A COUNTEREXAMPLE CONCERNING THE MAXIMUM
AND MINIMUM OF A SUBHARMONIC FUNCTION

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Abstract. For every $\Delta > 0$ a function $u$ subharmonic in the plane is con-
structed such that $u$ has the order $\rho = 1 + \Delta$ and satisfies the condition

$$\min_{\varphi} u(re^{i\varphi})/\max_{\varphi} u(re^{i\varphi}) \leq -(C + 1) \quad \text{for every } r > 0,$$

where $C = C(\rho) > 0$. This example answers a question of W. K. Hayman.

Let $f$ be an entire function of finite order $\rho$, and let

$$M(r, f) = \max_{\varphi} |f(re^{i\varphi})|, \quad \mu(r, f) = \min_{\varphi} |f(re^{i\varphi})|. $$

The well-known $\cos \pi \rho$-theorem asserts that if $\rho \leq 1$, then

$$\limsup_{r \to \infty} \frac{\log \mu(r, f)}{\log M(r, f)} \geq \cos \pi \rho.$$

In 1952 Hayman [H2] proved that for every $\rho > \rho_0$, $\rho$ being positive and large
enough, there exists an entire function of order $\rho$ such that

$$\limsup_{r \to \infty} \frac{\log \mu(r, f)}{\log M(r, f)} < -c(\rho) < -1;$$

in fact, $c(\rho)$ is unbounded as $\rho \to \infty$. However, the question [H1] of what
happens when $\rho$ is close to one remained open up to the present time. The
purpose of this paper is to give a particular answer to this question.

In fact, for every $\rho > 1$ we construct a function $u$ subharmonic in $\mathbb{C}$ that
satisfies the conditions:

(i) $u(0) = 0$;

(ii) $B(r, u) = r^\rho, \ 0 < r < \infty$;

(iii) $L(u, r)/B(u, r) < -(1 + C(\rho))$ for every $r > 0$ and some $C(\rho) > 0$.

(Here $L$ and $B$ denote the lower and upper bounds of $u(z)$ on the circle $|z| = r$.)

In order to get a corresponding example of an entire function, it is sufficient
to use a suitable theorem on the approximation of a subharmonic function by
the logarithm of modulus of an entire one (see [A] or [Yu]). The details will be
given in Appendix 2.

TWO LEMMAS

To construct this example we need two lemmas.

**Lemma 1.** For every $T > 1$ and every $\tau$, $T > \tau > 1$, there exists a subharmonic
function $g$ such that for some $\alpha > 0$ the following relations are valid:

1. $B(g, r) = g(-r) = r$,

2. $g(r) < -(1 + \alpha)r$ \hspace{1cm} $\forall \ r \in \bigcup_{k=-\infty}^{\infty} [T^k, \tau T^k] = \mathcal{R}^+(T, \tau)$.

**Proof.** First we introduce some notation. Let $T, \tau$ be positive numbers such
that $1 < \tau < T$, and let $I_\varepsilon$ be the complex $\varepsilon$-neighborhood of the interval
$I = I_0 = [1, \tau]$. Let

$$\mathcal{R}_\varepsilon(T, \tau) = \bigcup_{k=-\infty}^{\infty} T^k I_\varepsilon.$$

Denote by $\mathcal{U}_\varepsilon$ the family of subharmonic functions that satisfy the inequality
$(z = r e^{i\theta})$

3. $u_\varepsilon(re^{i\theta}) \leq \begin{cases} r(1 + \cos \theta) & \text{for } z \notin \mathcal{R}_\varepsilon(T, \tau), \\ 0 & \text{for } z \in \mathcal{R}_\varepsilon(T, \tau). \end{cases}$

Let $u$ be the least upper bound of this family. Obviously, $u$ is a subharmonic
function. This function satisfies the relations

4. $u(Tz) = Tu(z)$

and

5. $u(-r) = 0$ \hspace{1cm} $\forall r \geq 0$.

