REPRESENTING CHARACTERISTIC HOMOLOGY
CLASSES OF $mCP^2\#n\overline{CP}^2$

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ABSTRACT. We prove the following theorems.

Theorem 1. If $m, n \geq 1$, $x \in H_2(mCP^2\#n\overline{CP}^2)$ is a characteristic homology class with $x^2 = 16l + m - n > 0$ and

1. $m < 3l + 1$ provided $l \geq 0$, or
2. $m < -19l + 1$ provided $l < 0$.

Suppose that the 11/8-conjecture is true. Then $x$ cannot be represented by a smoothly embedded 2-sphere.

Theorem 2. Let $m, n \geq 4l > 0$, $x \in H_2(mCP^2\#n\overline{CP}^2)$ be a primitive characteristic homology class with $x^2 = \pm 16l + m - n$. Then $x$ can be represented by a smoothly embedded 2-sphere.

1. INTRODUCTION

In this note we consider the problem of representing some 2-dimensional characteristic homology classes of $mCP^2\#n\overline{CP}^2$ by smoothly embedded 2-spheres.

In recent years, with the results of Donaldson [D1, D2] great progress has been made in determining whether a 2-dimensional homology class of a 4-manifold can be represented by a smoothly embedded 2-sphere (see [GG1, GG2, Go, K, L, Li, Lu]). Most of the results are negative. For the positive answers, the problem is often reduced by the results of Wall [W1, W2] to representing some special homology classes by constructing smooth embeddings.

For instance Li [Li] obtained partial results for $n(CP^2\#CP^2)$, $4 \leq n \leq 7$; he determined all the cases except for $x \in H_2(n(CP^2\#CP^2))$ being primitive characteristic with $|x^2| = 16l$, $l \geq 1$.

For $4 \leq n \leq 6$ and $l \geq 2$, it is relevant to the 11/8-conjecture, i.e., if the 11/8-conjecture is true, such representing does not exist. In fact, we wish to prove the following
Theorem 1. If \( m, n \geq 1, x \in H_2(m\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}) \) is a characteristic homology class with \( x^2 = 16l + m - n > 0 \) and one of the following conditions is satisfied:

1. \( m < 3l + 1 \) if \( l \geq 0 \),
2. \( m < -19l + 1 \) if \( l < 0 \).

Suppose that the 11/8-conjecture is true. Then \( x \) cannot be represented by a smoothly embedded 2-sphere.

Remark 1. If \( x \in H_2(m\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}) \) is a characteristic homology class and can be represented by a smoothly embedded 2-sphere, then by [KM] we have \( x^2 \equiv m - n \) mod 16, i.e., \( x^2 = 16l + m - n \) for some \( l \).

Remark 2. If \( x^2 = -16/ + m - n < 0 \), we only need to exchange \( m \) and \( n \) in Theorem 1.

Example 1. If \( n < 4, x \in H_2(n(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2})) \), and \( x \) is characteristic with \( x^2 = 16 \), then \( x \) cannot be represented by a smoothly embedded 2-sphere because the 11/8-conjecture is true for rank= signature = 2 or 4 by Donaldson [D2].

Example 2. If \( n < 7, x \in H_2(n(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2})) \), \( x \) is characteristic with \( x^2 = 32 \), and the 11/8-conjecture is true for rank= signature = 10, then \( x \) cannot be represented by a smoothly embedded 2-sphere.

Therefore, one of the most interesting cases is whether a primitive characteristic homology class \( x \in H_2(4(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2})) \) with \( x^2 = 16 \) is represented by a smoothly embedded 2-sphere. Using the method in [FK] we obtain results, such as

Theorem 2. Let \( m, n \geq 4l > 0, x \in H_2(m\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}) \) be a primitive characteristic homology class with \( x^2 = \pm 16l + m - n \). Then \( x \) can be represented by a smoothly embedded 2-sphere.

It is sufficient to prove the following special case of Theorem 2.

