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# A LINEAR RECURRENCE SYSTEM

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ABSTRACT. We look at a triangular system of $n$ equations and reinvestigate a related function introduced by Chen and Kuck. Our main contribution is to provide a new proof of a result which forms the basis of their work.

We investigate a function which will be used to evaluate the linear recursion

$$x_i = a_{i,0} + \sum_{j=1}^{i-1} a_{i,j} x_j \quad \text{for } i = 1, 2, 3, \dots, n$$

where the $a_{ij}$'s are arbitrary numbers.

We can express this in matrix notation as $X = C + AX$ where

$$C = \begin{bmatrix} a_{10} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a_{n0} \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ a_{21} & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ a_{31} & a_{32} & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot & \cdot \\ a_{n1} & a_{n2} & a_{n3} & \cdot & \cdot & \cdot & 0 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}.$$

$A$ is referred to as the coefficient matrix, $C$ as the constant vector, and $X$ as the solution vector.

**Definition.** For $0 \le j < 2^n$, $1 \le i \le 2^n$, we define a sequence of functions $f_0(i, j)$, $f_1(i, j)$, $f_2(i, j), \dots, f_n(i, j)$ such that

$$f_{r+1}(i, j) = \begin{cases} f_r(i, j) + \sum_{k=j+2^r-b}^{j+2^{r+1}-b-1} f_r(i, k) f_r(k, j) & \text{if } j \equiv b \pmod{2^{r+1}}, \\ & \text{where } 0 \le b < 2^r, \\ f_r(i, j) \quad \text{otherwise} & \end{cases}$$

for $0 \le r < n$, with

$$f_0(i, j) = a_{i,j} \quad \text{and} \quad a_{i,j} = 0 \quad \text{for } i \le j.$$

*Remark.* $a_{i,j} = 0$ for $i \le j$ implies $f_r(i, j) = 0$ for $i \le j$.

By repeatedly applying this recursive definition we can express any $f_{r+1}(i, j)$ in terms of $f_r$, $f_{r-1}, \dots$, and finally $f_0$, and thus express $f_{r+1}(i, j)$ as a function of $a$'s only.

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**Examples.**

$$f_1(4,0) = f_0(4,0) + f_0(4,1)f_0(1,0) = a_{4,0} + a_{4,1}a_{1,0},$$
$$f_1(5,1) = f_0(5,1) = a_{5,1},$$
$$\begin{aligned} f_2(3,0) &= f_1(3,0) + f_1(3,2)f_1(2,0) \\ &= f_0(3,0) + f_0(3,1)f_0(1,0) + f_0(3,2)[f_0(2,0) + f_0(2,1)f_0(1,0)] \\ &= a_{3,0} + a_{3,1}a_{1,0} + a_{3,2}(a_{2,0} + a_{2,1}a_{1,0}). \end{aligned}$$

**Theorem.** *Let* $j \equiv b \pmod{2^r}$, $0 \le b < 2^r$; *then*

$$f_r(i,j) = a_{i,j} + \sum a_{i,j(1)}a_{j(1),j(2)} \cdots a_{j(u),j}$$

*where the sum is over all* $u$*-tuples* $(j(1), \dots, j(u))$ *satisfying* $j < j(u) < \cdots < j(1) < \min\{j + 2^r - b, i\}$, *where* $0 < u < \min\{2^r - b, i - j\}$.

*Proof.* Our proof is by induction on $r$. We observe that

$$f_0(i,j) = a_{i,j},$$
$$f_1(i,j) = \begin{cases} a_{i,j} + a_{i,j+1}a_{j+1,j} & \text{if } j \equiv 0 \pmod 2, \\ a_{i,j} & \text{otherwise.} \end{cases}$$

So the result is verified for $r = 0$ and $r = 1$.

