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REPRESENTING CHARACTERISTIC HOMOLOGY CLASSES OF $mCP^2\#n\overline{CP}^2$

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ABSTRACT. We prove the following theorems.

Theorem 1. *If $m, n \geq 1$, $x \in H_2(mCP^2\#n\overline{CP}^2)$ is a characteristic homology class with $x^2 = 16l + m - n > 0$ and*
(1) $m < 3l + 1$ provided $l \geq 0$, or
(2) $m < -19l + 1$ provided $l < 0$.
Suppose that the 11/8-conjecture is true. Then x cannot be represented by a smoothly embedded 2-sphere.

Theorem 2. *Let $m, n \geq 4l > 0$, $x \in H_2(mCP^2\#n\overline{CP}^2)$ be a primitive characteristic homology class with $x^2 = \pm 16l + m - n$. Then x can be represented by a smoothly embedded 2-sphere.*

1. INTRODUCTION

In this note we consider the problem of representing some 2-dimensional characteristic homology classes of $mCP^2\#n\overline{CP}^2$ by smoothly embedded 2-spheres.

In recent years, with the results of Donaldson [D1, D2] great progress has been made in determining whether a 2-dimensional homology class of a 4-manifold can be represented by a smoothly embedded 2-sphere (see [GG1, GG2, Go, K, L, Li, Lu]). Most of the results are negative. For the positive answers, the problem is often reduced by the results of Wall [W1, W2] to representing some special homology classes by constructing smooth embeddings.

For instance Li [Li] obtained partial results for $n(CP^2\#\overline{CP}^2)$, $4 \leq n \leq 7$; he determined all the cases except for $x \in H_2(n(CP^2\#\overline{CP}^2))$ being primitive characteristic with $|x^2| = 16l$, $l \geq 1$.

For $4 \leq n \leq 6$ and $l \geq 2$, it is relevant to the 11/8-conjecture, i.e., if the 11/8-conjecture is true, such representing does not exist. In fact, we wish to prove the following

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Theorem 1. *If $m, n \geq 1$, $x \in H_2(mCP^2\#n\overline{CP}^2)$ is a characteristic homology class with $x^2 = 16l + m - n > 0$ and one of the following conditions is satisfied*:

(1) $m < 3l + 1$ *if* $l \geq 0$,
(2) $m < -19l + 1$ *if* $l < 0$.

Suppose that the $11/8$*-conjecture is true. Then* x *cannot be represented by a smoothly embedded 2-sphere.*

Remark 1. If $x \in H_2(mCP^2\#n\overline{CP}^2)$ is a characteristic homology class and can be represented by a smoothly embedded 2-sphere, then by [KM] we have $x^2 \equiv m - n \mod 16$, i.e., $x^2 = 16l + m - n$ for some l.

Remark 2. If $x^2 = -16l + m - n < 0$, we only need to exchange m and n in Theorem 1.

Example 1. If $n < 4$, $x \in H_2(n(CP^2\#\overline{CP}^2))$, and x is characteristic with $x^2 = 16$, then x cannot be represented by a smoothly embedded 2-sphere because the $11/8$-conjecture is true for rank$-|$signature$| = 2$ or 4 by Donaldson [D2].

Example 2. If $n < 7$, $x \in H_2(n(CP^2\#\overline{CP}^2))$, x is characteristic with $x^2 = 32$, and the $11/8$-conjecture is true for rank $-|$signature$| \leq 10$, then x cannot be represented by a smoothly embedded 2-sphere.

Therefore, one of the most interesting cases is whether a primitive characteristic homology class $x \in H_2(4(CP^2\#\overline{CP}^2))$ with $x^2 = 16$ is represented by a smoothly embedded 2-sphere. Using the method in [FK] we obtain results, such as

Theorem 2. *Let* $m, n \geq 4l > 0$, $x \in H_2(mCP^2\#n\overline{CP}^2)$ *be a primitive characteristic homology class with* $x^2 = \pm 16l + m - n$. *Then* x *can be represented by a smoothly embedded 2-sphere.*

It is sufficient to prove the following special case of Theorem 2.

Theorem $2'$. *Let* $x \in H_2(4(CP^2\#\overline{CP}^2))$ *be a primitive characteristic homology class with* $|x^2| = 16$. *Then* x *can be represented by a smoothly embedded 2-sphere.*

Theorem $2'$ implies Theorem 2. By [W1] and [W2] it is sufficient to find a special primitive characteristic homology class $y \in H_2(mCP^2\#n\overline{CP}^2)$ with $y^2 = \pm 16l + m - n$, which can be represented by a smoothly embedded 2-sphere. Write $mCP^2\#n\overline{CP}^2 = l[4(CP^2\#\overline{CP}^2)]\#(m-4l)CP^2\#(n-4l)\overline{CP}^2$ and set

$$y = x_1 + x_2 + \cdots + x_l + \gamma_1 + \gamma_2 + \cdots + \gamma_{m-4l} + \delta_1 + \delta_2 + \cdots + \delta_{n-4l}$$

where x_i's are the images of $x \in H_2(4(CP^2\#\overline{CP}^2))$ in Theorem $2'$, γ_j's the images of the standard generator of $H_2(CP^2)$, and δ_k's the images of the standard generator of $H_2(\overline{CP}^2)$, in $H_2(l[4(CP^2\#\overline{CP}^2)]\#(m-4l)CP^2\#(n-4l)\overline{CP}^2)$. y is primitive characteristic with $y^2 = \pm 16l + m - n$. Tube the 2-spheres which represent x_i's, γ_j's, and δ_k's, one obtains a smoothly embedded 2-sphere which represents y.