Theorem 2'. Let \( x \in H_2(4(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2})) \) be a primitive characteristic homology class with \( |x^2| = 16 \). Then \( x \) can be represented by a smoothly embedded 2-sphere.

Theorem 2' implies Theorem 2. By [W1] and [W2] it is sufficient to find a special primitive characteristic homology class \( y \in H_2(m\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}) \) with \( y^2 = \pm 16l + m - n \), which can be represented by a smoothly embedded 2-sphere. Write \( m\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2} = (4(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2})) \# (m - 4l)\mathbb{CP}^2 \# (n - 4l)\overline{\mathbb{CP}^2} \) and set

\[
y = x_1 + x_2 + \cdots + x_l + y_1 + y_2 + \cdots + y_{m-4l} + \delta_1 + \delta_2 + \cdots + \delta_{n-4l}
\]

where \( x_i \)'s are the images of \( x \in H_2(4(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2})) \) in Theorem 2', \( y_j \)'s are the images of the standard generator of \( H_2(\mathbb{CP}^2) \), and \( \delta_k \)'s are the images of the standard generator of \( H_2(\overline{\mathbb{CP}^2}) \), in \( H_2((4(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2})) \# (m - 4l)\mathbb{CP}^2 \# (n - 4l)\overline{\mathbb{CP}^2}) \). \( y \) is primitive characteristic with \( y^2 = \pm 16l + m - n \). Tube the 2-spheres which represent \( x_i \)'s, \( y_j \)'s, and \( \delta_k \)'s, one obtains a smoothly embedded 2-sphere which represents \( y \).
2. Proof of Theorem 1

Set

\[ M = m\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2} \# Q_1 \# \cdots \# Q_{16l+m-n-1}, \]

where the \( Q \)'s are copies of \( \overline{\mathbb{CP}^2} \). The image of \( x \) in \( H_2(M) \) is still denoted by \( x \) and the image of the standard generator of \( H_2(\overline{\mathbb{CP}^2}) \) in \( H_2(M) \) corresponding to \( Q_i \) by \( \delta_i \). Let

\[ y = x + \sum_{i=1}^{16l+m-n-1} \delta_i; \]

then \( y \) is a characteristic homology class with \( y^2 = 1 \) in \( H_2(M) \).

Suppose that \( x \) can be represented by a smoothly embedded 2-sphere. Then \( y \) can also be represented by a smoothly embedded 2-sphere \( S \). Since \( y^2 = 1 \), surger the tubular neighbourhood \( N \) of \( S \) from \( M \), one obtains a simply connected smooth 4-manifold

\[ M_1 = (M \setminus N) \cup D^4 \]

and gets

\[ M = M_1 \# \mathbb{CP}^2. \]

Since \( y \) is characteristic, \( M_1 \) is spin \((w_2 = 0)\) and has even intersection form.

The 11/8-conjecture says that

\[ 8 \operatorname{rank}(M_1) \geq 11|\operatorname{sign}(M_1)|. \]

But

\[ 8 \operatorname{rank}(M_1) = 16m + 128l - 16 \quad \text{and} \quad 11|\operatorname{sign}(M_1)| = 176|l|. \]

Thus \( 8 \operatorname{rank}(M_1) < 11 \times |\operatorname{sign}(M_1)| \) is equivalent to the fact that either (1) or (2) holds. This completes the proof of Theorem 1.

3. Proof of Theorem 2

Let \( N \) be an oriented smooth 4-manifold, \( K \subset N \) an orientable closed surface, and \( A \) an oriented simple closed curve representing an element of a symplectic basis of \( H_1(K) \) and spanning an oriented smoothly embedded disk \( B \) transversely to \( K \) in \( N \) such that \( B \cap K = A \) and \( \text{Int} B \cap K = \emptyset \). Then there is an obstruction, the Euler class \( \chi(B, A) \in H^2(B, A) \), to extending a section of \( \nu_{A\to K} \), the normal bundle of \( A \) in \( K \), to a section of \( \nu_{B\to N} \), the normal bundle of \( B \) in \( N \) (see [FK]).

