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GROWTH PROPERTY FOR THE MINIMAL SURFACE EQUATION IN UNBOUNDED DOMAINS

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ABSTRACT. Here we prove that if u satisfies the minimal surface equation in an unbounded domain Ω which is properly contained in a half plane, then the growth rate of u is of the same order as the shape of Ω and $u|_{\partial\Omega}$.

1. INTRODUCTION

The purpose of this paper is to improve a Phragmèn-Lindelöf Theorem for the minimal surface equation in $\mathbb{R}^2$. Hwang has proved that if u satisfies the minimal surface equation in an unbounded domain Ω, which is properly contained in a half plane, the growth property of u depends on Ω and $u|_{\partial\Omega}$ only, without requiring any restriction for u [3]. In this respect, the Phragmèn-Lindelöf Theorem for the minimal surface equation is better than that of the Laplace equation. We remark that if u satisfies the Laplace equation in an unbounded domain Ω, the growth property of u cannot be determined completely by the shape of Ω and $u|_{\partial\Omega}$ alone [7].

But the estimate in [3] is not good enough. For example, let $\Omega = \{-\cosh y < x < \cosh y | y > 0\}$ and $\operatorname{div} Tu = 0$ in Ω, $u|_{\partial\Omega} = \sqrt{(\cosh y)^2 - x^2}|_{\partial\Omega}$. Then, by Example 3.4 of [3], we know that $u = O(ye^y)$ as $y \to \infty$, but the growth rate of the solution $\sqrt{(\cosh y)^2 - x^2}$ (catenoid) is $O(e^y)$.

The purpose of this paper is to improve the estimate of [3]. We will prove that the growth rate of u is of the same order as the shape of Ω and $u|_{\partial\Omega}$ (Theorems 2.12 and 2.13). In fact, let $\Omega = \{-\cosh y < x < \cosh y\}$. We will prove that a catenoid is the maximum solution among the solutions with vanishing boundary value (Corollary 2.3).

2. PHRAGMÉN-LINDELÖF THEOREMS FOR $\mathbb{R}^2$

Throughout the paper, Ω will be a connected domain (bounded or unbounded) in $\mathbb{R}^2$ and, for any function $u \in C^2(\Omega)$, Tu will denote the vector $Du/\sqrt{1+|Du|^2}$, where Du is the gradient vector of u and the minimal surface

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operator $\mathfrak{M}$ is given by

$$\mathfrak{M}u = (1+|Du|^2)\Delta u - D_i u D_j u D_{ij} u = (1+|Du|^2)^{3/2} \operatorname{div} Tu,$$

where Δu is the Laplacian of u.

We will use functions of the form

$$F(x, y) = (G(x, y))^{1/2} g(y) + h(y)$$

as comparison functions, and we compute $\mathfrak{M}F$ in the following lemma.

Lemma 2.1. *Let*

$$F(x, y) = (G(x, y))^{1/2} g(y) + h(y),$$

where $F, G: \mathbb{R}^2 \to \mathbb{R}^1$, $g, h: \mathbb{R}^1 \to \mathbb{R}^1$, $F, G, g, h \in C^2$, *and* $G > 0$. *Then* $G^{3/2}\mathfrak{M}F = I + II + III$, *where*

$$\begin{aligned}
I &= g^3 \left(\frac{1}{4} G_x^2 \left(\frac{1}{2} G_{yy} + G\frac{g''}{g} - 3G\left(\frac{g'}{g}\right)^2 \right) \right. \\
&\qquad \left. + \frac{1}{2} G_{xx} \left(\frac{1}{4} G y^2 + G_y G \frac{g'}{g} + G^2 \left(\frac{g'}{g}\right)^2 \right) - \frac{1}{2} G_x G_{xy} \left(\frac{1}{2} G_y + G\frac{g'}{g} \right) \right), \\
II &= g \left(-\frac{1}{4} G_y^2 + \frac{1}{2} G G_{yy} + G G_y \frac{g'}{g} + G^2 \frac{g''}{g} - \frac{1}{4} G_x^2 + \frac{1}{2} G_{xx} G \right), \\
III &= G^{1/2} h'' \left(G + \frac{1}{4} G_x^2 g^2 \right) \\
&\quad + G^{1/2} h' \left(-\frac{1}{2} G_x G_{xy} g^2 - G_x^2 g g' + \frac{1}{2} G_y G_{xx} g^2 + G G_{xx} g g' \right) \\
&\quad + h'^2 \left(-\frac{1}{4} G_x^2 g + \frac{1}{2} G G_{xx} g \right).
\end{aligned}$$

Proof. By simple computation we have

$$\begin{aligned}
(1+F_x^2)F_{yy} &= \left(1 + \frac{1}{4} G^{-1} G_x^2 g^2\right) \\
&\qquad \times \left(-\frac{1}{4} G^{-3/2} G_y^2 g + \frac{1}{2} G^{-1/2} G_{yy} g + G^{-1/2} G_y g' + G^{1/2} g'' + h'' \right),
\end{aligned}$$

$$\begin{aligned}
-2F_x F_y F_{xy} &= -2\left(\frac{1}{2} G^{-1/2} G_x g\right)\left(\frac{1}{2} G^{-1/2} G_y g + G^{1/2} g' + h'\right) \\
&\qquad \times \left(-\frac{1}{4} G^{-3/2} G_x G_y g + \frac{1}{2} G^{-1/2} G_{xy} g + \frac{1}{2} G^{-1/2} G_x g' \right),
\end{aligned}$$

$$\begin{aligned}
(1+F_y^2)F_{xx} &= \left(1 + \left(\frac{1}{2} G^{-1/2} G_y g + G^{1/2} g' + h'\right)^2\right) \\
&\qquad \times \left(-\frac{1}{4} G^{-3/2} G_x^2 g + \frac{1}{2} G^{-1/2} G_{xx} g \right).
\end{aligned}$$

