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CORRIGENDUM TO "ON HAUSDORFF DIMENSION OF RECURRENT NET FRACTALS"

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In the proof of Theorem 3.1 in [6] a characterization of Larma's finite-dimensional metric spaces due to Rogers [5] was taken for granted and used. Since this characterization is not true in general, in this note we add a further hypothesis on the complete metric space treated in [6] which guarantees the validity of Theorem 3.1. Moreover, some natural conditions tacitly assumed in [6] are made explicit, thus extending the geometric analysis. Proposition 1.1 is correctly stated and improved.

By defining in [6] net fractals in a complete metric space, we intended to provide a procedure for generating sets which look like fractals. They are expected to be, topologically, at least perfect subsets of the given metric space and in particular uncountable sets. In [6] we, tacitly, assumed (without explicitly stating) that

$$\mathcal{N} \cap \operatorname{int}(A_{\mathbf{i}|n}) \neq \varnothing \quad \forall \mathbf{i}|n, \tag{A}$$

which guarantees that $\mathcal{N}$ has the above topological properties. Further (A) is a natural condition and necessary to avoid that the geometric procedure described in [6] collapses. In fact, if (A) is not assumed the set $\mathcal{N}$ might be contained in the boundary set $B := \bigcup_{\mathbf{i}|n} A_{\mathbf{i}|n} \setminus \operatorname{int}(A_{\mathbf{i}|n})$; thus, by standard results (see [3]), in that case we will have $\mathcal{H}_d^s(\mathcal{N}) = 0$, for any dimension s for which $\mathcal{H}_D^s(\mathcal{N}) < \infty$ (here $\mathcal{H}_d$ and $\mathcal{H}_D$ represent the Hausdorff measures with respect to the metrics d and D used in [6]). Therefore, conditions (2) and (3) in [6] would become insignificant within the scope of the dimension estimate of $\mathcal{N}$.

See also the discussion about net fractals generated by 'proper constructions' in [1], which is the reference given in [6] for the original definition.

An analogous assumption was made in [6, §4] for the self-similar set K. It is natural to assume, as it happens in most concrete examples, that K is not completely contained in the boundary of the open set O, for otherwise, K would be the kernel of a geometric *scheme does not satisfy condition* (A); consequently, by the above remarks, it would be $\mathcal{H}_d^s(K) = 0$ for any s such that $\mathcal{H}_D^s(K) < \infty$. In particular, this is the case when s is the similarity dimension (see [2]) of K. Hence, when $K \subseteq \overline{O} \setminus O$, in general we have

$$\dim_H(K) < \text{ similarity dimension of } K.$$

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Although Proposition 1.1 in [6] is not used to get the main results, it must be noted that as proved in [6] the topology of (Ω, D) is always finer than that of (Ω, d) but that the topological equivalence, there stated, may fail on the boundary set $B \cap \mathcal{N}$. However, for any dimension s for which $\mathcal{H}_D^s(\mathcal{N}) < \infty$, applying again the remarks above, we get the topological equivalence of d and D modulo a subset of zero $\mathcal{H}_D^s$-measure and thus of zero $\mathcal{H}_d^s$-measure. In particular, when $\mathcal{H}_D^s(\mathcal{N}) < \infty$, and this is the case for net fractals satisfying the hypothesis of Theorems 2.2 and 3.1, the space $(\mathcal{N}, d)$ is the disjoint union of an ultrametric subspace and a subset of zero $\mathcal{H}_d^s$-measure. In particular,

$\mathcal{N}$ has the same Hausdorff measure of an

ultametric, topologically zero-dimensional, totally disconnected subspace,

showing that from a measure-theoretic point of view net fractals satisfying the hypothesis of Theorem 3.1 are always ultrametric net fractals.

In the proof of Theorem 3.1 we employed a characterization of Larman's finite dimensionality due to Rogers [5, p. 104 1.1 and p. 122, Theorem 57 condition (b)], taking for granted that if $\mathcal{H}^n(A) = 0$ for some positive integer n, then A is finite dimensional in the sense of Larman [4].

