A LINEAR RECURRENCE SYSTEM

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Abstract. We look at a triangular system of \( n \) equations and reinvestigate a related function introduced by Chen and Kuck. Our main contribution is to provide a new proof of a result which forms the basis of their work.

We investigate a function which will be used to evaluate the linear recursion

\[
x_i = a_{i,0} + \sum_{j=1}^{i-1} a_{i,j} x_j \quad \text{for } i = 1, 2, 3, \ldots, n
\]

where the \( a_{ij} \)'s are arbitrary numbers.

We can express this in matrix notation as \( X = C + AX \) where

\[
C = \begin{bmatrix}
a_{10} \\
\vdots \\
a_{n0}
\end{bmatrix}, \quad A = \begin{bmatrix}
a_{21} & 0 & \cdots & 0 \\
0 & a_{22} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{n,n-1}
\end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\
\vdots \\
x_n \end{bmatrix}
\]

\( A \) is referred to as the coefficient matrix, \( C \) as the constant vector, and \( X \) as the solution vector.

Definition. For \( 0 < j < 2^n, \ 1 < i < 2^n \), we define a sequence of functions \( f_0(i, j), f_1(i, j), f_2(i, j), \ldots, f_n(i, j) \) such that

\[
f_{r+1}(i, j) = \begin{cases} 
    f_r(i, j) + \sum_{k=j+2^r-b}^{j+2^r-1} f_r(i, k) f_r(k, j) & \text{if } j \equiv b \pmod{2^{r+1}}, \\
    f_r(i, j) & \text{otherwise}
\end{cases}
\]

for \( 0 \leq r < n \), with

\[
f_0(i, j) = a_{i,j} \quad \text{and} \quad a_{i,j} = 0 \quad \text{for } i < j.
\]

Remark. \( a_{i,j} = 0 \) for \( i \leq j \) implies \( f_r(i, j) = 0 \) for \( i \leq j \).

By repeatedly applying this recursive definition we can express any \( f_{r+1}(i, j) \) in terms of \( f_r, f_{r-1}, \ldots, \) and finally \( f_0 \), and thus express \( f_{r+1}(i, j) \) as a function of \( a \)'s only.

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Examples.

\[ f_1(4,0) = f_0(4,0) + f_0(4,1) f_0(1,0) = a_{4,0} + a_{4,1} a_{1,0}, \]
\[ f_1(5,1) = f_6(5,1) = a_{5,1}, \]
\[ f_2(3,0) = f_1(3,0) + f_1(3,2) f_1(2,0) \]
\[ = f_0(3,0) + f_0(3,1) f_0(1,0) + f_0(3,2) [f_0(2,0) + f_0(2,1) f_0(1,0)] \]
\[ = a_{3,0} + a_{3,1} a_{1,0} + a_{3,2} (a_{2,0} + a_{2,1} a_{1,0}). \]

Theorem. Let \( j \equiv b \pmod{2^r}, 0 < b < 2^r \); then

\[ f_r(i, j) = a_{i,j} + \sum a_{i,j(1)} a_{j(1), j(2)} \cdots a_{j(u), j} \]

where the sum is over all \( u \)-tuples \((j(1), \ldots, j(u))\) satisfying \( j < j(u) < \cdots < j(1) < \min\{j + 2^r - b, i\}\), where \( 0 < u < \min\{2^r - b, i - j\}\).

Proof. Our proof is by induction on \( r \). We observe that

\[ f_0(i, j) = a_{i,j}, \]
\[ f_1(i, j) = \begin{cases} a_{i,j} + a_{i,j+1} a_{j+1,j} & \text{if } j \equiv 0 \pmod{2}, \\ a_{i,j} & \text{otherwise.} \end{cases} \]

So the result is verified for \( r = 0 \) and \( r = 1 \).

Assume the result is valid for \( r = s \). Now using this, we will deduce the corresponding result for \( r + 1 \):

\[ f_{r+1}(i, j) = a_{i,j} + \sum a_{i,j(1)} a_{j(1), j(2)} \cdots a_{j(u), j} \]

where the sum is over all \( u \)-tuples \((j(1), \ldots, j(u))\) satisfying \( j < j(u) < \cdots < j(1) < \min\{j + 2^{s+1} - b, i\}\), where \( 0 < u < \min\{2^{s+1} - b, i - j\}\) and \( j \equiv b \pmod{2^{s+1}}, 0 < b < 2^{s+1} \).

We consider three cases.

Case 1. \( j \equiv b \pmod{2^{s+1}}, 2^s \leq b < 2^{s+1} \). In this case \( f_{s+1}(i, j) = f_s(i, j) \) by definition, and by the induction hypothesis

\[ f_{s+1}(i, j) = a_{i,j} + \sum a_{i,j(1)} a_{j(1), j(2)} \cdots a_{j(u), j}, \]

where \( 0 < u < \min\{2^s - b', i - j\} \) and \( j \equiv b' \pmod{2^s}, 0 < b' < 2^s \).