Lemma 1. If \( \chi(A) = \chi(B, A)[B, A] = \pm 2 \), then one can surger on \( K \) in \( N \# S^2 \times S^2 \) and obtain an orientable surface \( K' \) with genus reduced by 1: \( \text{genus}(K') = \text{genus}(K) - 1 \).

Proof. Let \( S_1 = \{(x, x) \in S^2 \times S^2 \} \) and \( S_2 = \{(x, -x) \in S^2 \times S^2 \} \) in \( S^2 \times S^2 \). Then \( [S_1] \cdot [S_1] = 2 \) and \( [S_2] \cdot [S_2] = -2 \), i.e., the Euler number \( \chi(\nu_{S_1\to S_2 \times S^2}) = 2 \) and \( \chi(\nu_{S_2\to S_2 \times S^2}) = -2 \). (There is an error in [FK, p. 94]. The Euler numbers of both \( \Delta(S^2) \) and \( -\Delta(S^2) \) should be 2.) Take the connected sum of \( (N, B) \) with \( (S^2 \times S^2, S_1 \) or \( S_2) \) according to \( \chi(A) = -2 \) or 2. Then we obtain

\[ \chi(\nu_{B\#S_1\to N\#S^2 \times S^2}) = 0 \]
and can surger on \( K \) along \( B \# S_i \). The result of the surgery is a surface \( K' \) with genus(\( K' \)) = genus(\( K \)) - 1.

Let \( N \) be a smooth 4-manifold, \( K \subset N \) an oriented closed surface, and \( A_1, \ldots, A_h \subset K \) a family of simple closed curves. Let \( B_1, \ldots, B_h \subset N \) be a family of 2-disks such that \( B_i \cap K = A_i \), \( \text{Int} B_i \cap K = \emptyset \), \( B_i \cap B_j = A_i \cap A_j \), and \( \text{Int} B_i \cap \text{Int} B_j = \emptyset \), \( i \neq j \). We say that family \( A_1, \ldots, A_h \) simply spans family \( B_1, \ldots, B_h \) in \( N \).

**Lemma 2.** Let \( \gamma \) be the standard generator of \( H_2(\mathbb{CP}^2) \). Then \( 3\gamma \) can be represented by an oriented smoothly embedded torus \( T \), and there are two oriented simple closed curves \( A_1, A_2 \subset T \) with \( A_1 \cap A_2 = \text{single point} \), which represent a symplectic basis of \( H_1(T) \), simply span 2-disks, and have Euler obstructions \( \pm 1 \).

**Proof.** See the proof of Lemma 6 in [FK].

**Lemma 3.** Let \( N \) and \( N' \) be two oriented smooth 4-manifolds, \( T \subset N \) and \( T' \subset N' \) two oriented tori, and \( A_1, A_2 \subset T \) and \( A'_1, A'_2 \subset T' \) oriented simple closed curves representing symplectic bases \( a_1, a_2 \) of \( H_1(T) \) and \( a'_1, a'_2 \) of \( H_1(T') \) respectively. If family \( A_1, A_2 \) and family \( A'_1, A'_2 \) simply span family \( B_1, B_2 \) in \( N \) and family \( B'_1, B'_2 \) in \( N' \) respectively, then in \( T \# T' \subset N \# N' \) there are two oriented simple closed curves \( C_1 \) and \( C_2 \) which simply span oriented 2-disks \( D_1 \) and \( D_2 \) in \( N \# N' \) with

\[
\chi(C_1) = \chi(A_1) + \chi(A'_1) \quad \text{and} \quad \chi(C_2) = \chi(A_2) + \chi(A'_2),
\]

where \( C_1 \) and \( C_2 \) represent \( a_1 + a'_1 \) and \( a_2 - a'_2 \in H_1(T \# T') \), respectively, and \( a_1, a'_1, a_2, a'_2 \) represented by \( A_1, A'_1, A_2, A'_2 \) form a symplectic basis of \( H_1(T \# T') \) with \( a_1 \cdot a_2 = 1 = a'_1 \cdot a'_2 \) and \( a_1 \cdot a'_1 = a_1 \cdot a'_2 = a'_1 \cdot a_2 = a_2 \cdot a'_2 = 0 \).

