COMMON SUBSPACES OF \( L_p \)-SPACES

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Abstract. For \( n \geq 2, p < 2, \) and \( q > 2 \) does there exist an \( n \)-dimensional Banach space different from Hilbert spaces which is isometric to subspaces of both \( L_p \) and \( L_q \)? Generalizing the construction from the paper Zonoids whose polars are zonoids by R. Schneider (Proc. Amer. Math. Soc. 50 (1975), 365-368) we give examples of such spaces. Moreover, for any compact subset \( Q \) of \( (0, \infty) \setminus \{2k, k \in \mathbb{N}\} \) we can construct a space isometric to subspaces of \( L_q \) for all \( q \in Q \) simultaneously.

1. Introduction

This work started with the following question: for given \( n \geq 2, p \in (0, 2), \) and \( q > 2 \) does there exist an \( n \)-dimensional Banach space which is different from Hilbert spaces and isometric to subspaces of both \( L_p \) and \( L_q \)?

It is a well-known fact first noticed by P. Levy that Hilbert spaces are isometric to subspaces of \( L_q \) for all \( q > 0 \). On the other hand, it was proved in [4] that for \( n \geq 3, q > 2, \) and \( p > 0 \) the function \( \exp(-\|x\|_p^p) \) is not positive definite where \( \|x\|_q = (|x_1|^q + \cdots + |x_n|^q)^{1/q} \). (This result gave an answer to a question posed by Schoenberg [12] in 1938.) In 1966 Bretagnolle, Dacunha-Castelle, and Krivine [1] proved that for \( 0 < p < 2 \) a space \( (E, \| \cdot \|) \) is isometric to a subspace of \( L_p \) if and only if the function \( \exp(-\|x\|^p) \) is positive definite. Thus, in the language of isometries, the above-mentioned result from [4] means that for every \( n \geq 3, q > 2, \) and \( p \in (0, 2) \) the space \( L_p^n \) is not isometric to a subspace of \( L_q \). (For \( p \geq 1 \) this fact was first proved in [2].)

The initial purpose of this work was to find a non-Hilbertian subspace \( (E, \| \cdot \|) \) of \( L_q \) with \( q > 2 \) of dimension at least 3 such that the function \( \exp(-\|x\|^p) \) is positive definite. The latter problem is equivalent to that at the beginning of the paper.

We prove, however, a more general fact: for every \( n \geq 2 \) and every compact subset \( Q \) of \( (0, \infty) \setminus \{2k, k \in \mathbb{N}\} \) there exists an \( n \)-dimensional Banach space different from Hilbert spaces which is isometric to subspaces of \( L_q \) for all \( q \in Q \) simultaneously.

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In 1975 Schneider [11] proved that there exist nontrivial zonoids whose polars are zonoids or, in other words, there exist non-Hilbertian Banach spaces $X$ such that $X^\ast$ are isometric to subspaces of $L_1$. It turns out that Schneider’s construction of special subspaces of $L_1$ can be extended to all numbers $q > 0$ which are not even integers, and in this way we obtain our main result.

2. Some properties of spherical harmonics

We start with some properties of spherical harmonics (see [6] for details).

Let $P_m$ denote the space of spherical harmonics of degree $m$ on the unit sphere $\Omega_n$ in $\mathbb{R}^n$. Recall that spherical harmonics of degree $m$ are restrictions to the sphere of harmonic homogeneous polynomials of degree $m$. We consider spherical harmonics as functions from the space $L_2(\Omega_n)$. Any two spherical harmonics of different degrees are orthogonal on $L_2(\Omega_n)$.

The dimension $N(n, m)$ of the space $P_m$ can easily be calculated [6, p. 4]:

$$N(n, m) = \frac{(2m + n - 2)\Gamma(n + m - 2)}{\Gamma(m + 1)\Gamma(n - 1)} .$$

Let $\{Y_{mj}: j = 1, \ldots, N(n, m)\}$ be an orthonormal basis of the space $P_m$. By the Addition Theorem [6, p. 9], for every $x \in \Omega_n$,

$$\sum_{j=1}^{N(n, m)} Y_{mj}^2(x) = \frac{N(n, m)}{\omega_n} ,$$

where $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of the sphere $\Omega_n$.