Hence

$$
\begin{aligned}
G^{5/2}\mathfrak{M}F &= (G+\tfrac{1}{4}G_x^2g^2)(-\tfrac{1}{4}G_y^2g+\tfrac{1}{2}GG_{yy}g+GG_yg'+G^2g''+G^{3/2}h'') \\
&\quad -2(\tfrac{1}{2}G_xg)(\tfrac{1}{2}G_yg+Gg'+G^{1/2}h')(-\tfrac{1}{4}G_xG_yg+\tfrac{1}{2}GG_{xy}g+\tfrac{1}{2}GG_xg') \\
&\quad +(G+(\tfrac{1}{2}G_yg+Gg'+h'G^{1/2})^2)(-\tfrac{1}{4}G_x^2g+\tfrac{1}{2}GG_{xx}g) \\
&= \tfrac{1}{4}G_x^2g^2(-\tfrac{1}{4}G_y^2g+\tfrac{1}{2}GG_{yy}g+GG_yg'+G^2g'') \\
&\quad -2(\tfrac{1}{2}G_xg)(\tfrac{1}{2}G_yg+Gg')(-\tfrac{1}{4}G_xG_yg+\tfrac{1}{2}GG_{xy}g+\tfrac{1}{2}GG_xg') \\
&\quad +(\tfrac{1}{2}G_yg+Gg')^2(-\tfrac{1}{4}G_x^2g+\tfrac{1}{2}GG_{xx}g) \\
&\quad +G(-\tfrac{1}{4}G_y^2g+\tfrac{1}{2}GG_{yy}g+GG_yg'+G^2g''-\tfrac{1}{4}G_x^2g+\tfrac{1}{2}G_{xx}gG) \\
&\quad +G^{3/2}h''(G+\tfrac{1}{4}G_x^2g^2) \\
&\quad +G^{1/2}h'(-G_xg)(-\tfrac{1}{4}G_xG_yg+\tfrac{1}{2}GG_{xy}g+\tfrac{1}{2}GG_xg') \\
&\quad +2h'G^{1/2}(\tfrac{1}{2}G_yg+Gg')(-\tfrac{1}{4}G_x^2g+\tfrac{1}{2}GG_{xx}g) \\
&\quad +h'^2G(-\tfrac{1}{4}G_x^2g+\tfrac{1}{2}GG_{xx}g) \\
&= g^3(\tfrac{1}{4}G_x^2(\tfrac{1}{2}GG_{yy}+G^2\tfrac{g''}{g}-3G^2(\tfrac{g'}{g})^2) \\
&\qquad +\tfrac{1}{2}GG_{xx}(\tfrac{1}{2}G_y+G\tfrac{g'}{g})^2-\tfrac{1}{2}GG_xG_{xy}(\tfrac{1}{2}G_y+G\tfrac{g'}{g})) \\
&\quad +Gg(-\tfrac{1}{4}G_y^2+\tfrac{1}{2}GG_{yy}+GG_y\tfrac{g'}{g}+G^2\tfrac{g''}{g}-\tfrac{1}{4}G_x^2+\tfrac{1}{2}G_{xx}G) \\
&\quad +G(G^{1/2}h''(G+\tfrac{1}{4}G_x^2g^2) \\
&\qquad +G^{1/2}h'(-\tfrac{1}{2}G_xG_{xy}g^2-G_x^2gg'+\tfrac{1}{2}G_yG_{xx}g^2+GG_{xx}gg') \\
&\qquad\qquad +h'^2(-\tfrac{1}{4}G_x^2g+\tfrac{1}{2}GG_{xx}g)) \\
&= G(I+II+III).
\end{aligned}
$$

The lemma follows.

Now we treat the Phragmèn-Lindelöf Theorem for comparison functions $H(x,y)$ with faster growth. Since, in a half plane, the bound of the solutions with vanishing boundary value does not even exist, the domain must be properly contained in a half plane.

Lemma 2.2. *Let* $H(x,y)=a\sqrt{f^2(y)-x^2}$ *and* $\Omega\subset\{-f_1(y)<x<f_1(y),\,y>0\}$, *where* $f,f_1:[0,\infty)\to[0,\infty)$, $f\in C^2$, $f_1\in C^0$, $f\geq f_1>0$ *for* $y>0$, $f'\equiv\frac{df}{dy}>0$, *and* a *is a positive constant. Let* $u\in C^2(\Omega)\cap C^0(\bar{\Omega})$, *and suppose that*

(i) $\operatorname{div}TH-\operatorname{div}Tu\leq 0$ *in* Ω,

(ii) $(u-H)|_{\partial\Omega}\leq 0$,

(iii) $\liminf_{y\to\infty}f_1(y)/f'^2=0$.

