REPRESENTING CHARACTERISTIC HOMOLOGY CLASSES OF $mCP^2#nCP^2$

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Abstract. We prove the following theorems.

Theorem 1. If $m, n \geq 1$, $x \in H_2(mCP^2#nCP^2)$ is a characteristic homology class with $x^2 = 16l + m - n > 0$ and

1. $m < 3l + 1$ provided $l \geq 0$, or
2. $m < -9l + 1$ provided $l < 0$.

Suppose that the 11/8-conjecture is true. Then $x$ cannot be represented by a smoothly embedded 2-sphere.

Theorem 2. Let $m, n \geq 4l > 0$, $x \in H_2(mCP^2#nCP^2)$ be a primitive characteristic homology class with $x^2 = \pm 16l + m - n$. Then $x$ can be represented by a smoothly embedded 2-sphere.

1. Introduction

In this note we consider the problem of representing some 2-dimensional characteristic homology classes of $mCP^2#nCP^2$ by smoothly embedded 2-spheres.

In recent years, with the results of Donaldson [D1, D2] great progress has been made in determining whether a 2-dimensional homology class of a 4-manifold can be represented by a smoothly embedded 2-sphere (see [GG1, GG2, Go, K, L, Li, Lu]). Most of the results are negative. For the positive answers, the problem is often reduced by the results of Wall [W1, W2] to representing some special homology classes by constructing smooth embeddings.

For instance Li [Li] obtained partial results for $n(CP^2#CP^2)$, $4 \leq n \leq 7$; he determined all the cases except for $x \in H_2(n(CP^2#CP^2))$ being primitive characteristic with $|x^2| = 16l$, $l \geq 1$.

For $4 \leq n \leq 6$ and $l \geq 2$, it is relevant to the 11/8-conjecture, i.e., if the 11/8-conjecture is true, such representing does not exist. In fact, we wish to prove the following

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Theorem 1. If \( m, n \geq 1 \), \( x \in H_2(m \mathbb{C}P^2 \# n \overline{\mathbb{C}P^2}) \) is a characteristic homology class with \( x^2 = 16l + m - n > 0 \) and one of the following conditions is satisfied:

1. \( m < 3l + 1 \) if \( l \geq 0 \),
2. \( m < -19l + 1 \) if \( l < 0 \).

Suppose that the 11/8-conjecture is true. Then \( x \) cannot be represented by a smoothly embedded 2-sphere.

Remark 1. If \( x \in H_2(m \mathbb{C}P^2 \# n \overline{\mathbb{C}P^2}) \) is a characteristic homology class and can be represented by a smoothly embedded 2-sphere, then by [KM] we have \( x^2 \equiv m - n \mod 16 \), i.e., \( x^2 = 16l + m - n \) for some \( l \).

Remark 2. If \( x^2 = -16l + m - n < 0 \), we only need to exchange \( m \) and \( n \) in Theorem 1.

Example 1. If \( n < 4 \), \( x \in H_2(n(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2})) \), and \( x \) is characteristic with \( x^2 = 16 \), then \( x \) cannot be represented by a smoothly embedded 2-sphere because the 11/8-conjecture is true for rank \( \sigma \) signature \( |\sigma| = 2 \) or 4 by Donaldson [D2].

Example 2. If \( n < 7 \), \( x \in H_2(n(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2})) \), \( x \) is characteristic with \( x^2 = 32 \), and the 11/8-conjecture is true for rank \( \sigma \) signature \( |\sigma| \leq 10 \), then \( x \) cannot be represented by a smoothly embedded 2-sphere.

Therefore, one of the most interesting cases is whether a primitive characteristic homology class \( x \in H_2(4(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2})) \) with \( x^2 = 16 \) is represented by a smoothly embedded 2-sphere. Using the method in [FK] we obtain results, such as

**Theorem 2.** Let \( m, n \geq 4l > 0 \), \( x \in H_2(m \mathbb{C}P^2 \# n \overline{\mathbb{C}P^2}) \) be a primitive characteristic homology class with \( x^2 = \pm 16l + m - n \). Then \( x \) can be represented by a smoothly embedded 2-sphere.