Linear combinations of functions $Y_{mj}$ are dense in the space $L_2(\Omega_n)$ [6, p. 43]. Therefore, if $F$ is a continuous function on $\Omega_n$ and $(F, Y_{mj}) = 0$ for every $m = 0, 1, 2, \ldots$ and every $j = 1, \ldots, N(n, m)$, then $F \equiv 0$ on $\Omega_n$. Here $(F, Y)$ stands for the scalar product in $L_2(\Omega_n)$.

Let $\Delta$ be the Laplace-Beltrami operator on the sphere $\Omega_n$. Then for every $Y_m \in P_m$ we have [6, p. 39]

$$\Delta Y_m + m(m + n - 2)Y_m \equiv 0 .$$

An immediate consequence of (3) (and a well-known fact) is that $\Delta$ is a symmetric operator, and we can apply Green’s formula: for every function $H$ from the class $C^{2r}$, $r \in \mathbb{N}$, of functions on $\Omega_n$ having continuous partial derivatives of order $2r$ and for every $Y_m \in P_m$, $m \geq 1$,

$$(-m(m + n - 2))^{r}(H, Y_m) = (H, \Delta' Y_m) = (\Delta' H, Y_m) .$$

We also need the Funk-Henke formula [6, p. 20]: for every $Y_m \in P_m$, every continuous function $f$ on $[-1, 1]$, and every $x \in \Omega_n$

$$\int_{\Omega_n} f((x, \xi)) Y_m(\xi) d\xi = \lambda_m Y_m(x) ,$$

where $\langle \cdot, \cdot \rangle$ stands for the scalar product in $\mathbb{R}^n$ and

$$\lambda_m = \frac{(-1)^m \pi^{(n-1)/2}}{2^{m-1}\Gamma(m + (n - 1)/2)} \int_{-1}^{1} f(t) \frac{d^m}{dt^m} (1 - t^2)^{m+(n-3)/2} \, dt .$$

Let us calculate $\lambda_m$ in the case where $f(t) = |t|^q$, $q > 0$. The numbers $\lambda_m$ have already been calculated by Richards [8]. However, we present a simple proof to make this paper complete.
Lemma 1. If $m$ is an even integer, $q > 0$, $q \neq 2k$, $k \in N$, and $f(t) = |t|^q$, then

$$
\lambda_m = \frac{\pi^{n/2-1} \Gamma(q + 1) \sin(\pi(m - q)/2) \Gamma((m - q)/2)}{2^{q-1} \Gamma((m + n + q)/2)}.
$$

Note that $\lambda_m = 0$ for odd $m$.

Proof. First assume that $q > m$, and calculate the integral from (6) by parts $m$ times. Then use the formula $\int_{-1}^{1} |t|^{a-1}(1 - t^2)^{b-1} \, dt = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha + \beta)$ and formulas for the $\Gamma$-function: $\Gamma(2x) = 2^{2x-1}\Gamma(x)\Gamma(x + 1/2)/\pi^{1/2}$ and $\Gamma(1 - x)\Gamma(x) = \pi/\sin(\pi x)$. We get (7) for $q > m$. Note that both sides of (7) are analytic functions of $q$ in the domain $\Re q > 0$, $q \neq 2k$, $k \in N$. Because of the uniqueness of analytic extension, (7) holds for every $q$ from this domain. We are done. \(\Box\)