Then $u\leq H$ *in* Ω.

Remark. It is easy to see that if $\lim_{y\to\infty}f(y)/f'^2=0$, then the rate of growth of f must be faster than y^2 as $y\to\infty$.

Proof of Lemma 2.2. If $\{(x,y)\in\Omega|u(x,y)-H(x,y)>0\}$ is nonempty, there exists $\varepsilon>0$ such that $\Omega'=\{(x,y)\in\Omega|u(x,y)-H(x,y)>\varepsilon\}$ is non-empty and $\partial\Omega'\cap\Omega$ is smooth (Sard's Theorem). Since $(u-H)|_{\partial\Omega}\leq 0$, we have $\partial\Omega'\subset\Omega$ and $\partial\Omega'=\{(x,y)\in\Omega|u-H=\varepsilon\}$.

For every $y_0 > 0$, let $\Omega_{y_0} = \Omega' \cap \{y < y_0\}$ and $\Gamma_{y_0} = \partial\Omega_{y_0} \cap \{y = y_0\}$. By the divergence theorem, we have

$$\begin{aligned}
&\int_{\partial\Omega_{y_0}} \tan^{-1}(u - H - \varepsilon)(Tu - TH)\cdot\nu\, d\sigma \\
&\quad = \int\int_{\Omega_{y_0}} \frac{(Du - DH)}{1 + (u - H - \varepsilon)^2}(Tu - TH)\, dx \\
&\qquad + \int\int_{\Omega_{y_0}} (u - H - \varepsilon)(\operatorname{div} Tu - \operatorname{div} TH)\, dx,
\end{aligned}$$

where ν is the unit outer normal of $\partial\Omega_{y_0}$. Noticing that $\partial\Omega_{y_0}\backslash\Gamma_{y_0} \subset \partial\Omega'$, we have $u - H = \varepsilon$ on $\partial\Omega_{y_0}\backslash\Gamma_{y_0}$. Since $\tan^{-1}(u - H - \varepsilon)(\operatorname{div} Tu - \operatorname{div} TH) \geq 0$ in Ω_{y_0} and $Tu\cdot\nu \leq 1$,

$$\begin{aligned}
&\int\int_{\Omega_{y_0}} \frac{(Du - DH)}{1 + (u - H - \varepsilon)^2}(Tu - TH)dx \\
&\quad \leq \int_{\Gamma_{y_0}} \tan^{-1}(u - H - \varepsilon)(Tu - TH)\cdot\nu\, d\sigma \\
&\quad \leq \int_{\Gamma_{y_0}} \tan^{-1}(u - H - \varepsilon)(1 - TH\cdot\nu)\, d\sigma.
\end{aligned}$$

Since $\nu = \langle 0, 1\rangle$ on Γ_{y_0},

$$H = a\sqrt{f^2 - x^2}, \qquad H_y = a\frac{ff'}{\sqrt{f^2 - x^2}}, \qquad H_x = -\frac{ax}{\sqrt{f^2 - x^2}},$$

$$\begin{aligned}
TH\cdot\nu|_{\Gamma_{y_0}} &= \frac{H_y}{\sqrt{1 + H_x^2 + H_y^2}} = \frac{aff'/\sqrt{f^2 - x^2}}{\sqrt{1 + a^2x^2/(f^2 - x^2) + a^2f^2f'^2/(f^2 - x^2)}} \\
&= \frac{aff'}{\sqrt{f^2 - x^2 + a^2x^2 + a^2f^2f'^2}},
\end{aligned}$$

$$\begin{aligned}
&(1 - TH\cdot\nu)|_{\Gamma_{y_0}} \\
&\quad = \frac{\sqrt{f^2 - x^2 + a^2x^2 + a^2f^2f'^2} - aff'}{\sqrt{f^2 - x^2 + a^2x^2 + a^2f^2f'^2}} \\
&\qquad \cdot \frac{\sqrt{f^2 - x^2 + a^2x^2 + a^2f^2f'^2} + aff'}{\sqrt{f^2 - x^2 + a^2x^2 + a^2f^2f'^2} + aff'} \\
&\quad = \frac{f^2 - x^2 + a^2x^2}{\sqrt{f^2 - x^2 + a^2x^2 + a^2f^2f'^2}(\sqrt{f^2 - x^2 + a^2x^2 + a^2f^2f'^2} + aff')} \\
&\quad \leq \frac{(1 + a^2)f^2}{a^2f^2f'^2} = \frac{1 + a^2}{a^2f'^2}.
\end{aligned}$$

Now

$$\begin{aligned}\int\int_{\Omega_{y_0}} &\frac{(Du-DH)}{1+(u-H-\varepsilon)^2}(Tu-TH)\,dx \\ &\le \int_{\Gamma_{y_0}} \tan^{-1}(u-H-\varepsilon)(1-TH\cdot\nu)\,d\sigma \\ &\le \frac{\pi}{2}\int_{\Gamma_{y_0}} \frac{1+a^2}{a^2f'^2}\,d\sigma \le \pi\cdot\frac{(1+a^2)f_1}{a^2f'^2}(y_0).\end{aligned}$$

Let $y_0 \to \infty$. We have

$$0 \le \int\int_{\Omega'} \frac{(Du-DH)}{1+(u-H-\varepsilon)^2}(Tu-TH)\,dx \le 0.$$

Since $(Du-DH)\cdot(Tu-TH) \ge 0$ and the equality holds when $Du = DH$, we have $Du - DH \equiv 0$ in Ω'. Then $u \equiv H+\varepsilon$ in Ω', and, by definition, Ω' must be empty. This is impossible, and we conclude that $u(x) \le H(x)$ for all x in Ω.