It is sufficient to prove the following special case of Theorem 2.

**Theorem 2’.** Let \( x \in H_2(4(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2})) \) be a primitive characteristic homology class with \( |x^2| = 16 \). Then \( x \) can be represented by a smoothly embedded 2-sphere.

Theorem 2’ implies Theorem 2. By [W1] and [W2] it is sufficient to find a special primitive characteristic homology class \( y \in H_2(m \mathbb{C}P^2 \# n \overline{\mathbb{C}P^2}) \) with \( y^2 = \pm 16l + m - n \), which can be represented by a smoothly embedded 2-sphere. Write \( m \mathbb{C}P^2 \# n \overline{\mathbb{C}P^2} = l(4(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2})(m - 4l) \mathbb{C}P^2 \# (n - 4l) \overline{\mathbb{C}P^2} \) and set

\[
y = x_1 + x_2 + \cdots + x_l + \gamma_1 + \gamma_2 + \cdots + \gamma_{m-4l} + \delta_1 + \delta_2 + \cdots + \delta_{n-4l}
\]

where \( x_i \)'s are the images of \( x \in H_2(4(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2})) \) in Theorem 2’, \( \gamma_j \)'s the images of the standard generator of \( H_2(\mathbb{C}P^2) \), and \( \delta_k \)'s the images of the standard generator of \( H_2(\overline{\mathbb{C}P^2}) \), in \( H_2(l(4(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2})(m - 4l) \mathbb{C}P^2 \# (n - 4l) \overline{\mathbb{C}P^2}) \). \( y \) is primitive characteristic with \( y^2 = \pm 16l + m - n \). Tube the 2-spheres which represent \( x_i \)'s, \( \gamma_j \)'s, and \( \delta_k \)'s, one obtains a smoothly embedded 2-sphere which represents \( y \).
2. Proof of Theorem 1

Set

\[ M = mCP^2 \# n\overline{CP}^2 \# Q_1 \# \cdots \# Q_{16l+m-n-1}, \]

where the \( Q \) 's are copies of \( \overline{CP}^2 \). The image of \( x \) in \( H_2(M) \) is still denoted by \( x \) and the image of the standard generator of \( H_2(\overline{CP}^2) \) in \( H_2(M) \) corresponding to \( Q_i \) by \( \delta_i \). Let

\[ y = x + \sum_{i=1}^{16l+m-n-1} \delta_i; \]

then \( y \) is a characteristic homology class with \( y^2 = 1 \) in \( H_2(M) \).

Suppose that \( x \) can be represented by a smoothly embedded 2-sphere. Then \( y \) can also be represented by a smoothly embedded 2-sphere \( S \). Since \( y^2 = 1 \), surger the tubular neighbourhood \( N \) of \( S \) from \( M \), one obtains a simply connected smooth 4-manifold

\[ M_1 = (M\setminus N) \cup D^4 \]

and gets

\[ M = M_1 \# CP^2. \]

Since \( y \) is characteristic, \( M_1 \) is spin \( (w_2 = 0) \) and has even intersection form.

The 11/8-conjecture says that

\[ 8 \text{rank}(M_1) \geq 11|\text{sign}(M_1)|. \]

But

\[ 8 \text{rank}(M_1) = 16m + 128l - 16 \quad \text{and} \quad 11|\text{sign}(M_1)| = 176|l|. \]

Thus \( 8 \text{rank}(M_1) < 11 \times |\text{sign}(M_1)| \) is equivalent to the fact that either (1) or (2) holds. This completes the proof of Theorem 1.

