ON A COMPLEMENTARITY PROBLEM
IN BANACH SPACE

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1. Introduction and statement of the Theorem

Let $B$ be reflexive real Banach space, and let $B^*$ be its dual. Let the value of $u \in B^*$ at $x \in B$ be denoted by $(u, x)$. Let $C$ be a closed convex cone in $B$ with the vertex at 0. The polar of $C$ is the cone $C^*$ defined by

$$C^* = \{u \in B^*: (u, x) \geq 0 \text{ for each } x \in C\}.$$ 

For any $e \in C^*$ and $r > 0$ we write

$$D_r(e) = \{x \in C: 0 \leq (e, x) \leq r\},$$

$$D_r^0(e) = \{x \in C: 0 < (e, x) < r\},$$

$$S_r(e) = \{x \in C: (e, x) = r\}.$$ 

A mapping $T: C \rightarrow B^*$ is said to be monotone if

$$(Tx - Ty, x - y) \geq 0$$

for all $x, y \in C$ and strictly monotone if strict inequality holds whenever $x \neq y$. The mapping $T$ is said to be hemicontinuous on $C$ if for all $x, y \in C$ the map $t \mapsto T(ty + (1-t)x)$ of $[0, 1]$ to $B^*$ is continuous when $B$ is endowed with the weak* topology. The mapping $T$ is said to be bounded if $T$ maps bounded subsets of $C$ into bounded subsets of $B^*$.

The purpose of this note is to prove the following existence and uniqueness theorem for the nonlinear complementarity problem.
Theorem. Let $T: C \to B^*$ be hemicontinuous and monotone such that there is an $x \in C$ with $Tx \in \text{int} C^*$. Then there is an $x_0$ such that
\begin{equation}
  x_0 \in C, \quad Tx_0 \in C^*, \quad (Tx_0, x_0) = 0.
\end{equation}
Further, if $T$ is strictly monotone, then there is unique $x_0$ satisfying (1.1).

This work has been motivated by the work of Dash and Nanda [2], who have incorrectly proved the same result under an additional hypothesis of boundedness of the operator $T$ but under the weaker assumption that there exists an $x \in C$ with $Tx \in C^*$. We shall show that the result of Dash and Nanda [2] is not true (even with bounded $T$) and that the stronger assumption $Tx \in \text{int} C^*$ cannot be omitted.

To prove the theorem we need the following result of Browder (see Browder [1] and Mosco [3]).

Proposition. Let $T$ be a monotone hemicontinuous map of a closed, convex, bounded subset $K$ of $B$, with $0 \in K$, into $B^*$. Then there is an $x_0 \in K$ such that $(Tx_0, y - x_0) \geq 0$ for all $y \in K$.

2. Proof of the Theorem

For any $e \in \text{int} C^*$ and each $r > 0$, $D_r(e)$ is clearly convex. The function $f: C \to \mathbb{R}$ defined by $f(z) = (e, z)$ is obviously continuous. Since $D_r(e) = f^{-1}([0, r])$, the set $D_r(e)$ is closed. We now need to show that $D_r(e)$ is bounded. Suppose to the contrary that it is not. Then we can choose a sequence $\{z_n\}$ of isolated points in $D_r(e)$ satisfying
\begin{equation}
  ||z_n|| \to \infty \quad \text{as} \quad n \to \infty.
\end{equation}
Let $y_n = z_n/||z_n||$. Then $||y_n|| = 1$ and $y_n \in D_r(e)$ for all $n$. We therefore have a weakly convergent subsequence $y_{n_k} \rightharpoonup y$. Since $D_r(e)$ is closed, $y \in D_r(e)$. Moreover,
\begin{equation}
  (e, y) = \lim_{k \to \infty} (e, y_{n_k}) = \lim_{k \to \infty} (e, z_{n_k}/||z_{n_k}||).
\end{equation}
But
\begin{equation}
  (e, z_{n_k}/||z_{n_k}||) \leq r/||z_{n_k}|| \to 0 \quad \text{as} \quad k \to \infty.
\end{equation}
Thus $(e, y) = 0$. Since $e \in \text{int} C^*$, we conclude that $y = 0$. This is a contradiction in view of the fact that $||y_{n_k}|| = 1$ for every $k$.