Remark. The following well-known fact is used to prove Lemma 2.2: $Du/\sqrt{1+|Du|^2}$ has norm less than 1. It is a very important idea for the capillary surface equation (cf. [1, Theorem 5.1]).

Now we obtain the result: catenoid is the maximum solution among those surfaces on $\Omega = \{-\cosh y < x < \cosh y\}$ satisfying the minimal surface equation and with vanishing boundary value.

Corollary 2.3. *Let* $\Omega = \{-\cosh y < x < \cosh y\}$, *and let* $u \in C^2(\Omega)\cap C^0(\bar\Omega)$. *Suppose that*

(i) $\operatorname{div} Tu \ge 0$ *in* Ω,

(ii) $u|_{\partial\Omega} \le 0$.

Then $u \le \sqrt{(\cosh y)^2 - x^2}$.

Proof. The corollary can be proved by the fact that

$$\operatorname{div} Tu - \operatorname{div} T\sqrt{(\cosh y)^2 - x^2} \ge 0, \qquad \lim_{y\to\pm\infty}\frac{\cosh y}{[(\cosh y)']^2} = 0,$$

and a similar argument as in the proof of Lemma 2.2.

By Lemma 2.2, it is sufficient to show that $\operatorname{div} T(a\sqrt{f^2-x^2}) \le 0$ holds when $\liminf_{y\to\infty}(f_1/f'^2) = 0$. Let $f \in C^2$, $f \ge 0$, and $f' > 0$. Define $p(f) = (1/(\log f)')' = (f'^2 - ff'')/f'^2 = 1 - ff''/f'^2$; some properties of $p(f)$ will be discussed later. By setting $F = a\sqrt{f^2-x^2}$, $G = f^2 - x^2$, and $g = a$ in Lemma 2.1 where a is a positive constant, we get

$$\begin{aligned}G^{3/2}\mathfrak{M}F &= a^3(x^2f'^2 + x^2ff'' - f^2f'^2) + a(ff''(f^2-x^2) - x^2f'^2 - f^2) \\ &= f'^2a^3(-f^2 + (2-p(f))x^2) + f'^2a(f^2(1-p(f)) - (2-p(f))x^2) - af^2 \\ &= -(a-a^3)(2-p(f))x^2f'^2 + f^2f'^2a((1-p(f)) - a^2) - af^2.\end{aligned}$$

Lemma 2.4. *Let* $f, p(f), F, G$, *and* a *be defined as above, and let* $\Omega \subset \{(x,y) | -f_1(y) < x < f_1(y),\ y > 0\}$, *where* $f_1 > 0$ *is given. Suppose* $p(f) \ge p_0$, *where* p_0 *is a constant. Then*

(i) *if* $1 \ge p_0 \ge 0$, $a = \sqrt{1-p_0}$, *and* $f \ge f_1$, *or*

(ii) *if* $p_0 < 0$, $a > \sqrt{1-p_0}$, a *is a positive constant, and* $f^2 \geq (a^2-1)(2-p_0)f_1^2/(a^2-(1-p_0))$,

we have $G^{3/2}\mathfrak{M}F \leq 0$ *in* Ω.

Proof. (i) If $2 \geq p(f) \geq p_0$, it is easy to see that

$$G^{3/2}\mathfrak{M}F \leq 0.$$

If $p(f) \geq 2$, we have

$$G^{3/2}\mathfrak{M}F \leq (p(f)-2)\sqrt{1-p_0}p_0x^2f'^2 + f^2f'^2\sqrt{1-p_0}(-p(f)+p_0) \leq 0.$$

(ii) By assumption, it is easy to have the following:

$$\begin{aligned} G^{3/2}\mathfrak{M}F &= f'^2a^3(-f^2+(2-p(f))x^2) \\ &\quad + f'^2a(f^2(1-p(f))-(2-p(f))x^2) - af^2 \\ &= af'^2((a^2-1)(2-p(f))x^2 - f^2(a^2-(1-p(f)))) - af^2 \leq 0. \end{aligned}$$

Hence it is easy to derive the following theorem.

Theorem 2.5. *Let* f, f_1, $p(f)$, *and* Ω *be defined as in Lemma* 2.4. *Let* $\liminf_{y\to\infty}(f_1/f'^2) = 0$, $u \in C^2(\Omega)\cap C^0(\bar{\Omega})$, *and* $\operatorname{div} Tu \geq 0$ *in* Ω. *Then*

(i) *if* $p(f) \geq p_0 \geq 0$ *where* p_0 *is a constant,* $f \geq f_1$, $1 \geq p_0 \geq 0$, *and* $u|_{\partial\Omega} \leq \sqrt{1-p_0}\sqrt{f^2-x^2}|_{\partial\Omega}$, *then we have* $u \leq \sqrt{1-p_0}\sqrt{f^2-x^2}$ *in* Ω,

(ii) *if* $p(f) \geq p_0$, *where* p_0 *is a negative constant, and*

$$f^2 \geq \frac{(a^2-1)(2-p_0)}{a^2-(1-p_0)}f_1^2, \qquad u|_{\partial\Omega} \leq a\sqrt{f^2-x^2}|_{\partial\Omega},$$

where a *is a positive constant satisfying* $a^2-1+p_0 > 0$, *then we have* $u \leq a\sqrt{f^2-x^2}$ *in* Ω.

