SCHARLEMANN'S 4-MANIFOLDS
AND SMOOTH 2-KNOTS IN $S^2 \times S^2$

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Abstract. Scharlemann gave an example of a 4-manifold admitting a fake homotopy structure on $S^3 \times S^1 S^2 \times S^2$, which is homeomorphic to $S^3 \times S^1 S^2 \times S^2$ by a theorem of Freedman. We address the problem whether a Scharlemann's manifold is diffeomorphic to $S^3 \times S^1 S^2 \times S^2$ in terms of 2-knots in $S^2 \times S^2$.

1. Introduction

Since Milnor's discovery [18] of exotic 7-spheres which are homeomorphic to the standard 7-sphere but not diffeomorphic, the existence of exotic smooth structures in dimension 5 and higher has been well known. Meanwhile, in dimension 3 and less, there are no exotic smooth structures; that is, homeomorphic smooth manifolds are diffeomorphic.

The first example of an exotic smooth structure in dimension 4 is the fake $\mathbb{R}P^4$ of Cappell and Shaneson [2], which is homeomorphic to $\mathbb{R}P^4$ by a theorem of Freedman [8] but not diffeomorphic. This manifold, however, is nonorientable.

As for orientable 4-manifolds, in conjunction with Freedman's work [7], Donaldson's theorem [3] on the nonrepresentation of any nontrivial definite form by a smooth 4-manifold gives the remarkable result that $\mathbb{R}^4$ admits an exotic smooth structure [12]. Furthermore, Donaldson gave, in [4], the first example of exotic orientable closed 4-manifolds by using the $\Gamma$-invariant introduced by him: a certain Dolgachev surface is an exotic $CP^2 \# 9 CP^2$. The Dolgachev surface and $CP^2 \# 9 CP^2$ are $h$-cobordant nondiffeomorphic manifolds, so they give a counterexample to the $h$-cobordism conjecture for simply connected smooth 4-manifolds. Other related exotic manifolds have been found by using the $\Gamma$-invariant or another gauge theoretic invariant [9, 10].

Before these discoveries in the 1980s of exotic orientable closed 4-manifolds, Scharlemann found a family of strange orientable closed 4-manifolds in 1976 [25]. Every Scharlemann's manifold gives a fake homotopy structure for $S^3 \times S^1 S^2 \times S^2$. Meanwhile, since a Scharlemann's manifold and $S^3 \times S^1 S^2 \times S^2$ are...
s-cobordant, the 5-dimensional topological s-cobordism theorem for \( \pi_1 \cong \mathbb{Z} \) proved by Freedman [8] states that they are homeomorphic. From a Scharlemann’s manifold and \( S^3 \times S^1 \# S^2 \times S^2 \), Gompf constructed two compact orientable 4-manifolds with boundary which are homeomorphic but not diffeomorphic [14]. Then a question comes to mind.

**Problem (Matsumoto [15, Problem 4.15]).** Is a Scharlemann’s manifold diffeomorphic to \( S^3 \times S^1 \# S^2 \times S^2 \)?

This problem asks whether a Scharlemann’s manifold gives an exotic structure on \( S^3 \times S^1 \# S^2 \times S^2 \) and whether they give a counterexample to the s-cobordism conjecture for smooth 4-manifolds with \( \pi_1 \cong \mathbb{Z} \). In this note, we address this problem. We note that after connected sum with \( S^2 \times S^2 \) a Scharlemann’s manifold becomes diffeomorphic to \( S^3 \times S^1 \# 2(S^2 \times S^2) \), which Fintushel and Pao proved [6]. (In fact, Gompf proved that compact orientable homeomorphic 4-manifolds must become diffeomorphic after connected sum with an unspecified number of copies of \( S^2 \times S^2 \) [13].)

We show that for each Scharlemann’s manifold \( X \) there exists a nullhomologous smooth 2-knot in \( S^2 \times S^2 \) which is topologically trivial and has the following property: This 2-knot is smoothly trivial if and only if \( X \) is diffeomorphic to \( S^3 \times S^1 \# S^2 \times S^2 \).

Also, by using our 2-knots one may show that after connected sum of the twisted \( S^2 \)-bundle \( S^2 \times S^2 \) over \( S^2 \) a Scharlemann’s manifold becomes diffeomorphic to \( S^3 \times S^1 \# S^2 \times S^2 \# S^2 \times S^2 \).