Assume the result is valid for $r = s$. Now using this, we will deduce the corresponding result for $s + 1$:

$$f_{s+1}(i,j) = a_{i,j} + \sum a_{i,j(1)}a_{j(1),j(2)} \cdots a_{j(u),j}$$

where the sum is over all $u$-tuples $(j(1), \dots, j(u))$ satisfying $j < j(u) < \cdots < j(1) < \min\{j + 2^{s+1} - b, i\}$, where $0 < u < \min\{2^{s+1} - b, i - j\}$ and $j \equiv b \pmod{2^{s+1}}$, $0 \le b < 2^{s+1}$.

We consider three cases.

*Case* 1. $j \equiv b \pmod{2^{s+1}}$, $2^s \le b < 2^{s+1}$. In this case $f_{s+1}(i,j) = f_s(i,j)$ by definition, and by the induction hypothesis

$$f_{s+1}(i,j) = a_{i,j} + \sum a_{i,j(1)}a_{j(1),j(2)} \cdots a_{j(u),j},$$
$$j < j(u) < \cdots < j(1) < \min\{j + 2^s - b', i\},$$

where $0 < u < \min\{2^s - b', i - j\}$ and $j \equiv b' \pmod{2^s}$, $0 \le b' < 2^s$.

But $j \equiv b \pmod{2^{s+1}}$, $2^s \le b < 2^{s+1}$, implies $j \equiv b - 2^s \pmod{2^s}$, $0 \le b < 2^s$, and $b' = b - 2^s$. Therefore

$$f_{s+1}(i,j) = a_{i,j} + \sum a_{i,j(1)}a_{j(1),j(2)} \cdots a_{j(u),j},$$
$$j < j(u) < \cdots < j(1) < \min\{j + 2^{s+1} - b, i\},$$

where $0 < u < \min\{2^{s+1} - b, i - j\}$ and $j \equiv b \pmod{2^{s+1}}$, $2^s \le b < 2^{s+1}$.

*Case* 2. $j \equiv b \pmod{2^{s+1}}$, $0 \le b < 2^s$, and $i \le j + 2^s - b$. It follows by definition that $f_{s+1}(i,j) = f_s(i,j)$ and by the induction hypothesis

$$f_{s+1}(i,j) = a_{i,j} + \sum a_{i,j(1)}a_{j(1),j(2)} \cdots a_{j(u),j},$$
$$j < j(u) < \cdots < j(1) < \min\{j + 2^s - b', i\},$$

where $0 < u < \min\{2^s - b', i - j\}$ and $j \equiv b' \pmod{2^s}$, $0 \le b' < 2^s$. Since $j \equiv b \pmod{2^{s+1}}$, $0 \le b < 2^s$, implies $j \equiv b \pmod{2^s}$, $0 \le b < 2^s$, we have

$b' = b$. Then $\min\{j+2^s-b, i\} = i = \min\{j+2^{s+1}-b, i\}$, since $i \le j+2^s-b$. Therefore

$$f_{s+1}(i, j) = a_{i,j} + \sum a_{i,j(1)}a_{j(1),j(2)} \cdots a_{j(u),j},$$
$$j < j(u) < \cdots < j(1) < \min\{j + 2^{s+1} - b, i\},$$

where $0 < u < \min\{2^{s+1} - b, i - j\}$ and $j \equiv b \pmod{2^{s+1}}$, $0 \le b < 2^s$, and $i \le j + 2^s - b$.