But \( j \equiv b \pmod{2^{s+1}}, 2^s \leq b < 2^{s+1} \), implies \( j \equiv b - 2^s \pmod{2^s}, 0 \leq b < 2^s \), and \( b' = b - 2^s \). Therefore

\[ f_{s+1}(i, j) = a_{i,j} + \sum a_{i,j(1)} a_{j(1), j(2)} \cdots a_{j(u), j}, \]

where \( 0 < u < \min\{2^{s+1} - b, i - j\} \) and \( j \equiv b \pmod{2^{s+1}}, 2^s \leq b < 2^{s+1} \).

Case 2. \( j \equiv b \pmod{2^{s+1}}, 0 \leq b < 2^s \), and \( i \leq j + 2^s - b \). It follows by definition that \( f_{s+1}(i, j) = f_s(i, j) \) and by the induction hypothesis

\[ f_{s+1}(i, j) = a_{i,j} + \sum a_{i,j(1)} a_{j(1), j(2)} \cdots a_{j(u), j}, \]

where \( 0 < u < \min\{2^s - b', i - j\} \) and \( j \equiv b' \pmod{2^s}, 0 < b' < 2^s \). Since \( j \equiv b \pmod{2^{s+1}}, 0 \leq b < 2^s \), implies \( j \equiv b \pmod{2^s}, 0 \leq b < 2^s \), we have
$b' = b$. Then $\min\{j + 2^s - b, i\} = i = \min\{j + 2^{s+1} - b, i\}$, since $i \leq j + 2^s - b$.

Therefore

$$
f_{s+1}(i, j) = a_{i, j} + \sum a_{i, j(1)}a_{j(1), j(2)} \cdots a_{j(u), j},
j < j(u) < \cdots < j(1) < \min\{j + 2^{s+1} - b, i\},$$

where $0 < u < \min\{2^{s+1} - b, i - j\}$ and $j \equiv b \pmod{2^{s+1}}$, $0 \leq b < 2^s$, and $i \leq j + 2^s - b$.

**Case 3.** $j \equiv b \pmod{2^{s+1}}$, $0 \leq b < 2^s$, and $i > j + 2^s - b$. By definition

$$
f_{s+1}(i, j) = f_s(i, j) + \sum_{k = j + 2^s - b}^{j + 2^{s+1} - b - 1} f_s(i, k)f_s(k, j),
\min\{j + 2^s + X - b, i\} - 1
$$

From the induction hypothesis

$$
f_s(i, k) = a_{i, k} + \sum a_{i, k(1)}a_{k(1), k(2)} \cdots a_{k(u), k},
k < k(u) < \cdots < k(1) < \min\{k + 2^s - c, i\},$$

where $0 < u < \min\{2^s - c, i - k\}$ and $k \equiv c \pmod{2^s}$, $0 \leq c < 2^s$. When $j + 2^s - b \leq k < j + 2^{s+1} - b$, $k = j + 2^s - b + c$ and $k + 2^s - c = j + 2^{s+1} - b$. Therefore, for all $k$ such that $j + 2^s - b \leq k < j + 2^{s+1} - b$ we have

(i) $$f_s(i, k) = a_{i, k} + \sum a_{i, k(1)}a_{k(1), k(2)} \cdots a_{k(u), k},
k < k(u) < \cdots < k(1) < \min\{j + 2^{s+1} - b, i\},$$

where $0 < u < \min\{j + 2^{s+1} - b, i\} - k$ and $j \equiv b \pmod{2^{s+1}}$, $0 \leq b < 2^s$. Also, by the induction hypothesis

$$
f_s(k, j) = a_{k, j} + \sum a_{k, j(1)}a_{j(1), j(2)} \cdots a_{j(u), j},
\min\{j + 2^s - b', k\},$$

where $0 < u < \min\{2^s - b', k - j\}$ and $j \equiv b' \pmod{2^s}$, $0 \leq b' < 2^s$. But since $j \equiv b \pmod{2^{s+1}}$, $0 \leq b < 2^s$, it follows that $b' = b$, and when $j + 2^s - b \leq k < j + 2^{s+1} - b$ we have $\min\{j + 2^s - b, k\} = j + 2^s - b$. Therefore, for all $k$ such that $j + 2^s - b \leq k < j + 2^{s+1} - b$