If $u(z) < r(1 + \cos \theta)$ then $u$ is harmonic in some $\delta$-neighborhood of $z$;
indeed, otherwise the function $u_\delta$ continuous in $\mathbb{C}$, coinciding with $u$ outside
this neighborhood and harmonic inside, belongs to $\mathcal{U}_\varepsilon$. The same argument
shows that if $u \neq 0$ then $u > 0$ outside $(-\infty, 0] \cup \mathcal{R}_\varepsilon(T, \tau)$. We will prove
later that $u \neq 0$. Assuming this let us finish the proof of the lemma. If $\nu > 0$
is small enough then

$$u(z) < r(1 + \cos \theta), \quad z = re^{i\theta} \in I_{\varepsilon + \nu},$$

and, hence, $u$ is harmonic in $I_{\varepsilon + \nu}\setminus \overline{I_\varepsilon}$. Since $u$ is positive and harmonic in
$I_{\varepsilon + \nu}\setminus \overline{I_\varepsilon}$ and $u(z) = 0$, $z \in \overline{I_\varepsilon}$, the normal derivative of $u$ (in the direction of
outward normal to the boundary of $\overline{I_\varepsilon}$) has positive infimum on $\partial \overline{I_\varepsilon}$. Denote
by $G_{I_\varepsilon}(z, 1)$ the Green function of $I_\varepsilon$ with the pole at 1. We remark that its
normal derivative is bounded since the boundary of $I_\varepsilon$ is smooth. Hence, the
function

$$v(z) = \begin{cases} u(z) & \text{for } z \in \mathbb{C}\setminus \mathcal{R}_\varepsilon(T, \tau), \\ -\gamma T^k G_{I_\varepsilon}(z/T^k, 1) & \text{for } z \in T^k I_\varepsilon \end{cases}$$
is subharmonic if \( \gamma > 0 \) is sufficiently small. It is clear that

\[
(6) \quad v(r) \leq -\alpha r, \quad r \in \mathcal{R}(T^+, \tau),
\]

for some positive \( \alpha \).

The function

\[
g(re^{i\theta}) = -r \cos \theta + v(re^{i\theta})
\]
satisfies all conditions of Lemma 1 due to properties (3)–(6).

It remains to prove that \( u \neq 0 \) with an appropriate choice of \( \epsilon \). Consider the domains

\[
\Omega_{\epsilon, \delta} = \{ z : |\arg z| < \pi - \delta, \ z \notin \mathcal{R}_\epsilon^+(T, \tau) \}
\]

for some \( \delta \geq 0 \).

There exists a function \( u_{\epsilon, \delta} \) positive and harmonic inside the domain \( \Omega_{\epsilon, \delta} \) and vanishing on its boundary. This function is unique up to a multiplicative constant (see Appendix 1). As the region \( \Omega_{\epsilon, \delta} \) is invariant under the transformation \( z \mapsto Tz \), we conclude from the uniqueness of \( u_{\epsilon, \delta} \) that

\[
(7) \quad u_{\epsilon, \delta}(Tz) = T^\rho u_{\epsilon, \delta}(z)
\]

for some positive \( \rho = \rho(\epsilon, \delta) \). We always extend \( u_{\epsilon, \delta} \) to a subharmonic function in the plane by putting it equal to zero outside \( \Omega_{\epsilon, \delta} \). To complete the proof we need to show that for every fixed \( T \) and \( \tau \in (1, T) \) there exist \( \epsilon \) and \( \delta \) such that

\[
(8) \quad \rho(\epsilon, \delta) = 1.
\]

In this case the function \( u_{\epsilon, \delta}(z) \) after multiplication by a suitable positive constant will belong to \( \mathcal{H} \).

To prove (8) we show that \( \rho(\epsilon, \delta) \) is continuous for all admissible (so small that \( \Omega_{\epsilon, \delta} \) is connected) values of \( \epsilon \) and \( \delta \) and takes values greater and smaller than 1.

First, we show that \( \rho(0, 0) < 1 \). Indeed, \( w_0 \) is positive and harmonic in the upper halfplane. So

\[
(9) \quad \int_{\mathbb{R}} \frac{u_{0, 0}(x)}{1 + x^2} dx < \infty,
\]

which is consistent with (7) only if \( \rho < 1 \).

Second, it follows from the Phragmén-Lindelöf Principle that \( \rho(\epsilon, \delta) > 1 \) if \( \delta > \pi/2 \).

Since the continuity of \( \rho(\epsilon, \delta) \) is proved in Appendix 1, this completes the proof of Lemma 1.

We need also the following lemma.