2. Proof of Theorem 1

Set

$$M = mCP^2\#n\overline{CP}^2\#Q_1\#\cdots\#Q_{16l+m-n-1},$$

where the Q's are copies of $\overline{CP}^2$. The image of x in $H_2(M)$ is still denoted by x and the image of the standard generator of $H_2(\overline{CP}^2)$ in $H_2(M)$ corresponding to Q_i by δ_i. Let

$$y = x + \sum_{i=1}^{16l+m-n-1} \delta_i;$$

then y is a characteristic homology class with $y^2 = 1$ in $H_2(M)$.

Suppose that x can be represented by a smoothly embedded 2-sphere. Then y can also be represented by a smoothly embedded 2-sphere S. Since $y^2 = 1$, surger the tubular neighbourhood N of S from M, one obtains a simply connected smooth 4-manifold

$$M_1 = (M\backslash N) \cup D^4$$

and gets

$$M = M_1\#CP^2.$$

Since y is characteristic, M_1 is spin $(w_2 = 0)$ and has even intersection form.

The $11/8$-conjecture says that

$$8\,\mathrm{rank}(M_1) \geq 11|\mathrm{sign}(M_1)|.$$

But

$$8\,\mathrm{rank}(M_1) = 16m + 128l - 16 \quad \text{and} \quad 11|\mathrm{sign}(M_1)| = 176|l|.$$

Thus $8\,\mathrm{rank}(M_1) < 11 \times |\mathrm{sign}(M_1)|$ is equivalent to the fact that either (1) or (2) holds. This completes the proof of Theorem 1.

3. Proof of Theorem 2′

Let N be an oriented smooth 4-manifold, $K \subset N$ an orientable closed surface, and A an oriented simple closed curve representing an element of a symplectic basis of $H_1(K)$ and spanning an oriented smoothly embedded disk B transversely to K in N such that $B \cap K = A$ and $\mathrm{Int}\,B \cap K = \varnothing$. Then there is an obstruction, the Euler class $\chi(B, A) \in H^2(B, A)$, to extending a section of $\nu_{A\hookrightarrow K}$, the normal bundle of A in K, to a section of $\nu_{B\hookrightarrow N}$, the normal bundle of B in N (see [FK]).

Lemma 1. *If $\chi(A) = \chi(B, A)[B, A] = \pm 2$, then one can surger on K in $N\#S^2 \times S^2$ and obtain an orientable surface K' with genus reduced by* 1: $\mathrm{genus}(K') = \mathrm{genus}(K) - 1$.

Proof. Let $S_1 = \{(x, x) \in S^2 \times S^2\}$ and $S_2 = \{(x, -x) \in S^2 \times S^2\}$ in $S^2 \times S^2$. Then $[S_1]\cdot[S_1] = 2$ and $[S_2]\cdot[S_2] = -2$, i.e., the Euler number $\chi(\nu_{S_1\hookrightarrow S^2\times S^2}) = 2$ and $\chi(\nu_{S_2\hookrightarrow S^2\times S^2}) = -2$. (There is an error in [FK, p. 94]. The Euler numbers of both $\Delta(S^2)$ and $-\Delta(S^2)$ should be 2.) Take the connected sum of (N, B) with $(S^2 \times S^2, S_1$ or $S_2)$ according to $\chi(A) = -2$ or 2. Then we obtain

$$\chi(\nu_{B\#S_i\hookrightarrow N\#S^2\times S^2}) = 0$$

and can surger on K along $B\#S_i$. The result of the surgery is a surface K' with $\text{genus}(K') = \text{genus}(K) - 1$.

Let N be a smooth 4-manifold, $K \subset N$ an oriented closed surface, and $A_1, \dots, A_h \subset K$ a family of simple closed curves. Let $B_1, \dots, B_h \subset N$ be a family of 2-disks such that $B_i \cap K = A_i$, $\text{Int}\, B_i \cap K = \varnothing$, $B_i \cap B_j = A_i \cap A_j$, and $\text{Int}\, B_i \cap \text{Int}\, B_j = \varnothing$, $i \neq j$. We say that family $A_1, \dots, A_h$ simply spans family $B_1, \dots, B_h$ in N.