3. Main result

Let $X$ be an $n$-dimensional subspace of $L_q = L_q([0, 1])$ with $q > 0$. Let $f_1, \ldots, f_n$ be a basis in $X$ and $\mu$ the joint distribution of the functions $f_1, \ldots, f_n$ with respect to Lebesgue measure ($\mu$ is a finite measure on $R^n$). Then for every $x \in R^n$

$$
\|x\|^q = \left\| \sum_{k=1}^{n} x_k f_k \right\|^q = \int_0^1 \left| \sum_{k=1}^{n} x_k f_k(t) \right|^q \, dt = \int_{R^n} |\langle x, \xi \rangle|^q \, d\mu(\xi) = \int_{\Omega_n} |\langle x, \xi \rangle|^q \, d\nu(\xi),
$$

where $\nu$ is the projection of $\mu$ to the sphere. (For every Borel subset $A$ of $\Omega_n$, $\nu(A) = \int_{\{x \in R^n \mid \|x\|^2 \in A \}} \|x\|^2 \, d\mu(x)$.) Representation (8) of the norm is usually called the Levy representation. It is clear now that a norm in an $n$-dimensional Banach space admits the Levy representation with a probability measure on the sphere if and only if this space is isometric to a subspace of $L_q$. (Given the Levy representation we can choose functions $f_1, \ldots, f_n$ on $[0, 1]$ with the joint distribution $\nu$ and define an isometry by $x \rightarrow \sum_{k=1}^{n} x_k f_k$, $x \in R^n$.)

If we replace the measure $\nu$ by an arbitrary continuous (not necessarily nonnegative) function on the sphere, then a representation similar to the Levy representation is possible for a large class of Banach spaces (see [5] for the Levy representation with distributions instead of measures; such a representation is possible for any Banach space and any $q$ which is not an even integer). This is an idea going back to Blaschke that any smooth enough function on the sphere can be represented in the form (8) with a continuous function instead of a measure on the sphere. However, Blaschke and then Schneider [10] restricted themselves to the case $q = 1$ which is particularly important in the theory of convex bodies. The following theorem is an extension of Schneider’s results from [10, p. 77] and [11, p. 367] to all positive numbers $q$ which are not even integers. Note that for $q \in (0, 2)$ representation (9) was obtained and applied to determining spectral measures of stable laws by Richards in [9].

Theorem 1. Let $q > 0$, $q \neq 2k$, $k \in N$, and let $H$ be an even function of the class $C^{2r}$ on $\Omega_n$, where $r \in N$ and $2r > n + q$. Then there exists a continuous
function $b_H$ on the sphere $\Omega_n$ such that for every $x \in \Omega_n$

\begin{equation}
H(x) = \int_{\Omega_n} |(x, \xi)|^q b_H(\xi) \, d\xi.
\end{equation}

Besides that, there exist constants $K(q)$ and $L(q)$ depending on $n$ and $q$ only such that for every $x \in \Omega_n$

\begin{equation}
|b_H(x)| \leq K(q) \|H\|_{L_2(\Omega_n)} + L(q) \|\Delta' H\|_{L_2(\Omega_n)}.
\end{equation}

Proof. Define a function $b_H$ on $\Omega_n$ by

\begin{equation}
b_H(x) = \sum_{m=0;2|m}^{\infty} \lambda_m^{-1} \sum_{j=1}^{N(n,m)} (H, Y_{mj}) Y_{mj}(x).
\end{equation}

Let us prove that the series in the right-hand side of (11) converges uniformly on $\Omega_n$. By the Cauchy-Schwartz inequality, (2), and the fact that $Y_{mj}$ form an orthonormal basis in $P_m$, we get

\[
\left| \sum_{j=1}^{N(n,m)} (\Delta' H, Y_{mj}) Y_{mj}(x) \right| \leq \left( \sum_{j=1}^{N(n,m)} (\Delta' H, Y_{mj})^2 \right)^{1/2} \left( \sum_{j=1}^{N(n,m)} Y_{mj}^2(x) \right)^{1/2} \\
\leq \|\Delta' H\|_{L_2(\Omega_n)} \left( \frac{N(n,m)}{\omega_n} \right)^{1/2} 
\]