*Case* 3. $j \equiv b \pmod{2^{s+1}}$, $0 \le b < 2^s$, and $i > j + 2^s - b$. By definition

$$f_{s+1}(i, j) = f_s(i, j) + \sum_{k=j+2^s-b}^{j+2^{s+1}-b-1} f_s(i, k) f_s(k, j)$$
$$= f_s(i, j) + \sum_{k=j+2^s-b}^{\min\{j+2^{s+1}-b, i\}-1} f_s(i, k) f_s(k, j).$$

From the induction hypothesis

$$f_s(i, k) = a_{i,k} + \sum a_{i,k(1)}a_{k(1),k(2)} \cdots a_{k(u),k},$$
$$k < k(u) < \cdots < k(1) < \min\{k + 2^s - c, i\},$$

where $0 < u < \min\{2^s - c, i - k\}$ and $k \equiv c \pmod{2^s}$, $0 \le c < 2^s$. When $j + 2^s - b \le k < j + 2^{s+1} - b$, $k = j + 2^s - b + c$ and $k + 2^s - c = j + 2^{s+1} - b$. Therefore, for all $k$ such that $j + 2^s - b \le k < j + 2^{s+1} - b$ we have

(i) $$f_s(i, k) = a_{i,k} + \sum a_{i,k(1)}a_{k(1),k(2)} \cdots a_{k(u),k},$$
$$k < k(u) < \cdots < k(1) < \min\{j + 2^{s+1} - b, i\},$$

where $0 < u < \min\{j + 2^{s+1} - b, i\} - k$ and $j \equiv b \pmod{2^{s+1}}$, $0 \le b < 2^s$. Also, by the induction hypothesis

$$f_s(k, j) = a_{k,j} + \sum a_{k,j(1)}a_{j(1),j(2)} \cdots a_{j(v),j},$$
$$j < j(v) < \cdots < j(1) < \min\{j + 2^s - b', k\},$$

where $0 < v < \min\{2^s - b', k - j\}$ and $j \equiv b' \pmod{2^s}$, $0 \le b' < 2^s$. But since $j \equiv b \pmod{2^{s+1}}$, $0 \le b < 2^s$, it follows that $b' = b$, and when $j + 2^s - b \le k < j + 2^{s+1} - b$ we have $\min\{j + 2^s - b, k\} = j + 2^s - b$. Therefore, for all $k$ such that $j + 2^s - b \le k < j + 2^{s+1} - b$

(ii) $$f_s(k, j) = a_{k,j} + \sum a_{k,j(1)}a_{j(1),j(2)} \cdots a_{j(v),j},$$
$$j < j(v) < \cdots < j(1) < j + 2^s - b,$$

where $0 < v < 2^s - b$ and $j \equiv b \pmod{2^{s+1}}$, $0 \le b < 2^s$.

It follows from (i) and (ii) that

$$\sum_{k=j+2^s-b}^{\min\{j+2^{s+1}-b,\,i\}-1} f_s(i,k)f_s(k,j)$$
$$= \sum_{k=j+2^s-b}^{\min\{j+2^{s+1}-b,\,i\}-1} \Bigg( a_{i,k}a_{k,j} + \sum_{j(v)} a_{i,k}a_{k,j(1)}\cdots a_{j(v),j} + \sum_{k(u)} a_{i,k(1)}\dots a_{k(u),k}a_{k,j} + \sum_{k(u),\,j(v)} a_{i,k(1)}\cdots a_{k(u),k}a_{k,j(1)}\cdots a_{j(v),j}\Bigg)$$

where $k < k(u) < \cdots < k(1) < \min\{j+2^{s+1}-b,\, i\}$,

$j < j(v) < \cdots < j(1) < j + 2^s - b$,

$0 < u < \min\{j + 2^{s+1} - b,\, i\} - k$,

$0 < v < 2^s - b$, and $j \equiv b \pmod{2^{s+1}}$, $0 \le b < 2^s$.