We will show later that the above theorem still remains valid without the condition $\liminf_{y\to\infty}(f_1/f'^2) = 0$. We first investigate the properties of $p(f)$. Let $f \in C^2$, $f \geq 0$, $f' > 0$, a be a positive constant, and $\alpha \neq 0$ be a constant. Since $p(af^\alpha) = (1/(\log(af^\alpha))')' = \frac{1}{\alpha}(1/(\log f)')'$, we have $p(af^\alpha) = \frac{1}{\alpha}p(f)$. Then we prove the following lemma.

Lemma 2.6. *Let* f, $f_1 \in C^2$, *and* f, f', f_1, $f_1' > 0$. *Then*

(i) *if* $p(f)$, $p(f_1) \geq p_0 > 0$, *where* $0 < p_0 \leq 1$, *is a constant, then* $p(ff_1) \geq p_0/2 > 0$;

(ii) *if* $p(f) \geq 0$, *then* $p(e^{ay}f) \geq 0$ *for any positive constant* a;

(iii) *if* $p(f) \geq p_0$, *where* p_0 *is a negative constant, then* $p(e^{ay}f) \geq p_0$ *for any positive constant* a.

Proof. (i) Let $h = (\log f)' = f'/f > 0$ and let $h_1 = (\log f_1)' = f_1'/f_1 > 0$. Since $p(f) = (1/h)' = -h'/h^2 \geq p_0 > 0$ and $p(f_1) = (1/h_1)' = -h_1'/h_1^2 \geq p_0 > 0$, we have $p(ff_1) = (1/(h+h_1))' = -(h'+h_1')/(h+h_1)^2 \geq -(h'+h_1')/2(h^2+h_1^2) \geq p_0/2$.

(ii) Since $p(f) = -h'/h^2 \geq 0$, we have $p(e^{ay}f) = (1/((\log e^{ay})' + (\log f)'))' = (1/(a+h))' = -h'/(a+h)^2 \geq 0$.

(iii) Since $p(f) = -h'/h^2$ and $p(e^{ay}f) = -h'/(a+h)^2$ have the same sign and $|p(e^{ay}f)| \leq |p(f)|$, the result follows immediately.

Remark 2.7. (i) If $f = y^m$, where m is a positive constant, then $p(f) = \frac{1}{m}$.

(ii) $p(e^y) = 0$.

(iii) If we set $f = e^{y^\alpha}$, where $\alpha > 1$ is a constant, then $p(f) = (1/(y^\alpha)')' = (y^{1-\alpha}/\alpha)' = (1-\alpha)y^{-\alpha}/\alpha$. This implies that $p(f) \to 0^-$ as $y \to \infty$, so, for sufficiently large y, we have $p(f) > -\varepsilon$ for some small positive number ε.

Similarly, if f increases faster than the exponential function, then we can assume $p(f) \geq -\varepsilon$ for some small positive constant ε essentially.

Lemma 2.8. *Let* $f \in C^1$, $f, f' > 0$, *and* $\limsup_{y\to\infty}(f(y)/y^2) = +\infty$. *Then* $\liminf_{y\to\infty}(f/f'^2) = 0$.

Proof. Suppose not; then there exist positive constants y_0 and C such that, for every $y > y_0$, we have $f(y)/f'^2(y) \geq C$, so $f = O(y^2)$. Contradiction arises and the lemma follows.

We are now ready to remove the condition $\liminf_{y\to\infty}(f/f'^2) = 0$ in Theorem 2.5.

We will start with a theorem.

Theorem 2.9. *Let* $f \in C^0[0,\infty) \cap C^2(0,\infty)$, $f \geq 0$, $f' > 0$, $\Omega \subset \{(x,y)| -f(y) < x < f(y), y > 0\}$, $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$, $\operatorname{div} Tu \geq 0$, *and* $p(f) \geq p_0 \geq 0$, *where* p_0 *is a constant. Then*

(i) *if* $p_0 = 0$ *and* $(u - \sqrt{f^2 - x^2})|_{\partial\Omega} \leq 0$, *we have* $u \leq \sqrt{f^2 - x^2}$ *in* Ω,

(ii) *if* $1 \geq p_0 > 0$ *and* $(u - \sqrt{1-q_0}\sqrt{f^2 - x^2})|_{\partial\Omega} \leq 0$, *where* $q_0 = \min(1/8, p_0/2)$, *we have* $u \leq \sqrt{1-q_0}\sqrt{f^2 - x^2}$ *in* Ω.

Proof. (i) Let $h = e^{ay}f$, where a is a positive constant. Then by Lemma 2.6 $p(h) \geq 0$. So $\mathfrak{M}(\sqrt{h^2 - x^2}) \leq 0$ in Ω. Since $f' > 0$, $\limsup_{y\to\infty}(h/y^2) = +\infty$. By Lemma 2.8, $\liminf_{y\to\infty}(h/h'^2) = 0$. Since $u|_{\partial\Omega} \leq \sqrt{f^2 - x^2}|_{\partial\Omega} \leq \sqrt{h^2 - x^2}|_{\partial\Omega}$, by Theorem 2.5, $u \leq \sqrt{h^2 - x^2} = \sqrt{(e^{ay}f)^2 - x^2}$ in Ω. Setting $a \to 0$, the result follows.

