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LINKED PAIRS OF CONTRACTIBLE POLYHEDRA IN S^n

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ABSTRACT. B. Mazur has described a geometrically linked pair of compact contractible polyhedra in S^4. In this note we exhibit an even more extreme type of linking between compact contractible polyhedra in S^n, $n \geq 5$.

1. INTRODUCTION

Disjoint compacta $A_1, A_2 \subset S^n$ are *geometrically unlinked* if there is a PL embedding $f: S^{n-1} \to S^n$ so that $f(S^{n-1})$ separates S^n into components V_1 and V_2 with $A_1 \subset V_1$ and $A_2 \subset V_2$. In this case, $\bar{V}_1$ and $\bar{V}_2$ are contractible polyhedra (see (2) from §2), so by taking interiors of sufficiently small regular neighborhoods of $\bar{V}_1$ and $\bar{V}_2$ we see that if A_1 and A_2 are geometrically unlinked they also satisfy

Definition. Disjoint compacta $A_1, A_2 \subset S^n$ are *fundamentally unlinked* if there is a cover $\{U_1, U_2\}$ of S^n by contractible open sets so that $A_i \subset U_i$ for $i = 1, 2$ and $A_i \cap U_j = \varnothing$ when $i \neq j$.

If A_1 and A_2 are disjoint compact contractible polyhedra in S^n and $n \leq 3$, then they are geometrically unlinked. Indeed, if $N(A_1)$ is a regular neighborhood of A_1 disjoint from A_2, then $\partial N(A_1)$ is a PL $(n-1)$-sphere separating A_1 from A_2. In [Ma] Mazur made the surprising observation that, in S^4, a disjoint pair of compact contractible polyhedra may be geometrically linked. To do this, he constructed a compact contractible 4-manifold M (now known as a "Mazur manifold") which has nonsimply connected boundary and may be viewed as a regular neighborhood of a contractible 2-complex D contained in its interior. He then observes that the double, $M_1 \cup_\partial M_2$, of M is a PL 4-sphere and D_1 and D_2 are geometrically linked therein. Notice, however, that D_1 and D_2 are fundamentally unlinked.

A strategy similar to Mazur's may be used to produce pairs of geometrically linked, but fundamentally unlinked, compact contractible polyhedra in S^n for all $n \geq 4$. In this note we show that for $n \geq 5$ there exist fundamentally linked pairs of compact contractible polyhedra in S^n.

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2. Preliminaries

Throughout this paper we work in the PL category; all complexes are simplicial, manifolds are combinatorial, and maps are piecewise linear. All homology is with $\mathbb{Z}$-coefficients.

A group G is *perfect* if its abelianization, $G/[G, G]$, is the trivial group. A space X is *acyclic* if $\widetilde{H}_k(X) = 0$ for all k. A compact acyclic n-manifold is called a *homology n-cell*. An n-manifold with homology groups isomorphic to those of S^n is called a *homology n-sphere*.

The following facts are well known. They follow from standard results of algebraic topology including the VanKampen, Mayer-Vietoris, and Universal Coefficient theorems, as well as duality, the Hurewicz Theorem, and a theorem of Whitehead. We list them here for easy reference.

(1) The boundary of a homology n-cell is a homology $(n-1)$-sphere.

(2) If $\Sigma^{n-1} \subset S^n$ is a homology $(n-1)$-sphere and V_1 and V_2 are the components of $S^n - \Sigma^{n-1}$, then $\bar{V}_1$ and $\bar{V}_2$ are acyclic. If Σ^{n-1} is simply connected, then $\bar{V}_1$ and $\bar{V}_2$ are simply connected and thus contractible. If Σ^{n-1} is locally flat, then $\bar{V}_1$ and $\bar{V}_2$ are homology n-cells.

(3) The union of two homology n-cells among a common boundary is a homology n-sphere.

3. Main result

Theorem 3.1. *For any $n \geq 5$, there exists a fundamentally linked pair of compact contractible polyhedra in S^n.*

We will need the following lemmas. Both are tailored to the proof of Theorem 3.1 and could be stated in greater generality if so desired.

Lemma 3.2. *Let K be a finite acyclic 2-complex with fundamental group G. Then, for any $n \geq 5$, there exists a homology n-sphere Σ^n with $\pi_1(\Sigma^n) \cong G \times G$.*

Proof. For $n \geq 8$, we may embed $K \times K$ in $\mathbb{R}^{n+1}$. A regular neighborhood N of this embedding is a homology $(n+1)$-cell, so, by (1), ∂N is a homology n-sphere; moreover, by general position, $\pi_1(\partial N) \cong \pi_1(N) \cong G \times G$. Now, since $G \times G$ is the fundamental group of some high-dimensional homology sphere, the proof of Theorem 1 in [Ke], together with the remarks that precede it, show implicitly that there is an acyclic 3-complex, L, with $\pi_1(L) \cong G \times G$. Hence, for $n \geq 6$, we may use the same strategy as above. Finally, for $n = 5$, apply [St] to obtain a 3-complex $L' \subset \mathbb{R}^6$ which is simple homotopy equivalent to L, and let Σ^n be the boundary of a regular neighborhood of L'. □

Remark. Nonsimply connected, acyclic 2-complexes are plentiful. For example, removing the interior of a 3-ball from a nonsimply connected homology 3-sphere produces a homology 3-cell with the same fundamental group. This homology cell may then be collapsed onto a 2-dimensional subcomplex.