By removing the outer summation symbol and making a change of variables, we obtain

$$\sum_{k=j+2^s-b}^{\min\{j+2^{s+1}-b,\,i\}-1} f_s(i,k)f_s(k,j)$$
$$= \sum_{C(1)} a_{i,j(1)}a_{j(1),j} + \sum_{C(2)} a_{i,j(1)}a_{j(1),j(2)}\cdots a_{j(v+1),j} + \sum_{C(3)} a_{i,j(1)}\cdots a_{j(u),j(u+1)}a_{j(u+1),j} + \sum_{C(4)} a_{i,j(1)}\cdots a_{j(u),j(u+1)}a_{j(u+1),j(u+2)}\cdots a_{j(u+v+1),j}$$

where $C(1)$, $C(2)$, $C(3)$, $C(4)$ are the summation conditions:

$C(1)$: $j + 2^s - b \le j(1) < \min\{j + 2^{s+1} - b,\, i\}$ where $j \equiv b \pmod{2^{s+1}}$, $0 \le b < 2^s$,

$C(2)$: $j < j(v+1) < \cdots < j(2) < j + 2^s - b \le \min\{j + 2^{s+1} - b,\, i\}$ where $0 < v < 2^s - b$ and $j \equiv b \pmod{2^{s+1}}$, $0 \le b < 2^s$,

$C(3)$: $j + 2^s - b \le j(u+1) < j(u) < \cdots < j(1) < \min\{j + 2^{s+1} - b,\, i\}$ where $0 < u < \min\{j + 2^{s+1} - b,\, i\} - j(u+1)$ and $j \equiv b \pmod{2^{s+1}}$, $0 \le b < 2^s$,

$C(4)$: $j < j(u+v+1) < \cdots < j(u+2) < j + 2^s - b \le j(u+1) < j(u) < \cdots < j(1) < \min\{j+2^{s+1}-b,\, i\}$ where $0 < u < \min\{j+2^{s+1}-b,\, i\}-j(u+1)$, $0 < v < 2^s - b$, and $j \equiv b \pmod{2^{s+1}}$, $0 \le b < 2^s$.

*Remarks.* 1. $u + v + 1 < \min\{2^{s+1} - b,\, i - j\}$.

2. $C(1)$, $C(2)$, $C(3)$, $C(4)$ describe a partition of the condition

$$j + 2^s - b \le j(1) < \min\{j + 2^{s+1} - b,\, i\}, \qquad j < j(w) < \cdots < j(2) < j(1),$$

where $0 < w < \min\{2^{s+1} - b, i - j\}$ and $j \equiv b \pmod{2^{s+1}}$, $0 \le b < 2^s$.

We now conclude

$$\text{(iii)} \qquad \sum_{k=j+2^s-b}^{\min\{j+2^{s+1}-b,i\}-1} f_s(i,k)f_s(k,j) = \sum a_{i,j(1)} \cdots a_{j(w),j},$$

$$j + 2^s - b \le j(1) < \min\{j + 2^{s+1} - b, i\},$$
$$j < j(w) < \cdots < j(2) < j(1),$$

where $0 < w < \min\{2^{s+1} - b, i - j\}$ and $j \equiv b \pmod{2^{s+1}}$, $0 \le b < 2^s$. It follows from the induction hypothesis that

$$\text{(iv)} \qquad f_s(i,j) = a_{i,j} + \sum a_{i,j(1)} \cdots a_{j(x),j},$$
$$j < j(x) < \cdots < j(1) < \min\{j + 2^s - b', i\},$$

where $0 < x < \min\{2^s - b', i - j\}$ and $j \equiv b' \pmod{2^s}$, $0 \le b' < 2^s$.

We have $j \equiv b \pmod{2^{s+1}}$, $0 \le b < 2^s$, therefore $b' = b$, and since $i > j + 2^s - b$ the $\min\{j + 2^s - b, i\} = j + 2^s - b$.