**Proof.** Let \( \alpha \) be an oriented arc in \( T \# T' \) joining \( A_1 \) and \( A'_1 \) such that the intersection numbers of \( \alpha \cap A_1 \) and \( \alpha \cap A'_1 \) are +1 and -1, respectively. The tubular neighborhood \( V(\alpha) \) of \( \alpha \) in \( T \# T' \) is a band connecting \( B_1 \) and \( B'_1 \). Let \( C_1 \) be the boundary of \( B_1 \cup V(\alpha) \cup B'_1 \). Pushing the interior of \( V(\alpha) \) off \( T \# T' \) slightly, one obtains an oriented embedded 2-disk \( D_1 \cong B_1 \cup V(\alpha) \cup B'_1 \) spanned by \( \partial D_1 = C_1 = A_1 \# A'_1 \). Do the same with \( A_2 \) and \( A'_2 \), but use the oriented arc which joins \( A_2 \) and \( A'_2 \) being the segment of \( C_1 \) cut by \( A_2 \) and \( A'_2 \) with both intersection numbers +1. One obtains \( C_2 = A_2 \# A'_2 \) and a 2-disk \( D_2 \) with \( \partial D_2 = C_2 \). We get

\[
\chi(C_1) = \chi(A_1) + \chi(A'_1) \quad \text{and} \quad \chi(C_2) = \chi(A_2) + \chi(A'_2).
\]

Now we are going to prove Theorem 2'. By [W1] and [W2] it is sufficient to find a special primitive characteristic homology class \( x \in H_2(4(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2})) \) with \( x^2 = 16 \) which can be represented by a smoothly embedded 2-sphere.

Note that \( 4(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}) \) is diffeomorphic to \( 2(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}) \# 2(S^2 \times S^2) \) and \( y = 3\gamma_1 + 3\gamma_2 + \delta_1 + \delta_2 \in H_2(2(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}) \# 2(S^2 \times S^2)) \) is a primitive characteristic homology class with \( y^2 = 16 \), where \( \gamma_1, \gamma_2 \) are the images of the standard generator of \( H_2(\mathbb{CP}^2) \) and \( \delta_1, \delta_2 \) the images of the standard generator of \( H_2(\overline{\mathbb{CP}^2}) \) in \( H_2(2(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}) \# 2(S^2 \times S^2)) \).

Applying Lemmas 2 and 3 we can represent \( 3\gamma_1 + 3\gamma_2 \) by an oriented smoothly embedded double torus \( K \) in \( 2\mathbb{CP}^2 \) and obtain two oriented simple closed
curves $C_1$ and $C_2$ simply spanning 2-disks $D_1$ and $D_2$ such that
\[ \chi(C_1) = \pm 2 \quad \text{and} \quad \chi(C_2) = \pm 2. \]

By Lemma 1, $3\gamma_1 + 3\gamma_2$ can be represented by a smoothly embedded 2-sphere in $2CP^2\#2(S^2 \times S^2)$. Since the standard generator of $H_2(\mathbb{C}P^2)$ can be represented by $CP^1 = S^2$, $3\gamma_1 + 3\gamma_2 + \delta_1 + \delta_2 \in H_2(2\mathbb{C}P^2\#2(S^2 \times S^2))$ can be represented by a smoothly embedded 2-sphere in $2(\mathbb{C}P^2\#\mathbb{C}P^2)\#2(S^2 \times S^2)$. This completes the proof.

REFERENCES


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