(ii) $$f_s(k, j) = a_{k, j} + \sum a_{k, j(1)}a_{j(1), j(2)} \cdots a_{j(u), j},
\min\{j + 2^s - b, k\},$$

where $0 < u < 2^s - b$ and $j \equiv b \pmod{2^{s+1}}$, $0 \leq b < 2^s$. 
It follows from (i) and (ii) that

\[
\min\{j+2^{s+1} - b, i\} - 1 \sum_{k=j+2^s-b}^{\min\{j+2^{s+1} - b, i\} - 1} f_s(i, k) f_s(k, j) = \min\{j+2^{s+1} - b, i\} - 1 \sum_{k=j+2^s-b}^{\min\{j+2^{s+1} - b, i\} - 1} \left( \sum_{k\neq j(v)} a_i, k a_k, j \sum_{j(1)} a_i, k a_k, j(1) \cdots a_j(v), j \right.
\]

\[
+ \sum_{k(u)} a_i, k(1) \cdots a_k(u), k a_k, j
\]

\[
+ \sum_{k(u), j(v)} a_i, k(1) \cdots a_k(u), k a_k, j(1) \cdots a_j(v), j \right)
\]

where \( k < k(u) < \cdots < k(1) < \min\{j + 2^{s+1} - b, i\} \),

\[ \quad \begin{align*}
  j &< j(v) < \cdots < j(1) < j + 2^s - b, \\
  0 &< u < \min\{j + 2^{s+1} - b, i\} - k, \\
  0 &< v < 2^s - b, \text{ and } j \equiv b \pmod{2^{s+1}}, 0 < b < 2^s.
\end{align*}\]

By removing the outer summation symbol and making a change of variables, we obtain

\[
\min\{j+2^{s+1} - b, i\} - 1 \sum_{k=j+2^s-b}^{\min\{j+2^{s+1} - b, i\} - 1} f_s(i, k) f_s(k, j) = \sum_{C(1)} a_i, j(1) a_j(1), j + \sum_{C(2)} a_i, j(1) a_j(1), j(2) \cdots a_j(v+1), j
\]

\[
+ \sum_{C(3)} a_i, j(1) \cdots a_j(u), j(u+1) a_j(u+1), j
\]

\[
+ \sum_{C(4)} a_i, j(1) \cdots a_j(u), j(u+1) a_j(u+1), j(u+2) \cdots a_j(u+v+1), j
\]

where \( C(1), C(2), C(3), C(4) \) are the summation conditions:

- \( C(1) \): \( j + 2^s - b \leq j(1) < \min\{j + 2^{s+1} - b, i\} \) where \( j \equiv b \pmod{2^{s+1}}, 0 \leq b < 2^s \),

- \( C(2) \): \( j < j(v+1) < \cdots < j(2) < j + 2^s - b \leq \min\{j + 2^{s+1} - b, i\} \) where \( 0 < v < 2^s - b \) and \( j \equiv b \pmod{2^{s+1}}, 0 \leq b < 2^s \),

- \( C(3) \): \( j + 2^s - b \leq j(u+1) < j(u) < \cdots < j(1) < \min\{j + 2^{s+1} - b, i\} \) where \( 0 < u < \min\{j + 2^{s+1} - b, i\} - j(u+1) \) and \( j \equiv b \pmod{2^{s+1}}, 0 \leq b < 2^s \),

- \( C(4) \): \( j < j(u+v+1) < \cdots < j(u+2) < j + 2^s - b \leq j(u+1) < j(u) < \cdots < j(1) < \min\{j+2^{s+1} - b, i\} \) where \( 0 < u < \min\{j+2^{s+1} - b, i\} - j(u+1) \), \( 0 < v < 2^s - b \), and \( j \equiv b \pmod{2^{s+1}}, 0 \leq b < 2^s \).

**Remarks.**

1. \( u + v + 1 < \min\{2^{s+1} - b, i - j\} \).
2. \( C(1), C(2), C(3), C(4) \) describe a partition of the condition

\[ \quad j + 2^s - b \leq j(1) < \min\{j + 2^{s+1} - b, i\}, \quad j < j(w) < \cdots < j(2) < j(1), \]
where \(0 < w < \min\{2^{s+1} - b, i - j\}\) and \(j \equiv b \pmod{2^{s+1}}\), \(0 \leq b < 2^s\).

We now conclude
\[
\sum_{k=j+2^s-b}^{\min\{j+2^{s+1}-b, i\}-1} f_s(i, k)f_s(k, j) = \sum_{j+2^s-b}^{j+2^s-b} a_i,j(1) \cdots a_j(w), j,
\]
where \(0 < w < \min\{2^{s+1} - b, i - j\}\) and \(j \equiv b \pmod{2^{s+1}}\), \(0 \leq b < 2^s\). It follows from the induction hypothesis that
\[
f_s(i, j) = a_i,j + \sum_{j+2^s-b}^{j+2^s-b} a_i,j(1) \cdots a_j(x), j,
\]
where \(0 < x < \min\{2^s - b', i - j\}\) and \(j \equiv b' \pmod{2^s}\), \(0 \leq b' < 2^s\).

