KÄHLER-EINSTEIN SURFACES
WITH NONPOSITIVE BISECTIONAL CURVATURE

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ABSTRACT. In this note we show that, for a Kähler-Einstein surface $M$ with negative Ricci curvature and nonpositive bisectional curvature, if the cotangent bundle of $M$ is not quasi-ample then $M$ is a quotient of the bidisc.

1. INTRODUCTION AND STATEMENT OF RESULT

Let $(M, g)$ be a compact Kähler-Einstein surface with nonpositive bisectional curvature. Assume $c_1 < 0$ (otherwise $(M, g)$ is flat). Then the following was raised in [SY].

Conjecture. Under the above assumptions, $(M, g)$ is isometric to a locally hermitian symmetric surface, namely, a quotient of the ball $B^2$ or the bidisc $D \times D$.

For the ball quotient case, the first partial answer was obtained by Siu and Yang [SY] in 1981. We need some notation to describe their theorem.

Denote by $H_{\text{min}}$, $H_{\text{av}}$, and $H_{\text{max}}$ the minimum, average, and maximum values of the holomorphic sectional curvature in all directions at $p$ (note that $H_{\text{av}} = \frac{1}{3}$ scalar curvature is a constant). Let $a(p) = H_{\text{av}} - H_{\text{min}}$ and $b(p) = H_{\text{max}} - H_{\text{av}}$. Then for any K-E metric one always has $\frac{1}{2}b(p) \leq a(p) \leq 2b(p)$.

Let $\lambda_0 = 2/(1 + 3\sqrt{\frac{6}{11}})$ ($\approx 0.622$). Their theorem is

Theorem (Siu-Yang). Let $(M, g)$ be a compact Kähler-Einstein surface with nonpositive bisectional curvature and $c_1 < 0$. If, for any $p \in M$, $a(p) \leq \lambda b(p)$ for some $\lambda < \lambda_0$, then $M$ is a ball quotient.

In [P] Polombo improved this result by enlarging $\lambda_0$ to $\frac{48}{32} \approx 0.923$. (The results in [P] are actually more general.)

Note that for the bidisc case $a(p) \equiv 2b(p) > 0$ identically, so it is the other end of the story.
For a compact Kähler manifold $M$, let $\pi : P = P(T_M) \to M$ be the projectivized tangent bundle and $L$ the tautological line bundle (such that $\pi_*(L) = \Omega_M$, the cotangent bundle). Then by definition $\Omega_M$ is ample (nef) if the line bundle $L$ is ample. Closely related to this is the following:

**Definition.** $\Omega_M$ is said to be quasi-ample if $L$ is nef and $Y \cdot L^{\dim Y} > 0$ for any irreducible subvariety $Y \subseteq P$ with $\pi(Y) = M$.

Note that there are many examples of a compact Kähler manifold $(M, g)$ with nonpositive bisectional curvature such that $c_1(M) < 0$ but $\Omega_M$ is not quasi-ample. In the surface case, the ratios of the two Chern numbers $c_1^2/c_2$ of such surfaces can be any rational number between 1 and 2 (cf. [Z]).

In this note we shall give another partial answer to the conjecture:

**Theorem.** Let $(M, g)$ be a compact Kähler-Einstein surface with nonpositive bisectional curvature. Suppose $c_1 < 0$ and $M$ is not quasi-ample. Then it is a quotient of the bidisc (hence, $g$ is the canonical metric).

**2. Preliminaries**

In this section we shall analyze the quasi ampleness condition. For our purpose we will only consider the dimension-two case. However, the higher-dimensional situations are similar.

We shall always assume that $(M, g)$ is a general type Kähler surface with nonpositive bisectional curvature. Then $c_1(M) < 0$, as $M$ cannot contain any rational curves.

Denote by $\pi : P = P(T_M) \to M$ and $L$ the dual of the tautological line bundle on $P$ (such that $\pi_*(L) = \Omega_M$ is the cotangent bundle). Next let $\check{\gamma}$ be the hermitian metric on $L$ induced by $g$, and write $Z_g = \{(x, [v]) \in P | \exists w \neq 0 : R_{wvw} = 0\}$ for the zero locus of the bisectional curvature of $g$. It is the subset where the nonnegative curvature form $c_1(L, \check{\gamma})$ fails to be positive definite. A surface $Y \subseteq P$ will be called horizontal if $\pi(Y) = M$.

Since $c_1(L, \check{\gamma})$ is always positive in the fiber direction of $\pi$, it follows that:

**Lemma 1.** Let $(M, g)$ be a general type Kähler surface with nonpositive bisectional curvature. If $Y$ is a horizontal surface with $L^2 \cdot Y = 0$, then $Y \subset Z_g$.

We shall also need the following lemma. Let $C_1$, $C_2$ be the two Chern forms under $g$, and let $c_1$, $c_2$ be the Chern classes. The nonpositivity of the bisectional curvature implies that $C_1^2 - C_2 \geq 0$ on $M$. Write $V = V_g = \{x \in M | C_1^2(x) - C_2(x) > 0\}$. $V$ is not empty if and only if $c_1^2 > c_2$.