### 2. SCHARLEMANN’S MANIFOLDS AND 2-KNOTS IN \( S^2 \times S^2 \)

For a smooth manifold \( M \) and a submanifold \( A \) of \( M \), we denote a tubular neighborhood of \( A \) in \( M \) by \( N(A) \). In this note, we call a smoothly embedded 2-sphere in \( S^2 \times S^2 \) a 2-knot in \( S^2 \times S^2 \).

Let \( K \) be a 2-knot in \( S^4 \) with exterior \( E(K) \) and \( C \) a smooth circle in \( S^4 \) disjoint from \( K \). Since we may assume that \( C \) is standardly embedded in \( S^4 \) up to ambient isotopy, \( K \) is contained in \( S^4 - \text{int} N(C) = S^2 \cup D^2 \). This gives a 2-knot in \( S^2 \times S^2 = S^2 \cup D^2 \cup S^2 \times D^2 \) and is denoted by \( S(K, C) \). It follows from van Kampen’s theorem that the knot group of \( S(K, C) \), \( \pi_1(S^2 \times S^2 - S(K, C)) \), is isomorphic to \( \pi_1(S^4 - K)/H \), where \( H \) is the normal closure generated by the element represented by \( C \) in \( \pi_1(S^4 - K) \) [16, 23, 24].

**Definition.** Let \( S \) be a 2-knot in \( S^2 \times S^2 \). Then \( S \) is said to be smoothly (topologically, resp.) trivial if \( S \) bounds a smooth (topological, resp.) 3-ball in \( S^2 \times S^2 \).

We consider a nontrivial fibered 2-knot \( K \) in \( S^4 \) with closed fiber \( M \). Suppose that \( M \) is a homology 3-sphere such that \( \pi_1(M) \) has weight one. Then the exterior \( E(K) \) of \( K \) is a fiber bundle over \( S^1 \) with fiber \( M^\circ \) and monodromy map \( \sigma : M^\circ \to M^\circ \); i.e.,

\[
E(K) = M^\circ \times_{\sigma} S^1 = M^\circ \times I/(x, 0) \sim (\sigma(x), 1),
\]

where \( M^\circ \) is a punctured copy of \( M \). Let \( \alpha \) be a weight element for \( \pi_1(M^\circ) = \pi_1(M) \). We take a smooth circle in \( M^\circ \) representing \( \alpha \) and denote the circle by the same symbol \( \alpha \). (For a manifold \( X \), we shall sometimes not distinguish...
notationally between an element of \( \pi_1(X) \) and a smooth circle in \( X \) representing it.) Let \( \alpha_\ast \) denote a smooth circle \( \alpha \times \{\ast\} \) on a fiber \( M^\circ \times \{\ast\} \) in \( E(\mathcal{K}) = M^\circ \times_\sigma S^1 \). Then \( S(K, \alpha_\ast) \) is a null-homologous 2-knot in \( S^2 \times S^2 \) satisfying

\[
\pi_1(S^2 \times S^2 - S(K, \alpha_\ast)) \cong \mathbb{Z} \cong \pi_1(S^2 \times S^2 - \text{trivial 2-knot}).
\]

In fact, the knot group \( \pi_1(S^4 - K) \) has the presentation

\[
\langle \pi_1(M^\circ), \mu \mid \mu x \mu^{-1} = \sigma_t(x) \text{ for any } x \in \pi_1(M^\circ) \rangle.
\]

Since \( \alpha \in \pi_1(M^\circ) \) is a weight element, \( \pi_1(S^2 \times S^2 - S(K, \alpha_\ast)) \cong \langle \mu \rangle \cong \mathbb{Z} \).

The 2-knot \( S(K, \alpha_\ast) \) has a tubular neighborhood diffeomorphic to \( S^2 \times D^2 \), and so the self-intersection number of \( S(K, \alpha_\ast) \) in \( S^2 \times S^2 \) is zero. Hence, \( S(K, \alpha_\ast) \) represents \( p \zeta \) or \( p \eta \) for some integer \( p \), where \( \zeta \) and \( \eta \) are natural generators of \( H_2(S^2 \times S^2; \mathbb{Z}) \) with \( \zeta \cdot \zeta = \eta \cdot \eta = 0 \) and \( \zeta \cdot \eta = \eta \cdot \zeta = 1 \).

