\Phi(\lambda\zeta)) = \lambda, \quad \forall \lambda \in \mathbf{U}. \tag{16}$$

Since D_2 is star-shaped circular, $\lambda v \in D_2 \cap \Phi(V_1)$, $\forall \lambda \in \mathbf{U}$. Actually, (16) holds for all λ with $|\lambda| \leq t$ since g is holomorphic on D_2. This yields $g(v) = t > 1$, which contradicts $\|g\|_\infty = 1$.

(a) implies (b) since Φ is one-to-one.

Lemma 10. *If ζ is a primitive tripotent in V_1, then $\Phi(\zeta)$ is a primitive tripotent in V_2.*

Proof. By Lemma 8, Φ is an isometry of V_1 into V_2. Hence, $|\Phi(\zeta)| = |\zeta| = 1$. By Lemma 9(b), $\Phi(\zeta) \in \partial D_2$. Now we see that $\Phi(\zeta)$ is a primitive tripotent.

Finally, Lemmas 8 and 10 show that (i) implies (ii).

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