In summary, $D_r(e)$ is closed, convex, and bounded for each $r > 0$. Therefore, it follows from the above proposition that for each $r > 0$ there is an $x_r \in D_r(e)$ such that
\begin{equation}
  (Tx_r, y - x_r) \geq 0 \quad \text{for all} \quad y \in D_r(e).
\end{equation}
Since $0 \in D_r(e)$, it follows that $(Tx_r, x_r) \leq 0$. If there exist $e \in \text{int} C^*$ and $r > 0$ such that $x_r \in D^0_r(e)$, then there is some $\lambda \geq 1$ such that $\lambda x_r \in S_r(e) \subseteq D_r(e)$. Then from (2.1) we have that $(Tx_r, \lambda x_r) \leq (Tx_r, x_r) \leq (Tx_r, x_r)$. Since $(Tx_r, x_r) \leq 0$, it is impossible unless $(Tx_r, x_r) = 0$; thus $x_r$ satisfies (1.1). Now assume that $x_r \in S_r(e)$ for all $e \in \text{int} C^*$ and all $r > 0$. By the hypothesis there is an $x \in C$ with $Tx \in \text{int} C^*$. Set $e = Tx$. Choose $r > (Tx, x) \geq 0$. Now $x \in D^0_r(Tx)$, and since $T$ is monotone, we have
\begin{equation}
  (Tz, z - x) \geq (Tx, z - x) > 0 \quad \text{for all} \quad z \in S_r(Tx),
\end{equation}
but $x_r \in S_r(Tx)$; hence $(Tx_r, x_r - x) > 0$. 

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Since \( x \in D^0_0(Tx) \subseteq D_r(Tx) \), it follows from (2.1) that \( (Tx_r, x - x_r) \geq 0 \). Since this contradicts (2.2), the assumption that \( x_r \in S_r(e) \) for all \( r \) has thus been shown not to hold when \( e = Tx \). Thus the proof of the theorem is reduced to the previous case, the case where there exists \( e \in \text{int} C^* \) and \( r > 0 \) such that \( x \in D^0_0(e) \). If \( T \) is strictly monotone, it is easy to see that the solution is unique, and this completes the proof of the theorem.

Now observe that if \( e \in C^* \) but \( e \notin \text{int} C^* \), the sets \( D_r(e) \) need not be bounded. In this case we cannot conclude \( y = 0 \) from the fact that \( (e, y) = 0 \). Consider the case when \( B = \mathbb{R}^2 \), \( C = \mathbb{R}^2_+ \), and \( e = (1, 0) \). Then, for each \( r > 0 \), \( D_r(e) \) contains the positive \( y \)-axis and hence is unbounded. Therefore, Proposition (Browder) cannot be applied in this case. Also we cannot apply Lemma (Mosco) (see Mosco [4] as it was done in [2], because Lemma holds for nonempty closed convex and bounded sets in \( C \).

In fact, the following example shows that the theorem of [2] is false. The counterexample was suggested to the author by one of the referees of Zeitschrift Mathematische Operations-forschung und Statistik-Series Optimization in connection with some other paper and was communicated to the author by Professor Dr. K.-H. Elster, the editor.

Take \( B = \mathbb{R}^3 \) and \( C = \{ (x, y, z) \in \mathbb{R}^3 : x, z \geq 0, 2xz \geq y^2 \} \). Define \( T \) by \( T(x, y, z) = (x + 1, y + 1, 0) \). Then \( T \) is monotone and hemicontinuous (even bounded). The point \((1, -1, 1) \in C \) and \( T(1, -1, 1) = (2, 0, 0) \in C^* \). If \( u = (x, y, z) \in C \) with \( Tu \in C^* \), then \( y = -1 \), and hence \( x > 0 \). Hence, for any such \( u \), \( (Tu, u) = x(x + 1) > 0 \).

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