3. Proof of Theorem 2'

Let \( N \) be an oriented smooth 4-manifold, \( K \subset N \) an orientable closed surface, and \( A \) an oriented simple closed curve representing an element of a symplectic basis of \( H_1(K) \) and spanning an oriented smoothly embedded disk \( B \) transversely to \( K \) in \( N \) such that \( B \cap K = A \) and \( \text{Int} B \cap K = \emptyset \). Then there is an obstruction, the Euler class \( \chi(B, A) \in H^2(B, A) \), to extending a section of \( \nu_{A \to K} \), the normal bundle of \( A \) in \( K \), to a section of \( \nu_{B \to N} \), the normal bundle of \( B \) in \( N \) (see [FK]).

Lemma 1. If \( \chi(A) = x(B, A)[B, A] = \pm 2 \), then one can surger on \( K \) in \( N \# S^2 \times S^2 \) and obtain an orientable surface \( K' \) with genus reduced by 1: \( \text{genus}(K') = \text{genus}(K) - 1 \).

Proof. Let \( S_1 = \{(x, x) \in S^2 \times S^2\} \) and \( S_2 = \{(x, -x) \in S^2 \times S^2\} \) in \( S^2 \times S^2 \). Then \([S_1] \cdot [S_1] = 2 \) and \([S_1] \cdot [S_2] = -2 \), i.e., the Euler number \( \chi(\nu_{S_1 \to S^2 \times S^2}) = 2 \) and \( \chi(\nu_{S_2 \to S^2 \times S^2}) = -2 \). (There is an error in [FK, p. 94]. The Euler numbers of both \( \Delta(S^2) \) and \( -\Delta(S^2) \) should be 2.) Take the connected sum of \( (N, B) \) with \( (S^2 \times S^2, S_1 \) or \( S_2 \)) according to \( \chi(A) = -2 \) or 2. Then we obtain

\[ \chi(\nu_{B \# S_1 \to N \# S^2 \times S^2}) = 0 \]
and can surger on $K$ along $B\#S^1$. The result of the surgery is a surface $K'$ with genus($K'$) = genus($K$) - 1.

Let $N$ be a smooth 4-manifold, $K \subset N$ an oriented closed surface, and $A_1, \ldots, A_h \subset K$ a family of simple closed curves. Let $B_1, \ldots, B_h \subset N$ be a family of 2-disks such that $B_i \cap K = A_i$, $\text{Int} B_i \cap K = \varnothing$, $B_i \cap B_j = A_i \cap A_j$, and $\text{Int} B_i \cap \text{Int} B_j = \varnothing$, $i \neq j$. We say that family $A_1, \ldots, A_h$ simply spans family $B_1, \ldots, B_h$ in $N$.

**Lemma 2.** Let $\gamma$ be the standard generator of $H_2(CP^2)$. Then $3\gamma$ can be represented by an oriented smoothly embedded torus $T$, and there are two oriented simple closed curves $A_1$ and $A_2 \subset T$ with $A_1 \cap A_2 = \text{single point}$, which represent a symplectic basis of $H_1(T)$, simply span 2-disks, and have Euler obstructions $\pm 1$.

**Proof.** See the proof of Lemma 6 in [FK].

**Lemma 3.** Let $N$ and $N'$ be two oriented smooth 4-manifolds, $T \subset N$ and $T' \subset N'$ two oriented tori, and $A_1, A_2 \subset T$ and $A'_1, A'_2 \subset T'$ oriented simple closed curves representing symplectic bases $a_1, a_2$ of $H_1(T)$ and $a'_1, a'_2$ of $H_1(T')$ respectively. If family $A_1, A_2$ and family $A'_1, A'_2$ simply span family $B_1, B_2$ in $N$ and family $B'_1, B'_2$ in $N'$ respectively, then in $T \# T' \subset N \# N'$ there are two oriented simple closed curves $C_1$ and $C_2$ which simply span oriented 2-disks $D_1$ and $D_2$ in $N \# N'$ with

$$
\chi(C_1) = \chi(A_1) + \chi(A'_1) \quad \text{and} \quad \chi(C_2) = \chi(A_2) + \chi(A'_2),
$$

where $C_1$ and $C_2$ represent $a_1 + a'_1$ and $a_2 - a'_2 \in H_1(T \# T')$, respectively, and $a_1, a'_1, a_2, a'_2$ represented by $A_1, A'_1, A_2, A'_2$ form a symplectic basis of $H_1(T \# T')$ with $a_1 \cdot a_2 = 1 = a'_1 \cdot a'_2$ and $a_1 \cdot a'_1 = a_1 \cdot a'_2 = a'_2 \cdot a_2 = a_2 \cdot a'_2 = 0$.