Lemma 3.3. *Let K be a finite complex with perfect fundamental group G. If K may be written as $U \cup V$, where U and V are open (not necessarily connected) subsets of K, such that loops lying completely within either U or V contract in K, then K is simply connected.*

Proof. By [Wr, Lemma 7.2], G must be a free group, but the only perfect free group is trivial. □

Proof of Theorem 3.1. Let K be an acyclic 2-complex with nontrivial fundamental group G. By Lemma 3.2, we may choose a homology n-sphere, Σ^n with $\pi_1(\Sigma^n, q) \cong G \times G$. Let $G_1, G_2, G_3 < \pi_1(\Sigma^n, q)$ correspond to $G \times \{1\}$, $\{1\} \times G$, $\Delta_G = \{(g, g) | g \in G\} < G \times G$, respectively. Choose PL embeddings $e_i \colon (K, p) \to (\Sigma^n, q)$ for $i = 1, 2, 3$ so that image$((e_i)_\# \colon \pi_1(K, p) \to \pi_1(\Sigma^n, q)) = G_i$, for each i. By general position, we may homotope e_1 and e_2 to embeddings e_1' and e_2' so that e_1' and e_2', and e_3 have pairwise disjoint images which we will denote by K_1, K_2, and K_3. Choose regular neighborhoods N_1 and N_2 of K_1 and K_2 so that N_1, N_2, and K_3 are pairwise disjoint. Let $W = \Sigma^n - \operatorname{int}(N_1 \cup N_2)$, and choose embedded arcs α_1 and α_2 in W from q to points $q_1 \in \partial N_1$ and $q_2 \in \partial N_2$, respectively. Since G_1 is a normal subgroup of $\pi_1(\Sigma^n, q)$ (thus, invariant under conjugation), image$(\pi_1(N_i \cup \alpha_i)) = G_i$ for $i = 1, 2$. Furthermore, since K_i has codimension ≥ 3, the inclusions $\Sigma^n - (K_1 \cup K_2) \subset \Sigma^n$ and $N_i - K_i \subset N_i$ $(i = 1, 2)$ induce π_1-isomorphisms. Utilizing the collar structures on $N_i - K_i$, we may conclude that $W \subset \Sigma^n$ and $\partial N_i \subset N_i$ induce π_1-isomorphisms. By a slight abuse of notation, we write $\pi_1(W, q) = G_1 \times G_2$ with image$(\pi_1(\partial N_i \cup \alpha_i, q) \to \pi_1(W, q)) = G_i$, $i = 1, 2$.

By (1) of §2, ∂N_1 and ∂N_2 are homology $(n-1)$-spheres; so, by [Ke, p. 71], there exist (combinatorial) compact contractible manifolds C_1 and C_2 with $\partial C_i \approx \partial N_i$ for each i. If $W \cup_\partial C_i$ denotes the space obtained by gluing ∂C_i to W along ∂N_i, VanKampen's theorem gives an isomorphism $\pi_1(W \cup_\partial C_i, q) \to (G_1 \times G_2)/G_i$, for $i = 1, 2$. Furthermore, since the composition $G_3 \to G_1 \times G_2 \to (G_1 \times G_2)/G_i$ is an isomorphism for $i = 1, 2$, we have inclusion induced isomorphisms, $\pi_1(K_3) \to \pi_1(W \cup_\partial C_i)$.

Reasoning as above,

$$\pi_1(W \cup_\partial (C_1 \cup C_2), q) \cong (G_1 \times G_2)/\langle G_1 \cup G_2 \rangle = \{1\}.$$

Furthermore, by two applications of (3), $W \cup_\partial (C_1 \cup C_2)$ is a homology sphere. Hence, by the PL Generalized Poincaré Conjecture [Sm], $W \cup_\partial (C_1 \cup C_2) \approx S^n$.

Claim. C_1 and C_2 are fundamentally linked in $W \cup_\partial (C_1 \cup C_2) \approx S^n$.

Suppose there is an open cover $\{U_1, U_2\}$ of $W \cup_\partial (C_1 \cup C_2)$ by contractible sets with $C_i \subset U_i$ for $i = 1, 2$ and $C_i \cap U_j = \varnothing$ when $i \neq j$. Then $\{U_1 \cap K_3, U_2 \cap K_3\}$ is an open cover of K_3. By Lemma 3.3, we may assume without loss of generality that $U_1 \cap K_3$ contains a loop λ which is nontrivial in K_3. Now, U_1 is contractible, so λ contracts in $U_1 \subset W \cup_\partial C_1$. But, since $K_3 \subset W \cup_\partial C_1$ induces a π_1-isomorphism, this is impossible. □

Remark. In the above construction, the contractibility of U_i was only used to assert that a loop $\lambda \subset U_i$ contracts in U_i. Hence, we have actually shown that S^n cannot be covered by simply connected open sets U_1 and U_2 containing C_1 and C_2, respectively, and with $U_i \cap C_j = \varnothing$ for $i \neq j$.

Question. Does there exist a pair of fundamentally linked compact contractible polyhedra in S^4?

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