(ii) Let $h = (1+by)^4 f$, where b is a positive constant. Since $p((1+by)^4) = \frac{1}{4}$, by Lemma 2.6, $p(h) \geq \min(1/4, p_0)/2 = q_0 > 0$. By Lemma 2.4, $\mathfrak{M}(\sqrt{1-q_0}\sqrt{h^2 - x^2}) \leq 0$ in Ω. Since $f' > 0$, $\limsup_{y\to\infty}(h/y^2) = +\infty$. So $\liminf_{y\to\infty}(h/h'^2) = 0$. Since

$$u|_{\partial\Omega} \leq \sqrt{1-q_0}\sqrt{f^2 - x^2}|_{\partial\Omega} \leq \sqrt{1-q_0}\sqrt{h^2 - x^2}|_{\partial\Omega},$$

by Theorem 2.5, $u \leq \sqrt{1-q_0}\sqrt{h^2 - x^2} = \sqrt{1-q_0}\sqrt{(1+by)^8 f^2 - x^2}$ in Ω; the result then follows by letting $b \to 0$.

Theorem 2.10. *Let* f, Ω, *and* u *be as in Theorem* 2.9, $\lim_{y\to\infty} f(y) = +\infty$, *and* $p(f) \geq p_0 > 0$, *where* $1 \geq p_0 > 0$ *is a constant. Suppose* $u|_{\partial\Omega} \leq \sqrt{1-p_0}\sqrt{f^2 - x^2}$. *Then* $u \leq \sqrt{1-p_0}\sqrt{f^2 - x^2}$ *in* Ω.

Proof. *Case* 1: $0 < p_0 < 1$. Let $q_0 = \min(1/8, p_0/2)$. Then

$$u|_{\partial\Omega} \leq \sqrt{1-p_0}\sqrt{f^2 - x^2}|_{\partial\Omega} \leq \sqrt{1-q_0}\sqrt{f^2 - x^2}|_{\partial\Omega}.$$

By Theorem 2.9, $u \le \sqrt{1-q_0}\sqrt{f^2-x^2}$ in Ω. Let

$$F = \sqrt{f^2-x^2}\sqrt{1-bh(y)^{-\alpha}},$$

where α is some constant in $(0,1)$ to be determined later, ε is a constant such that $0<\varepsilon<p_0$ and $b=p_0-\varepsilon$, and $h(y)=f(y_0+y)$, where y_0 is a positive constant. Since $f'>0$ and $\lim_{y\to\infty} f=+\infty$, we can choose $y_0>0$ such that $h\ge 1$ for every $y>0$. Let

$$\begin{aligned}
G &= f^2-x^2, \qquad g=(1-bh^{-\alpha})^{1/2},\\
g' &= \tfrac{1}{2}(1-bh^{-\alpha})^{-1/2}(-b)(-\alpha)h^{-\alpha-1}h'\\
&= \tfrac{1}{2}(1-bh^{-\alpha})^{-1/2}b\alpha h^{-\alpha-1}h',\\
g'' &= -\tfrac{1}{4}(1-bh^{-\alpha})^{-3/2}(-b)(-\alpha)h^{-\alpha-1}h'b\alpha h^{-\alpha}h'\\
&\quad + \tfrac{b\alpha}{2}(1-bh^{-\alpha})^{-1/2}((-\alpha-1)h^{-\alpha-2}h'^2+h^{-\alpha-1}h'').
\end{aligned}$$

But $p(h)\ge p_0>0$, so $h'^2\ge hh''$. Hence $g''\le 0$. By Lemma 2.1,

$$\begin{aligned}
G^{3/2}\mathfrak{M}F &= g\Big(-f^2f'^2+(f^2-x^2)(f'^2+ff'')-x^2-(f^2-x^2)\\
&\qquad\qquad +(f^2-x^2)2ff'\frac{g'}{g}+G^2\frac{g''}{g}\Big)\\
&\quad +g^3\Big(x^2(f'^2+ff'')-f^2f'^2\\
&\qquad +x^2\Big(G\frac{g''}{g}-3G\Big(\frac{g'}{g}\Big)^2\Big)-\Big(G_yG\frac{g'}{g}+G^2\Big(\frac{g'}{g}\Big)^2\Big)\Big)\\
&\le g(-f^2f'^2+(f^2-x^2)(f'^2+ff'')\\
&\quad -x^2-(f^2-x^2)+2ff'\frac{g'}{g}(f^2-x^2))\\
&\quad +g^3(x^2(f'^2+ff'')-f^2f'^2)\\
&\le (-g+g^3)(2-p(f))x^2f'^2+f^2f'^2g((1-p(f))-g^2)\\
&\quad -gf^2+2ff'g'(f^2-x^2)\\
&\le -(g-g^3)(2-p(f))x^2f'^2-f^2f'^2g\max(p(f)-2,0)\\
&\quad +f^2f'^2g((1-\min(2,p(f))-g^2)-gf^2+2ff'g'(f^2-x^2).
\end{aligned}$$