Lemma 2. *Let γ be the standard generator of $H_2(CP^2)$. Then 3γ can be represented by an oriented smoothly embedded torus T, and there are two oriented simple closed curves A_1 and $A_2 \subset T$ with $A_1 \cap A_2 =$ single point, which represent a symplectic basis of $H_1(T)$, simply span 2-disks, and have Euler obstructions ± 1.*

Proof. See the proof of Lemma 6 in [FK].

Lemma 3. *Let N and N' be two oriented smooth 4-manifolds, $T \subset N$ and $T' \subset N'$ two oriented tori, and $A_1, A_2 \subset T$ and $A'_1, A'_2 \subset T$ oriented simple closed curves representing symplectic bases a_1, a_2 of $H_1(T)$ and a'_1, a'_2 of $H_1(T')$ respectively. If family A_1, A_2 and family A'_1, A'_2 simply span family B_1, B_2 in N and family B'_1, B'_2 in N' respectively, then in $T\#T' \subset N\#N'$ there are two oriented simple closed curves C_1 and C_2 which simply span oriented 2-disks D_1 and D_2 in $N\#N'$ with*

$$\chi(C_1) = \chi(A_1) + \chi(A'_1) \quad \textit{and} \quad \chi(C_2) = \chi(A_2) + \chi(A'_2),$$

where C_1 and C_2 represent $a_1 + a'_1$ and $a_2 - a'_2 \in H_1(T\#T')$, respectively, and a_1, a'_1, a_2, a'_2 represented by A_1, A'_1, A_2, A'_2 form a symplectic basis of $H_1(T\#T')$ with $a_1 \cdot a_2 = 1 = a'_1 \cdot a'_2$ and $a_1 \cdot a'_1 = a_1 \cdot a'_2 = a'_1 \cdot a_2 = a_2 \cdot a'_2 = 0$.

Proof. Let α be an oriented arc in $T\#T'$ joining A_1 and A'_1 such that the intersection numbers of $\alpha \cap A_1$ and $\alpha \cap A'_1$ are $+1$ and -1, respectively. The tubular neighbourhood $V(\alpha)$ of α in $T\#T'$ is a band connecting B_1 and B'_1. Let C_1 be the boundary of $B_1 \cup V(\alpha) \cup B'_1$. Pushing the interior of $V(\alpha)$ off $T\#T'$ slightly, one obtains an oriented embedded 2-disk $D_1 \cong B_1 \cup V(\alpha) \cup B'_1 \cong B_1 \natural B'_1$ spanned by $\partial D_1 = C_1 = A_1 \# A'_1$. Do the same with A_2 and A'_2, but use the oriented arc which joins A_2 and A'_2 being the segment of C_1 cut by A_2 and A'_2 with both intersection numbers $+1$. One obtains $C_2 = A_2 \# \overline{A'_2}$ and a 2-disk D_2 with $\partial D_2 = C_2$. We get

$$\chi(C_1) = \chi(A_1) + \chi(A'_1) \quad \text{and} \quad \chi(C_2) = \chi(A_2) + \chi(A'_2).$$

Now we are going to prove Theorem 2'. By [W1] and [W2] it is sufficient to find a special primitive characteristic homology class $x \in H_2(4(CP^2\#\overline{CP}^2))$ with $x^2 = 16$ which can be represented by a smoothly embedded 2-sphere.

Note that $4(CP^2\#\overline{CP}^2)$ is diffeomorphic to $2(CP^2\#\overline{CP}^2)\#2(S^2 \times S^2)$ and $y = 3\gamma_1 + 3\gamma_2 + \delta_1 + \delta_2 \in H_2(2(CP^2\#\overline{CP}^2)\#2(S^2 \times S^2))$ is a primitive characteristic homology class with $y^2 = 16$, where γ_1, γ_2 are the images of the standard generator of $H_2(CP^2)$ and δ_1, δ_2 the images of the standard generator of $H_2(\overline{CP}^2)$ in $H_2(2(CP^2\#\overline{CP}^2)\#2(S^2 \times S^2))$.

Applying Lemmas 2 and 3 we can represent $3\gamma_1 + 3\gamma_2$ by an oriented smoothly embedded double torus K in $2CP^2$ and obtain two oriented simple closed curves C_1 and C_2 simply spanning 2-disks D_1 and D_2 such that

$$\chi(C_1) = \pm 2 \quad \text{and} \quad \chi(C_2) = \pm 2.$$

By Lemma 1, $3\gamma_1 + 3\gamma_2$ can be represented by a smoothly embedded 2-sphere in $2CP^2\#2(S^2 \times S^2)$. Since the standard generator of $H_2(\overline{CP}^2)$ can be represented by $CP^1 = S^2$, $3\gamma_1 + 3\gamma_2 + \delta_1 + \delta_2 \in H_2(2(CP^2\#\overline{CP}^2)\#2(S^2 \times S^2))$ can be represented by a smoothly embedded 2-sphere in $2(CP^2\#\overline{CP}^2)\#2(S^2 \times S^2)$. This completes the proof.

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