While the converse is always true (see [4, corollary to Theorem 4]), this implication is in general false. In fact, for any Hausdorff function $h(t)$, any set $A \subseteq \Omega$ contains a countable subset A' with $h(A) = h(A')$, in Larman's notation. But A and hence A' may not be finite dimensional.

Let $\mathcal{N}' := \overline{\{x_{\mathbf{i}|n}\}_{\mathbf{i}|n}}$, where $\{x_{\mathbf{i}|n}\}_{\mathbf{i}|n}$ is the family of point centers of the open balls involved in condition (3) in [6], and call it the *expanded net fractal* associated with $\mathcal{N}$.

We have $\mathcal{N}' = \mathcal{N} \cup \{x_{\mathbf{i}|n}\}_{\mathbf{i}|n}$. In fact, clearly $\mathcal{N} \subseteq \mathcal{N}'$ and if x is an accumulation point of $\{x_{\mathbf{i}|n|}\}_{\mathbf{i}|n}$, we can find a sequence $(x_{\mathbf{i}|n(k)})_k$ with $\lim_{k\to\infty} x_{\mathbf{i}|n(k)} = x$. Since the coordinates i_j of the curtailed indexes $\mathbf{i}|n(k)$ can assume only a finite number of values, by a standard diagonal argument, we can determine an index, say $\mathbf{j}$, such that $(x_{\mathbf{j}|n(l)})_l$ is a subsequence of $(x_{\mathbf{i}|n(k)})_k$ and $\mathbf{j}|n(l+1)$ is an extension of $\mathbf{j}|n(l)$. It follows that $x = \lim_{l\to\infty} x_{\mathbf{j}|n(l)} = \bigcap_{n=1}^{\infty} A_{\mathbf{j}|n} \in \mathcal{N}$ and thus the claim also follows. Moreover, an analogous argument shows that $\mathcal{N}'$ is sequentially compact and thus compact. Further $\mathcal{H}^s(\mathcal{N}') = \mathcal{H}^s(\mathcal{N})$ since $\{x_{\mathbf{i}|n}\}_{\mathbf{i}|n}$ is countable.

The proof of Theorem 1.3 remains essentially the same if we can use the property

the expanded net fractal $\mathcal{N}'$ is a β-space.

In general complete metric spaces, in order to guarantee that $\mathcal{N}'$ is a β-space, we need a further condition concerning the relative position of the points $\{x_{\mathbf{i}|n}\}_{\mathbf{i}|n}$. However, we are not concerned here with a suitable modification of the basic requirements (1), (2), and (3) in [6] for a net fractal. A complete analysis will appear elsewhere. But we indicate a class of metric spaces, significant from a geometric point of view, in which the above property is automatically satisfied. It is the class of *locally finite-dimensional metric spaces*, i.e., the spaces in which every point admits a neighbourhood which is finite dimensional in the sense of Larman [4]. In fact, in these spaces, as we can see using Theorems 11 and 12 in [4], any compact subset is a β-space.

Among the spaces included in this large class, we find the Euclidean spaces and the Riemannian manifolds of class 2.

On page 397 replace lines 11–15 by:

If $U \cap A_{\mathbf{i}|n(i)} \neq \varnothing$, then we can find a ball $B(x, 2\rho)$ in $\mathcal{N}'$ such that U and $A_{\mathbf{i}|n(i)} \cap \mathcal{N}'$ are contained in it. Since $\mathcal{N}'$ is a β-space (for instance with triple (M, δ, α)), it follows that at most M^q disjoint balls of radius $\rho\alpha^k$ intersect $B(x, 2\rho)$ where q satisfies $(2\alpha)^q \leq \alpha^{k+1} < (2\alpha)^{q-1}$. Since $\alpha^k \rho \leq \lambda h \rho$, at most M^q balls of radius $\lambda h \rho$ can meet $B(x, 2\rho)$, hence at most M^q of the sets $\{A_{i_1, \dots, i_{n(i)}}\}$ can meet U.

On page 397 in lines 18 and 20 replace M^k by M^q.

On page 399 in line 35, replace 'O to be bounded' by 'O to be bounded and regular open'.

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