Since $1\ge g\ge\sqrt{1-b}$,

$$\begin{aligned}
G^{3/2}\mathfrak{M}F &\le f^2f'^2g(1-\min(2,p(f))-g^2)+2ff'g'f^2\\
&\le f^2f'^2g(-\min(2,p(f))+bh^{-\alpha})+2f^3f'\frac{1}{2}(1-bh^{-\alpha})^{-1/2}b\alpha h^{-\alpha-1}h'\\
&\le f^2f'^2\sqrt{1-b}(-\min(2,p(f))+bh^{-\alpha})+b\alpha h^{-\alpha-1}h'\frac{1}{\sqrt{1-b}}f^3f'\\
&\le f^2f'^2\sqrt{1-b}(-\min(2,p(f))+b)h^{-\alpha}+b\alpha h^{-\alpha-1}h'\frac{1}{\sqrt{1-b}}f^3f'\\
&\le h^{-\alpha}f^2f'^2\Big(-\sqrt{1-b}\varepsilon+\alpha bh^{-1}h'\frac{1}{\sqrt{1-b}}\frac{f}{f'}\Big).
\end{aligned}$$

Since $(f/f')' = p(f) > 0$, f/f' is monotone increasing. But $h(y) = f(y_0 + y)$, y_0 is a positive constant; therefore $\frac{h'}{h}\frac{f}{f'} \leq 1$. So when $0 < \alpha < (1-b)\varepsilon/b$, we have $G^{3/2}\mathfrak{M}F \leq 0$. Since

$$\lim_{y\to\infty} f(y) = +\infty, \qquad \lim_{y\to\infty} \sqrt{1 - bh(y)^{-\alpha}} \to 1,$$

we have $F(x, y) \geq \sqrt{1-q_0}\sqrt{f^2 - x^2}$ for sufficiently large y. Since $u \leq \sqrt{1-q_0}\sqrt{f^2-x^2}$ in Ω and $u|_{\partial\Omega} \leq F|_{\partial\Omega}$ by hypothesis, we have $u \leq F$ in Ω. So $u \leq \sqrt{1-bh^{-\alpha}}\sqrt{f^2-x^2}$ in Ω. Letting $\alpha \to 0$, $u \leq \sqrt{1-b}\sqrt{f^2-x^2} = \sqrt{1-(p_0-\varepsilon)}\sqrt{f^2-x^2}$ in Ω. Letting $\varepsilon \to 0$, we get $u \leq \sqrt{1-p_0}\sqrt{f^2-x^2}$ in Ω.

Case 2: $p_0 = 1$. Then for any constant p_1, $0 < p_1 < 1$, we have $p(f) \geq 1 \geq p_1$. Since $u|_{\partial\Omega} \leq \sqrt{1-1}\sqrt{f^2-x^2} \leq \sqrt{1-p_1}\sqrt{f^2-x^2}$, by Case 1, we have $u \leq \sqrt{1-p_1}\sqrt{f^2-x^2}$ in Ω. Letting $p_1 \to 1$, the result follows.

Since the case $\lim f < +\infty$ is not very important, we omit that case.

The case for a negative constant p_0 is studied in the following theorem.

Theorem 2.11. *Let $f \in C^0[0,\infty) \cap C^2(0,\infty)$, $f \geq 0$, $f' > 0$, $f_1 \in C^0[0,\infty)$, $f_1 > 0$, $\Omega \subset \{(x,y)|-f_1(y) < x < f_1(y),\ y > 0\}$, $u \in C^2(\Omega)\cap C^0(\bar{\Omega})$, $\operatorname{div} Tu \geq 0$ in Ω, and $p(f) \geq p_0$, where p_0 is a negative constant. Then if $f^2 \geq (a^2-1)(2-p_0)f_1^2/(a^2-(1-p_0))$ and $u|_{\partial\Omega} \leq a\sqrt{f^2-x^2}|_{\partial\Omega}$, where a is a positive constant satisfying $a^2 - 1 + p_0 > 0$, we have $u \leq a\sqrt{f^2-x^2}$ in Ω.*

Proof. Let $f_2 = e^{\alpha y} f$, where α is a positive constant. Then $f_2^2 \geq f^2 \geq (a^2-1)(2-p_0)f_1^2/(a^2-(1-p_0))$. By Lemma 2.6, $p(f_2) \geq p_0$. It is easy to see that $\limsup_{y\to\infty}(f_2/y^2) = +\infty$, by Lemma 2.8, and $\liminf_{y\to\infty}(f_2/f_2'^2) = 0$. Since $u|_{\partial\Omega} \leq a\sqrt{f^2-x^2}|_{\partial\Omega} \leq a\sqrt{f_2^2-x^2}|_{\partial\Omega}$, by Theorem 2.5, $u \leq a\sqrt{f_2^2-x^2} = a\sqrt{e^{2\alpha y}f^2-x^2}$ in Ω. The result then follows by letting $\alpha \to 0$.

Let $\Omega \subset \{-f_1(y) < x < f_1(y)|y > 0\}$ and $u|_{\partial\Omega} \leq af_1$, where a is a positive constant. The growing rate of u is stated in the following theorem.