**Lemma 2.** For $x \in V$ the set $\pi^{-1}(x) \cap Z_g$ consists of at most two points.

**Proof.** Consider the Nakano tensor $N$ defined by $N(\alpha \otimes \beta, \gamma \otimes \delta) = R_{\alpha \beta \gamma \delta}$ and extended linearly to $T_M \otimes T_M$. Since $g$ is Kählerian, $N$ is actually a hermitian bilinear form living in $S^2 T_M$. Since every element in $S^2 T_x, M$ is decomposable, the nonpositivity of bisectional curvature implies that $N \leq 0$. Let $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq 0$ be the three eigenvalues of $N$. It is straightforward to show that $\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1 = C_1^2 - C_2$. So, for $x \in V$, $\lambda_1 \leq \lambda_2 < 0$ at $x$. Hence, there can be at most one pair of directions at $x$ which gives zero bisectional curvature. Q.E.D.
3. Proof of the Theorem

From now on we will assume that $(M, g)$ is a compact Kähler-Einstein surface with $c_1 < 0$ and with nonpositive bisectional curvature. The two Chern numbers of $M$ satisfy $3c_2 \geq c_1^2 \geq c_1 > 0$.

**Lemma 3.** For the above $M$, $c_1^2 > c_2$.

*Proof.* Assume that $c_1^2 = c_2$. Then since the Ricci curvature is everywhere negative, by Theorem A of [YZ], $(M, g)$ is locally isometric to a hypersurface in $\mathbb{C}^3$. Since $g$ is Einstein, the theorem of Smyth [S] implies that $M$ is locally symmetric, which contradicts our assumption $c_1^2 = c_2$. Q.E.D.

**Corollary.** If $M$ is not quasi-ample, then for any $x \in V$ there exists a unique pair of directions $[\alpha], [\beta]$ at $x$ such that $R_{a\bar{a}b\bar{b}} = 0$. Moreover, $\alpha$ is perpendicular to $\beta$.

*Proof.* The existence of such a pair is guaranteed by $\pi(Z_g) = M$ (Lemma 1), while the uniqueness comes from Lemma 2. Now suppose $R_{a\bar{a}b\bar{b}} = 0$. Let $[\alpha']$ and $[\beta']$ be the directions at $x$ perpendicular to $[\alpha]$ and $[\beta]$, respectively. The Einstein condition implies that $R_{a\bar{a}'b\bar{b}'} = 0$. So for $x \in V$ these two pairs must coincide. Hence $\alpha \perp \beta$. Q.E.D.

Now we are ready to prove the theorem stated in §1.

*Proof of the Theorem.* Let $Y$ be a horizontal surface with $L^2 \cdot Y = 0$. Then $Y \subset Z_g$. So by Lemma 2 the degree $d$ of the restriction map $\pi|_Y \to M$ is one or two. First let us assume that $d = 1$.

Note that the metric $g$ is analytic, so $V$ is an open dense subset of $M$. Since $d = 1$, $Y$ is a blowing up of $M$ at finitely many points $E = \{p_1, \ldots, p_r\}$. For any $x \in M \setminus E$ choose a holomorphic tangent frame $(e_1, e_2)$ with $[e_1] \in Y$ in a neighborhood of $x$. Let $([\alpha], [\beta])$ be a unitary frame near $x$ with $[\alpha] = [e_1]$. Now $f = R_{a\bar{a}a\bar{a}} \equiv a$ is globally defined in $M \setminus E$, where $a$ is the Ricci curvature. By the computation in [SY], $\Delta f = |R_{a\bar{a}b\bar{b}}|^2$ ($f$ is $S_{11\bar{1}}$ in [SY]; here we used the fact $[\alpha] = [e_1]$). So $R_{a\bar{a}b\bar{b}} = 0$. Then it follows that the connection and hence, the holonomy split (note that $g$ is analytic). So $(M, g)$ is reducible. Each de Rham factor of the universal covering space is again Einstein with negative Ricci curvature and hence, the Poincaré disc.

Next let us assume $d = 2$. Again let $E = \{x \in M|\pi^{-1}(x) \subset Y\}$.

$E$ is finite as $Y$ is irreducible and horizontal. Since, for $x \in V$, $(\pi|_Y)^{-1}(x)$ consists of two perpendicular directions and $V$ is dense in $M$, it follows that $\pi|_Y$ cannot have any branch locus over $M \setminus E$. Hence it gives a 2-sheets unbranched covering over $M \setminus E$ and, hence, an unbranched double cover $M' \to M$. Now by the branched covering trick, in the pullback of the fiber bundle $\pi: P \to M$ by $M' \to M$, the inverse image of $Y$ consists of two irreducible components, each with degree one over $M'$, so we are in the first case (or it gives a holomorphic splitting of $T_{M'}$ in $M' \setminus E'$; therefore, $T_{M'}$ is not stable and $M' = D \times D/G$). Q.E.D.
References


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