**Lemma 2.** For every \( \Delta \in (0, 1) \) and \( T > 1 \) there exist \( \tau \in (1, T) \) and a subharmonic function \( v \) such that

\[
(10) \quad v(-r) = B(v, r) = r^\Delta \quad \text{for } r \in [0, \infty),
\]

\[
(11) \quad v(r) < -2r^\Delta \quad \text{for } r \in \bigcup_{k=-\infty}^{\infty} [\tau^{-1/\Delta}T^{k/\Delta}, T^{k/\Delta}].
\]
Proof. Let $M = T^{1/\Delta}$. Consider the function

\[
g(z) = \int_0^{+\infty} \log \left| 1 - \frac{z^2}{t^2} \right| d\mu(t),
\]

where $\mu$ is the measure supported by the set $\{M^k\}_{k=-\infty}^{\infty}$ and

\[
\mu\{M^k\} = M^{\Delta k}, \quad k \in \mathbb{N}.
\]

It is easy to see that integral (12) defines a subharmonic function with $g(Mz) = M^\Delta g(z), \quad z \in \mathbb{C}$. Now put

\[
f(re^{i\theta}) = f(re^{-i\theta}) = r^\Delta \cos(\Delta(\theta - \pi/2)), \quad 0 \leq \theta \leq \pi.
\]

The function $f$ is subharmonic in the plane. Then the function

\[
u_r(z) = \gamma g(z) + f(z)
\]
is subharmonic for every positive $\gamma$ and satisfies the relations

\[
r^\Delta(1 + \gamma A) \geq u_r(\pm ir) = B(r, u_r) \geq r^\Delta(1 - \gamma A), \quad r \in \mathbb{R},
\]

where $A = \sup\{g(ir)/r^\Delta : r > 0\}$ (it is clear that $0 < A < \infty$). Setting

\[
\tilde{u}_r(re^{i\theta}) = \begin{cases} u_r(z) & \text{for } |\theta| \leq \pi/2, \\ \max\{B(r, u_r), (1 + A\gamma)r^\Delta \cos(\Delta(\pi - \theta))\} & \text{for } |\pi - \theta| < \pi/2, \end{cases}
\]

we show that according to (13) this function is subharmonic if $\gamma > 0$ is small enough. Indeed, if $\gamma > 0$ is sufficiently small then the inequality

\[
(1 + \gamma A)\cos(\Delta\pi/2) < (1 - \gamma A) \leq B(r, u_r)r^{-\Delta}
\]
is valid, so $\tilde{u}_r$ is equal to $u_r$ in some angle $\{re^{i\theta} : |\theta| < \delta\}, \delta > \pi/2$, and $\tilde{u}_r(-r) = B(r, \tilde{u}_r) = r^\Delta(1 + \gamma A)$.

Using (13), (14), and the equality $\tilde{u}_r(1) = -\infty$ we conclude that the function

\[
u(z) = (1 + \gamma A)^{-1}\tilde{u}_r(z)
\]
satisfies all conditions of Lemma 2 for some suitable $\tau$. Lemma 2 is proved.

Construction of the example

Suppose $\rho \in (1, 2)$, and let $\Delta = \rho - 1$. We choose $T > 1$ arbitrarily; for example, put $T = 2$. According to Lemma 2 there exist $\tau \in (1, T)$ and a subharmonic function $v$ satisfying (10) and (11). According to Lemma 1 for this $\tau$ there exists a subharmonic function $g$ which satisfies (1) and (2) for some $\alpha > 0$. Define the function $u$ in the following way:

\[
u(re^{i\theta}) = \begin{cases} g(r^{1+\Delta}e^{i(1+\Delta)}\theta) & \text{for } |\theta| \leq \pi/(1 + \Delta), \\ v(-r^{1+\Delta}e^{i(1+\Delta)}\theta) & \text{for } |\pi - \theta| < \pi\Delta/(1 + \Delta). \end{cases}
\]

It follows from (1) and (10) that this function $u$ is subharmonic in the plane and satisfies (ii). Property (iii) follows from (2) and (11).

For $\rho \geq 2$ consider $u(z^n)$ with a suitable $n \in \mathbb{N}$.

The example is constructed.
THE MAXIMUM AND MINIMUM OF A SUBHARMONIC

APPENDIX 1

The results in this appendix are essentially known, but we could not find a convenient reference.