Theorem 2.12. *Let $f_1 \in C^0[0,\infty) \cap C^2(0,\infty)$, $f_1, f_1' > 0$, $\lim f_1 = \infty$, and $p(f_1) \geq p_0 \geq 0$, where $1 \geq p_0 \geq 0$ is a constant. Moreover, let $\Omega \subset \{(x,y)| -f_1(y) < x < f_1(y),\ y > 0\}$ and $f = af_1$, where $a \geq 0$ is a constant. Then if $\operatorname{div} Tu \geq 0$ in Ω and $u|_{\partial\Omega} \leq f$, we have $u \leq \sqrt{(a^2+1-p_0)f_1^2 + (1-p_0)x^2}$ in Ω.*

Proof. Case 1: $1 > p_0 \geq 0$. Since

$$u|_{\partial\Omega} \leq f = af_1 \leq \sqrt{1-p_0}\sqrt{\frac{a^2 f_1^2}{1-p_0} + f_1^2 - x^2}|_{\partial\Omega}$$

and $p(\sqrt{\frac{a^2}{1-p_0}+1}f_1) = p(f) \geq p_0$,

$$\begin{aligned} u &\leq \sqrt{1-p_0}\sqrt{\frac{a^2}{1-p_0}f_1^2 + f_1^2 - x^2} \\ &= \sqrt{(a^2+(1-p_0))f_1^2 - (1-p_0)x^2} \quad \text{in } \Omega \end{aligned}$$

by Theorem 2.10.

Case 2: $p_0 = 1$. For every constant b where $1 \geq b \geq 0$, by Case 1, we have $u \leq \sqrt{(a^2 + (1-b))f_1^2 - (1-b)x^2}$ in Ω. The result then follows by letting $b \to 0$.

Now we assume that $p(f_1) \geq p_0$ with p_0 being a negative constant, and set $f^2 = (a^2-1)(2-p_0)f_1^2/(a^2-(1-p_0))$, where $a > \sqrt{1-p_0}$ is a constant. If $u|_{\partial\Omega} \leq a\sqrt{f^2 - x^2}|_{\partial\Omega} \leq a\sqrt{(a^2-1)(2-p_0)/(a^2-(1-p_0)) - 1}f_1$, by Theorem 2.11, we have $u \leq a\sqrt{f^2 - x^2}$ in Ω. Now we want to compute the minimum of $a\sqrt{(a^2-1)(2-p_0)/(a^2-(1-p_0)) - 1}$. For convenience, let $b = a^2$ and $q_1 = 1 - p_0 > 1$, and we need to compute the minimum of

$$b\left(\frac{(b-1)(1+q_1)}{b-q_1} - 1\right) = b\frac{q_1 b - 1}{b - q_1}.$$

Take logarithms and differentiate the above function; we have $1/b + q_1/(q_1 b - 1) = 1/(b - q_1)$. From this, we get $b = q_1 \pm \sqrt{q_1^2 - 1}$. Since $b = a^2 > q_1$, we have $b = q_1 + \sqrt{q_1^2 - 1}$. So the minimum of $b(q_1 b - 1)/(b - q_1)$ is

$$(q_1 + \sqrt{q_1^2 - 1})\left(\frac{q_1^2 + q_1\sqrt{q_1^2 - 1} - 1}{\sqrt{q_1^2 - 1}}\right) = (q_1 + \sqrt{q_1^2 - 1})^2,$$

and the minimum of $a\sqrt{(a^2-1)(2-p_0)/(a^2-(1-p_0)) - 1}$ is $q_1 + \sqrt{q_1^2 - 1}$.

Theorem 2.13. *Assume that* $f_1, f_1' > 0$, $p(f_1) \geq p_0$, *where* p_0 *is a negative constant,* $\Omega \subset \{-f_1(y) < x < f_1(y) | y > 0\}$, *and* $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$. *If* $\operatorname{div} Tu \geq 0$ *in* Ω, $q_1 = 1 - p_0 > 1$, *and* $u|_{\partial\Omega} \leq bf_1$, *where* $b \geq q_1 + \sqrt{q_1^2 - 1}$ *is a positive constant, then* $u \leq \sqrt{b}\sqrt{(b+1)f^2 - x^2}$ *in* Ω.

Proof. By the assumption that $b \geq q_1 + \sqrt{q_1^2 - 1}$, we have

$$(b - (q_1 + \sqrt{q_1^2 - 1}))(b - (q_1 - \sqrt{q_1^2 - 1})) \geq 0,$$

$b^2 - 2q_1 b + 1 \geq 0$, and $(b+1)(b-q_1) \geq (b-1)(1+q_1)$. If we set $a = b^2$, then $(a^2+1)(a^2-q_1) \geq (a^2-1)(1+q_1)$ and we have $a^2+1 \geq (a^2-1)(1+q_1)/(a^2-q_1)$. Since $u|_{\partial\Omega} \leq a\sqrt{(a^2+1)f_1^2 - x^2}|_{\partial\Omega}$, by Theorem 2.11, $u \leq a\sqrt{(a^2+1)f_1^2 - x^2}$ in Ω and we have $u \leq \sqrt{b}\sqrt{(b+1)f^2 - x^2}$ in Ω.

By Theorems 2.12 and 2.13, we know that the growth rate of u is of the same order as the shape of Ω and $u|_{\partial\Omega}$.

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