We have \(j \equiv b \pmod{2^{s+1}}\), \(0 \leq b < 2^s\), therefore \(b' = b\), and since \(i > j + 2^s - b\) the \(\min\{j+2^s-b', i\} = j + 2^s - b\).

From (iii) and (iv) we conclude that when \(j \equiv b \pmod{2^{s+1}}\), \(0 \leq b < 2^s\), and \(i > j + 2^s - b\), then
\[
f_s(i, j) = a_i,j + \sum_{S(1)} a_i,j(1) \cdots a_j(x), j + \sum_{S(2)} a_i,j(1) \cdots a_j(w), j
\]
where \(S(1)\) and \(S(2)\) denote the summation conditions
\[
j < j(x) < \cdots < j(1) < j + 2^s - b, \quad 0 < x < 2^s - b,
\]
and \(j + 2^s - b < j(1) < \min\{j + 2^{s+1} - b, i\}\),
\[
j < j(w) < \cdots < j(2) < j(1), \quad 0 < w < \min\{2^{s+1} - b, i - j\},
\]
respectively. Hence we have
\[
f_s(i, j) = a_i,j + \sum_{S(1)} a_i,j(1) a_j(1), j(2) \cdots a_j(y), j,
\]
where \(0 < y < \min\{2^{s+1} - b, i - j\}\) and \(j \equiv b \pmod{2^{s+1}}\), \(0 \leq b < 2^s\), and \(i > j + 2^s - b\).

Together, Cases 1, 2, and 3 show that the theorem holds for \(r = s + 1\), and thus for all nonnegative integers less than or equal to \(n\).

\textbf{Corollary.} \(f_n(i, 0) = a_i,0 + \sum_{j=1}^{i-1} a_i,j x_j = x_i\) for \(1 \leq i \leq 2^n\).

\textbf{Proof.} Our proof is by induction on \(i\). The corollary is true for \(i = 1\) and \(i = 2\), since \(f_n(1, 0) = a_{1,0} = x_1\) \(\forall n \geq 0\) and \(f_n(2, 0) = a_{2,0} + a_{2,1} a_{1,0} = x_2\) \(\forall n \geq 1\). As an induction hypothesis, assume it holds for all \(i \leq k < 2^n\). Thus, by hypothesis we have
\[
f_n(i, 0) = a_i,0 + \sum_{j=1}^{i-1} a_i,j x_j = x_i \quad \text{for} \quad 1 \leq i \leq k.
\]

Now we prove the lemma for \(i = k + 1\). We know by definition that
\[
x_{k+1} = a_{k+1,0} + \sum_{j=1}^{k} a_{k+1,j} x_j,
\]
and by the induction hypothesis this can be written as

\[ x_{k+1} = a_{k+1, 0} + \sum_{j=1}^{k} a_{k+1, j} f_n(j, 0). \]

It follows from our theorem that

\[ x_{k+1} = a_{k+1, 0} + \sum_{j=1}^{k} a_{k+1, j} \left( a_j, 0 + \sum a_j, j(1) a_j(1), j(2) \cdots a_j(u), 0 \right), \]

\[ 0 < j(u) < \cdots < j(1) < \min\{2^n, j\}, \quad 0 < u < \min\{2^n, j\}, \]

and we obtain

\[ x_{k+1} = a_{k+1, 0} + \sum_{1 \leq j(1) \leq k} a_{k+1, j(1)} a_j(1), 0 + \sum a_{k+1, j(1)} \cdots a_{j(u+1), 0}, \]

\[ 0 < j(u + 1) < \cdots < j(2) < j(1) \leq k, \quad 0 < u < j(1). \]

Therefore,

\[ x_{k+1} = a_{k+1, 0} + \sum a_{k+1, j(1)} \cdots a_j(v), 0, \]

\[ 0 < j(v) < \cdots < j(1) \leq k, \quad 0 < v \leq k, \]

which can be written as

\[ x_{k+1} = a_{k+1, 0} + \sum a_{k+1, j(1)} \cdots a_j(v), 0, \]

\[ 0 < j(v) < \cdots < j(1) < \min\{2^n, k + 1\}, \quad 0 < v < \min\{2^n, k + 1\}, \]

and it follows by our theorem that

\[ x_{k+1} = f_n(k + 1, 0). \quad \square \]

Based on the result that \( f_n(i, 0) = x_i \) for \( 1 \leq i \leq 2^n \) Chen and Kuck [CK] give a parallel algorithm for evaluating \( x_i, \quad 1 \leq i \leq 2^n \). Time and processor bounds for solving the linear recurrence system are then obtained. Sameh and Brent [SB] later presented an alternate derivation of this algorithm.

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References