Meanwhile, since \( H_1(S^2 \times S^2 - S(K, \alpha_\ast); \mathbb{Z}) \cong \pi_1(S^2 \times S^2 - S(K, \alpha_\ast)) \cong \mathbb{Z} \), it follows that \( p = 0 \); namely, \( S(K, \alpha_\ast) \) is null-homologous.

Next we consider Scharlemann’s manifolds. These manifolds are constructed as follows: Let \( \Sigma \) be the Poincaré homology 3-sphere, which is the intersection of the unit 5-sphere in \( \mathbb{C}^3 \) with the complex variety \( z_0^5 + z_1^5 + z_2^5 = 0 \). The fundamental group \( \pi_1(\Sigma) \) is the binary dodecahedral group and has weight one. Let \( \alpha \) be a weight element. Let \( \alpha_\ast \) denote a smooth circle on a fiber \( \Sigma \times \{\ast\} \) in \( \Sigma \times S^1 \) representing \( \alpha \in \pi_1(\Sigma) \subset \pi_1(\Sigma \times S^1) \). Remove a tubular neighborhood of \( \alpha_\ast \) in \( \Sigma \times S^1 \), and attach \( S^2 \times D^2 \) to its boundary with the trivial framing. The resulting manifold \( X_\alpha \) obtained from \( \Sigma \times S^1 \) by this surgery is a Scharlemann’s manifold, which is a closed orientable smooth 4-manifold with \( \pi_1(X_\alpha) \cong \mathbb{Z} \).

The Poincaré homology 3-sphere \( \Sigma \) is smoothly embedded in \( S^5 \) as the link of the algebraic variety defined above. Then \( \pi_1(S^5 - \Sigma) \cong \mathbb{Z} \), and by Alexander duality \( S^5 - \Sigma \) is a homology circle. Thus, by doing surgery of \( S^5 - \text{int } N(\Sigma) \), we can obtain an \( h \)-cobordism \( W \) between \( X_\alpha \) and \( S^3 \times S^1 \# S^2 \times S^2 \). Furthermore, since the Whitehead group \( Wh(\mathbb{Z}) = 0 \), \( W \) is a \( s \)-cobordism between them. Hence, we have

**Lemma 1** [25]. Two manifolds \( X_\alpha \) and \( S^3 \times S^1 \# S^2 \times S^2 \) are \( s \)-cobordant.

We relate Scharlemann’s manifolds to certain null-homologous 2-knots in \( S^2 \times S^2 \):

**Proposition 2.** For any weight element \( \alpha \in \pi_1(\Sigma) \), there exists a null-homologous 2-knot \( S_\alpha \) in \( S^2 \times S^2 \) satisfying:

1. \( S_\alpha \) is topologically trivial in \( S^2 \times S^2 \) and
2. \( S_\alpha \) is smoothly trivial in \( S^2 \times S^2 \) if and only if the Scharlemann’s manifold \( X_\alpha \) is diffeomorphic to \( S^3 \times S^1 \# S^2 \times S^2 \).

Such 2-knots are constructed as follows: Let \( \beta \) be a weight element for \( \pi_1(\Sigma) \). In \( \Sigma \times S^1 \) let \( \beta t \) be a simple loop representing \( \beta \) times \( t \) in \( \pi_1(\Sigma \times S^1) \) such that \( \beta t \) meets each \( \Sigma \times \{\ast\} \) transversely in a single point, where \( t \) is an element represented by \( \{\ast\} \times S^1 \subset \Sigma \times S^1 \). Then \( E = \Sigma \times S^1 - \text{int } N(\beta t) \) is a fiber bundle over \( S^1 \) with fiber \( \Sigma^0 \). We consider a pair of spaces