**Proof.** Let $\alpha$ be an oriented arc in $T \# T'$ joining $A_1$ and $A'_1$ such that the intersection numbers of $\alpha \cap A_1$ and $\alpha \cap A'_1$ are $+1$ and $-1$, respectively. The tubular neighbourhood $V(\alpha)$ of $\alpha$ in $T \# T'$ is a band connecting $B_1$ and $B'_1$. Let $C_1$ be the boundary of $B_1 \cup V(\alpha) \cup B'_1$. Pushing the interior of $V(\alpha)$ off $T \# T'$ slightly, one obtains an oriented embedded 2-disk $D_1 \cong B_1 \cup V(\alpha) \cup B'_1 \cong B_1 \# B'_1$ spanned by $\partial D_1 = C_1 = A_1 \# A'_1$. Do the same with $A_2$ and $A'_2$, but use the oriented arc which joins $A_2$ and $A'_2$ being the segment of $C_1$ cut by $A_2$ and $A'_2$ with both intersection numbers $+1$. One obtains $C_2 = A_2 \# A'_2$ and a 2-disk $D_2$ with $\partial D_2 = C_2$. We get

$$
\chi(C_1) = \chi(A_1) + \chi(A'_1) \quad \text{and} \quad \chi(C_2) = \chi(A_2) + \chi(A'_2).
$$

Now we are going to prove Theorem 2'. By [W1] and [W2] it is sufficient to find a special primitive characteristic homology class $x \in H_2(4(CP^2 \# CP^2))$ with $x^2 = 16$ which can be represented by a smoothly embedded 2-sphere.

Note that $4(CP^2 \# CP^2)$ is diffeomorphic to $2(CP^2 \# CP^2) \# 2(S^2 \times S^2)$ and $y = 3\gamma_1 + 3\gamma_2 + \delta_1 + \delta_2 \in H_2(2(CP^2 \# CP^2) \# 2(S^2 \times S^2))$ is a primitive characteristic homology class with $y^2 = 16$, where $\gamma_1, \gamma_2$ are the images of the standard generator of $H_2(CP^2)$ and $\delta_1, \delta_2$ the images of the standard generator of $H_2(CP^2)$ in $H_2(2(CP^2 \# CP^2) \# 2(S^2 \times S^2))$.

Applying Lemmas 2 and 3 we can represent $3\gamma_1 + 3\gamma_2$ by an oriented smoothly embedded double torus $K$ in $2CP^2$ and obtain two oriented simple closed
curves \( C_1 \) and \( C_2 \) simply spanning 2-disks \( D_1 \) and \( D_2 \) such that
\[
\chi(C_1) = \pm 2 \quad \text{and} \quad \chi(C_2) = \pm 2.
\]
By Lemma 1, \( 3\gamma_1 + 3\gamma_2 \) can be represented by a smoothly embedded 2-sphere in \( 2C^2\mathbb{P}_2\#2(S^2 \times S^2) \). Since the standard generator of \( H_2(\mathbb{C}P^2) \) can be represented by \( C^1 = S^2, \ 3\gamma_1 + 3\gamma_2 + \delta_1 + \delta_2 \in H_2(2C^2\mathbb{P}_2\#2(S^2 \times S^2)) \) can be represented by a smoothly embedded 2-sphere in \( 2(C^2\mathbb{P}_2\#2(S^2 \times S^2)) \). This completes the proof.

REFERENCES


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