1. Let \( \Omega = \Omega_{\varepsilon, \delta} \) be the domain described in the proof of Lemma 1. We prove that there exists a function \( u \) positive and harmonic in \( \Omega \) and vanishing on \( \partial \Omega \). Denote by \( G \) the Green function of \( \Omega \); fix a point \( z_0 \in \Omega \); and choose a sequence \( z_k \to \infty, \; z_k \in \Omega \). Then the sequence of positive harmonic functions \( u_k(z) = G(z, z_k)/G(z_0, z_k) \) is equicontinuous at every point \( z \in \Omega \). If we extend \( u_k \) by putting \( u_k(z) = 0, \; z \in \mathbb{C} \setminus \Omega \), then the sequence \( u_k \) will be equicontinuous in \( \Omega \). This follows from a simple application of the two-constants theorem. Thus any limit function \( u \) of the sequence \( u_k \) is positive and harmonic in \( \Omega \) and vanishes on the boundary. We have \( u(z_0) = 1 \), so \( u \not\equiv 0 \).

2. Let us prove that a positive harmonic function in \( \Omega \) vanishing on the boundary is unique up to a multiplication by a positive constant. We use a modification of the argument in [Kj].

Let \( U \) be the set of all such functions. Without loss of generality we may suppose that \( \delta \) in the definition of the domain \( \Omega \) is equal to \( \pi/2 \) since the general case may be reduced to this by conformal mapping.

Let \( K = \{ r_0 e^{i\theta} : |\theta| \leq \pi/2 \} \), \( r_0 \in \Omega \). Let \( u \in U \). It is easy to see that the function \( g(z) = u(z)/\Re z \) defined in the right halfplane can be extended continuously to \( K \) and has a positive infimum on \( K \).

The basic assertion that we will use for the proof of the uniqueness is the following. There exists \( d > 0 \) such that for every \( u \in U \) we have

\[
\max_{z \in K} g(z) / \min_{z \in K} g(z) \leq d.
\]

Assume that (15) is false. Then there exists a sequence \( u_k(z) \in U \) such that

\[
\max_{z \in K} g(z) / \min_{z \in K} g(z) > k.
\]

We may suppose \( B(2r_0, u_k) = 1 \). By the Harnack inequality the sequence \( \{ u_k \} \) is a precompact family of harmonic functions on \( \Omega \), and by normalization no subsequence of this sequence tends to \( \infty \) or to identical \( 0 \). Applying the two-constants theorem we see that all limit functions vanish on the boundary, so all limit functions belong to \( U \). So (16) cannot occur and (15) holds.

We remark that (15) remains true (with the same constant \( d \)) if we take \( z \in T^k K \) instead of \( z \in K \) since \( U \) is invariant with respect to the transformations \( u(z) \leftrightarrow u(T^k z) \).

Let \( u, v \in U \) be two different functions. Let \( \alpha_k \) be the sequence defined as

\[ \alpha_k = \sup \{ \alpha : u(z) - \alpha_k v(z) \geq 0 \text{ for } z \in T^k K \}. \]

It is evident that \( \alpha_k \searrow \alpha \) for some nonnegative \( \alpha \). Then set \( \eta_k = \alpha_k - \alpha \to 0 \).

The sequence

\[ u(z) - \alpha_k v(z) = r_k(z) \]

converges to a nonnegative function \( r(z) = r_k(z) + \eta_k v(z) \), and either \( r(z) \in U \) or \( r(z) \equiv 0 \). According to the definition of \( \alpha_k \) the function \( r_k(z)/\Re z \) is zero
at some point of $T^kK$, since otherwise (15) implies that $(r_k(z) - \eta v_k(z))/Rz$ will be positive on $T^kK$ for some $\eta > 0$. Thus we have

$$r(z_k) = \eta v(z_k)$$

for some point $z_k \in T^kK$. Therefore, applying (15) to both the left and right sides of the last relation we obtain

$$r(z) < d^2 \eta v(z)$$

for every point $z \in K$. (This proof is valid when $r(z) \in U$. When $r(z) \equiv 0$ this inequality is trivial.) By the maximum principle

$$r(z) < d^2 \eta v(z) \quad \text{for } |z| < T^kK;$$

hence $r(z) \equiv 0$. So we get $u(z) = \alpha v(z)$, and hence the uniqueness is proved.