\[
(Y, K_\beta) = (E \cup_{\partial E} S^2 \times D^2, S^2 \times \{0\}).
\]
Since $\beta t$ is a weight element for $\pi_1(\Sigma \times S^1)$, $Y$ is a homotopy 4-sphere. In particular, if some power of $\beta$ lies in the center $Z(\pi_1(\Sigma))$ of $\pi_1(\Sigma)$, then the monodromy map $\sigma$ has finite order, so $Y$ is a diffeomorphic to $S^4$ [19, 21, 22]. Now we take such an element $\beta$. Thus $K_\beta$ is a fibered 2-knot in $S^4$ with exterior $E(K_\beta) = \Sigma \times S^1 - \text{int} N(\beta t) = \Sigma^0 \times S^1$. It follows that, for every weight element $\alpha \in \pi_1(\Sigma)$, $S(K_\beta, \alpha_*)$ is a null-homologous 2-knot in $S^2 \times S^2$ with $\pi_1(S^2 \times S^2 - S(K_\beta, \alpha_*)) \cong Z$. Let $E(S(K_\beta, \alpha_*))$ be the exterior of $S(K_\beta, \alpha_*)$ in $S^2 \times S^2$. We now consider the closed orientable smooth 4-manifold $M(S(K_\beta, \alpha_*))$ obtained from $S^2 \times S^2$ by surgery along $S(K_\beta, \alpha_*)$, i.e., $M(S(K_\beta, \alpha_*)) = D^3 \times S^1 \cup_{\partial E(S(K_\beta, \alpha_*))} E(S(K_\beta, \alpha_*))$. Noting that the surgery on $\beta t$ in $\Sigma \times S^1$ and the surgery on $K_\beta$ in $S^4$ simply undo each other, it follows that $M(S(K_\beta, \alpha_*))$ is diffeomorphic to $X_\alpha$. Our construction may be described by:

$$\begin{align*}
\Sigma \times S^1 & \xrightarrow{\text{surger} \beta t} S^4 \supset K_\beta \\
\vert & \quad \vert \\
X_\alpha & \xleftarrow{\text{surger} K_\beta} S^2 \times S^2 \supset S(K_\beta, \alpha_*)
\end{align*}$$

Proposition 2 then follows from the following lemma.

**Lemma 3.** Let $S$ be a null-homologous 2-knot in $S^2 \times S^2$ such that $\pi_1(S^2 \times S^2 - S) \cong Z$. Let $X$ be the resultant 4-manifold of surgery along $S$. Then $X$ is diffeomorphic to $S^3 \times S^1 \# S^2 \times S^2$ if and only if $S$ bounds a smooth 3-ball in $S^2 \times S^2$.

**Proof.** It is clear that if $S$ bounds a smooth 3-ball in $S^2 \times S^2$, then $X$ is diffeomorphic to $S^3 \times S^1 \# S^2 \times S^2$.

Suppose that $X$ is diffeomorphic to $S^3 \times S^1 \# S^2 \times S^2$. Then there exists a diffeomorphism $f : X \to S^3 \times S^1 \# S^2 \times S^2$. Let $D^3_+ \text{ and } D^3_-$ be 3-balls such that $D^3 = D^3_+ \cup D^3_-$ and $D^3_+ \cap D^3_- = \partial D^3_+ = \partial D^3_-$. We may assume that the 4-ball in $S^3 \times S^1$ of the connected sum $S^3 \times S^1 \# S^2 \times S^2$ lies in the interior of $D^3 \times S^1 \subset S^3 \times S^1$. Let $E(S)$ be the exterior of $S$. Then we have two smooth circles $C_0 = \{0\} \times S^1 \subset D^3 \times S^1 \subset D^3 \times S^1 \cup_{\partial E(S)} E(S) = X$ and $C_1 = \{0_+\} \times S^1 \subset D^3_+ \times S^1 \subset (D^3_+ \cup D^3_-) \times S^1 \# S^2 \times S^2 = S^3 \times S^1 \# S^2 \times S^2$ such that $N(C_0) = D^3 \times S^1 \subset D^3 \times S^1 \cup_{\partial E(S)} E(S) = X$ and $N(C_1) = D^3_+ \times S^1 \subset S^3 \times S^1 \# S^2 \times S^2$. The homotopy classes of $C_0$ and of $C_1$ generate $\pi_1(X) \cong Z$ and $\pi_1(S^3 \times S^1 \# S^2 \times S^2) \cong Z$ respectively. Since the diffeomorphism $f : X \to S^3 \times S^1 \# S^2 \times S^2$ induces an isomorphism $f_* : Z \cong \langle C_0 \rangle \to \langle C_1 \rangle \cong Z$, $f_*([C_0]) = [C_1]^{\pm 1}$. Hence, two smooth circles $f(C_0)$ and $C_1$ are free homotopic in $S^3 \times S^1 \# S^2 \times S^2$. Since $\dim f(C_0) = \dim C_1 = 1$ and $\dim(S^3 \times S^1 \# S^2 \times S^2) = 4$, there exists an ambient isotopy $\{h_t : S^3 \times S^1 \# S^2 \times S^2 \to S^3 \times S^1 \# S^2 \times S^2 \}_{t \in I}$ such that $h_1(f(C_0)) = C_1$. Thus the differentoism $f^* = h_1 \circ f : X \to S^3 \times S^1 \# S^2 \times S^2$ takes $C_0$ to $C_1$. Moreover, deforming $f^*$ via an ambient isotopy of $S^3 \times S^1 \# S^2 \times S^2$ if necessary, we obtain a diffeomorphism $\bar{f} : X \to S^3 \times S^1 \# S^2 \times S^2$ such that $\bar{f}(N(C_0)) = N(C_1)$ by the uniqueness of tubular neighborhoods. Therefore,
the restriction of \( \tilde{j} \) to \( E(S) \) gives a diffeomorphism
\[
g : E(S) \to D^3_\ast \times S^1 \# S^2 \times S^2.
\]
Identifying the boundaries of \( E(S) \) and \( D^3_\ast \times S^1 \# S^2 \times S^2 \) with \( S^2 \times S^1 \), we may assume, by [11], that, for some point \( \ast \in S^1 \), \( g(S^2 \times \{\ast\}) = S^2 \times \{\ast\} \). Hence, the 3-ball \( g^{-1}(D^3_\ast \times \{\ast\}) \) in \( E(S) \) provides a 3-ball in \( S^2 \times S^2 \) bounded by \( S \).