From (iii) and (iv) we conclude that when $j \equiv b \pmod{2^{s+1}}$, $0 \le b < 2^s$, and $i > j + 2^s - b$, then

$$f_{s+1}(i,j) = a_{i,j} + \sum_{S(1)} a_{i,j(1)} \cdots a_{j(x),j} + \sum_{S(2)} a_{i,j(1)} \cdots a_{j(w),j}$$

where $S(1)$ and $S(2)$ denote the summation conditions

$$j < j(x) < \cdots < j(1) < j + 2^s - b, \qquad 0 < x < 2^s - b,$$

and $j + 2^s - b \le j(1) < \min\{j + 2^{s+1} - b, i\}$,

$$j < j(w) < \cdots < j(2) < j(1), \qquad 0 < w < \min\{2^{s+1} - b, i - j\},$$

respectively. Hence we have

$$f_{s+1}(i,j) = a_{i,j} + \sum a_{i,j(1)} a_{j(1),j(2)} \cdots a_{j(y),j},$$
$$j < j(y) < \cdots < j(1) < \min\{j + 2^{s+1} - b, i\},$$

where $0 < y < \min\{2^{s+1} - b, i - j\}$ and $j \equiv b \pmod{2^{s+1}}$, $0 \le b < 2^s$, and $i > j + 2^s - b$.

Together, Cases 1, 2, and 3 show that the theorem holds for $r = s + 1$, and thus for all nonnegative integers less than or equal to $n$. □

**Corollary.** $f_n(i,0) = a_{i,0} + \sum_{j=1}^{i-1} a_{i,j}x_j = x_i$ *for* $1 \le i \le 2^n$.

*Proof*. Our proof is by induction on $i$. The corollary is true for $i = 1$ and $i = 2$, since $f_n(1,0) = a_{1,0} = x_1$ for $n \ge 0$ and $f_n(2,0) = a_{2,0} + a_{2,1}a_{1,0} = x_2$ for $n \ge 1$. As an induction hypothesis, assume it holds for all $i \le k < 2^n$. Thus, by hypothesis we have

$$f_n(i,0) = a_{i,0} + \sum_{j=1}^{i-1} a_{i,j}x_j = x_i \quad \text{for } 1 \le i \le k.$$

Now we prove the lemma for $i = k + 1$. We know by definition that

$$x_{k+1} = a_{k+1,0} + \sum_{j=1}^{k} a_{k+1,j}x_j,$$

and by the induction hypothesis this can be written as

$$x_{k+1} = a_{k+1,0} + \sum_{j=1}^{k} a_{k+1,j} f_n(j, 0).$$

It follows from our theorem that

$$x_{k+1} = a_{k+1,0} + \sum_{j=1}^{k} a_{k+1,j} \left( a_{j,0} + \sum a_{j,j(1)} a_{j(1),j(2)} \cdots a_{j(u),0} \right),$$
$$0 < j(u) < \cdots < j(1) < \min\{2^n, j\}, \qquad 0 < u < \min\{2^n, j\},$$

and we obtain

$$x_{k+1} = a_{k+1,0} + \sum_{1 \le j(1) \le k} a_{k+1,j(1)} a_{j(1),0} + \sum a_{k+1,j(1)} \cdots a_{j(u+1),0},$$
$$0 < j(u+1) < \cdots < j(2) < j(1) \le k, \qquad 0 < u < j(1).$$

Therefore,

$$x_{k+1} = a_{k+1,0} + \sum a_{(k+1),j(1)} \cdots a_{j(v),0},$$
$$0 < j(v) < \cdots < j(1) \le k, \qquad 0 < v \le k,$$

which can be written as

$$x_{k+1} = a_{k+1,0} + \sum a_{k+1,j(1)} \cdots a_{j(v),0},$$
$$0 < j(v) < \cdots < j(1) < \min\{2^n, k+1\}, \qquad 0 < v < \min\{2^n, k+1\},$$

and it follows by our theorem that

$$x_{k+1} = f_n(k+1, 0). \quad \square$$

Based on the result that $f_n(i, 0) = x_i$ for $1 \le i \le 2^n$ Chen and Kuck [CK] give a parallel algorithm for evaluating $x_i$, $1 \le i \le 2^n$. Time and processor bounds for solving the linear recurrence system are then obtained. Sameh and Brent [SB] later presented an alternate derivation of this algorithm.

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