3. Finally we prove that the order $\rho(\epsilon, \delta)$ of the positive harmonic function $u_{\epsilon, \delta}$ in $\Omega_{\epsilon, \delta}$ vanishing on the boundary is continuous for $0 \leq \delta \leq \pi/2$ and $0 \leq \epsilon \leq \epsilon_0$, where $\epsilon_0$ is small enough. It follows from the uniqueness of $u_{\epsilon, \delta}$ that $u_{\epsilon, \delta}(T_z) = C u_{\epsilon, \delta}(z)$, where $C = T^\rho$. So

$$\rho(\epsilon, \delta) = \frac{\log(u_{\epsilon, \delta}(T_{z_0})/u_{\epsilon, \delta}(z_0))}{\log T},$$

where $z_0$ is an arbitrary point in $\Omega_{\epsilon, \delta}$. Assume that $\epsilon_n \to \epsilon_0$, $\delta_n \to \delta_0$. We may suppose that $z_0 \in \Omega_{\epsilon_n, \delta_n}$ for all $n$. Then (after selection of a subsequence if necessary) we have $u_{\epsilon_n, \delta_n} \to v$, on compacts in $\Omega_{\epsilon_0, \delta_0}$, where $v$ is a positive harmonic function in $\Omega_{\epsilon_0, \delta_0}$, vanishing on the boundary. From the uniqueness it follows that $v = C u_{\epsilon_0, \delta_0}$, and we have

$$\rho(\epsilon_n, \delta_n) \to \rho(\epsilon_0, \delta_0)$$

in view of (17).

APPENDIX 2

In this appendix we show how to get the corresponding entire example from the subharmonic one. We need an approximation theorem of Azarin [A] and the notion of a $C_0$-set.

First we give the definition of a $C_0$-set. A set $E$ is called a $C_0$-set if it may be covered by disks $B(\rho_k, \xi_k)$ ($\rho_k$ are their radii, $\xi_k$ are the centers) such that the relation

$$\lim_{r \to \infty} \frac{1}{r} \sum_{|\xi_k| \leq r} \rho_k = 0$$

is valid.

Now we cite Azarin's theorem on the approximation of subharmonic functions.

Theorem A. Let $u(z)$ be a subharmonic function of finite proximate order $\rho(r)$. Then there exists an entire function $f(z)$ of the same proximate order $\rho(r)$ and the same type with respect to this order, and there exists a $C_0$-set $E$ such that the estimate

$$\log |f(z)| - u(z) = o(|z|^{\rho(|z|)}), \quad |z| \to \infty,$$

holds outside $E$. 
Let $D$ be an open set intersecting every circle $|z| = r$ such that $TD = D$, $T$ being taken from the definition of the subharmonic function $u(z)$. Let $E$ be a $C_0$-set. It is easy to see that there exists $r_0$ such that the set $D \setminus E$ intersects every circle of radii larger than $r_0$. Indeed, if this is not so then there exists a sequence $r_k \to \infty$ such that

$$\text{mes}\{\varphi : r_k e^{i\varphi} \in E\} \geq \text{mes}\{\varphi : r_k e^{i\varphi} \in D\} \geq \delta > 0, \quad r_k \to \infty.$$ 

But the latter inequality contradicts the definition of a $C_0$-set.

Now taking $\epsilon > 0$ small enough to satisfy $1 + \alpha - \epsilon > 1$ and setting $D = \{z : u(z) < -(1 + \alpha - \epsilon)|z|^\rho\}$ we can construct by Azarin's theorem an entire function $f(z)$ and a $C_0$-set $E$ such that

$$B(|z|, \log|f(z)|) - B(|z|, u(z)) = o(|z|^\rho), \quad |z| \to \infty, \quad z \in \mathbb{C},$$

and

$$\log|f(z)| - u(z) = o(|z|^\rho), \quad |z| \to \infty, \quad z \in \mathbb{C} \setminus D.$$ 

Since the set $D \setminus E$ intersects every circle $|z| = r$ for all sufficiently large $r > r_0$, we have

$$B(|z|, \log|f(z)|) = (1 + o(1))|z|^\rho, \quad |z| \to \infty,$$

and

$$L(|z|, \log|f(z)|) < -(1 + \alpha - \epsilon)(1 + o(1))|z|^\rho, \quad |z| \to \infty,$$

and hence the function $f(z)$ gives the needed example.

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**References**


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