This implies that \( S \) is smoothly trivial in \( S^2 \times S^2 \). □

Thus \( S(K_\beta, \alpha_\ast) \) satisfies property (2).

By Lemma 1, two manifolds \( X_\alpha \) and \( S^3 \times S^1 \# S^2 \times S^2 \) are s-cobordant, so it follows from the 5-dimensional topological s-cobordism theorem for \( \pi_1 \cong \mathbb{Z} \) [8] that they are homeomorphic, that is, there is a homeomorphism \( g : X_\alpha \to S^3 \times S^1 \# S^2 \times S^2 \). Hence \( S(K_\beta, \alpha_\ast) \) is topologically trivial in \( S^2 \times S^2 \) for a similar reason as above. This completes the proof of Proposition 2. □

Remark. Proposition 2 implies that if a Scharlemann’s manifold \( X_\alpha \) is not diffeomorphic to \( S^3 \times S^1 \# S^2 \times S^2 \), then the 2-knot \( S \) in Proposition 2 will give an exotic knotting of the trivial 2-knot in \( S^2 \times S^2 \). Any example of exotic knottings of \( S^2 \) into a 4-manifold has not been known yet, but some examples of exotic knottings of 2-disc or nonorientable surfaces into a 4-manifold are known [5, 26].

Matumoto proved in [17] that for a smooth simply connected 4-manifold \( Y \) if \( S \) is a null-homologous 2-knot in \( Y \) with \( \pi_1(Y - S) \cong \mathbb{Z} \), then \( S \) is smoothly trivial in \( Y \# n(S^2 \times S^2) \) for some nonnegative integer \( n \); namely, \( S \) bounds a smooth 3-ball in it. Hence, it follows from Proposition 2 that the 2-knot \( S_\alpha \) is smoothly trivial in \( S^2 \times S^2 \# n(S^2 \times S^2) \) for some nonnegative integer \( n \) and \( X_\alpha \# n(S^2 \times S^2) \) is diffeomorphic to \( S^3 \times S^1 \# S^2 \times S^2 \# n(S^2 \times S^2) \).

Fintushel and Pao proved that \( X_\alpha \# S^2 \times S^2 \) is diffeomorphic to \( S^3 \times S^1 \# S^2 \times S^2 \# S^2 \times S^2 \) [6], so the 2-knot \( S_\alpha \) is smoothly trivial in \( S^2 \times S^2 \# S^2 \times S^2 \). In a use of a 2-knot \( S_\alpha \) in \( S^2 \times S^2 \) one may show that, for the twisted \( S^2 \)-bundle \( S^2 \times S^2 \) over \( S^2 \), \( X_\alpha \# S^2 \times S^2 \) is diffeomorphic to \( S^3 \times S^1 \# S^2 \times S^2 \# S^2 \times S^2 \). But this follows from a better result proved by Akbulut [1]: \( X_\alpha \# CP^2 \) is diffeomorphic to \( S^3 \times S^1 \# S^2 \times S^2 \# CP^2 \).

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References

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