It follows from (4) and the latter inequality that

\[
|b_H(x)| \leq |\lambda_0^{-1}(H, Y_0) Y_0(x)| \\
+ \sum_{m=2;2|m}^{\infty} \lambda_m^{-1} \left( \frac{-1}{m(m+n-2)} \right)^r \left| \sum_{j=1}^{N(n,m)} (\Delta' H, Y_{mj}) Y_{mj}(x) \right| \\
\leq |\lambda_0^{-1}\omega_n^{-1/2}\|H\|_{L_2(\Omega_n)} + \sum_{m=2;2|m}^{\infty} \lambda_m^{-1} m^{-2r} \left( \frac{N(n,m)}{\omega_n} \right)^{1/2} \|\Delta' H\|_{L_2(\Omega_n)}. 
\]

Let us show that the series $\sum_{m=2;2|m}^{\infty} \lambda_m^{-1} m^{-2r} (N(n,m)/\omega_n)^{1/2}$ converges. In fact, it follows from (1) that $N(n,m) = O(m^{n-2})$, and it is an easy consequence of (7) and the Stirling formula that $\lambda_m^{-1} = O(m^{-(n+2q)/2})$. Since $2r > n + q = (n+2q)/2 + (n-2)/2 + 1$, we get $\lambda_m^{-1} m^{-2r} (N(n,m)/\omega_n)^{1/2} = o(m^{-1+\epsilon})$ for some $\epsilon > 0$, and the series is convergent. We denote the sum of this series by $L(q)$ and put $K(q) = |\lambda_0|^{-1}\omega_n^{-1/2}$, so we get (10).

We have proved that the series in (11) converges uniformly and defines a continuous function on $\Omega_n$. It follows from (5) and the fact that all functions $Y_{mj}$ are orthogonal that $(H, Y_{mj}) = (\int_{\Omega_n} |(x, \xi)|^q b_H(\xi) \, d\xi, \ Y_{mj}(x))$ for every $m = 0, 1, 2, \ldots$ and $j = 1, \ldots, N(n,m)$. Hence, the function $b_H$ satisfies (9). □

Let $X$ be an $n$-dimensional Banach space, and let $q > 0$, $q \neq 2k$, $k \in N$. Let $c(q) = \Gamma((n+q)/2)/(2\Gamma((q+1)/2)\pi^{(n-1)/2})$ be a constant such that $1 = c(q) \int_{\Omega_n} |(x, \xi)|^q \, d\xi$ for every $x \in \Omega_n$. (The latter integral does not depend on the choice of $x \in \Omega_n$; it means that the norm of the space $l_2^n$ admits the Levy representation with the uniform measure on the sphere and the space $l_2^n$ is isometric to a subspace of $L_q$ for every $q$.)
Denote by $H(x), x \in \Omega_n$, the restriction of the function $\|x\|^q$ to the sphere $\Omega_n$. Assume that the function $H$ belongs to the class $C^{2r}$ on $\Omega_n$, where $2r > n + q$, $r \in \mathbb{N}$. Let $b_H$ be the function corresponding to $H$ by Theorem 1.

**Lemma 2.** If the number $K(q)\|H - 1\|_{L^2(\Omega_n)} + L(q)\|\Delta^r H\|_{L^2(\Omega_n)}$ is less than $c(q)$, then the space $X$ is isometric to a subspace of $L_q$.

**Proof.** By (9) and definition of the number $c(q)$,
\[
H(x) - 1 = \int_{\Omega_n} |\langle x, \xi \rangle|^q (b_H(\xi) - c(q)) \, d\xi
\]
for every $x \in \Omega_n$. By (9), $|b_H(x) - c(q)| < c(q)$ for every $x \in \Omega_n$. It means that the function $b_H$ is positive on the sphere. Equality (9) implies that the space $X$ admits the Levy representation with a nonnegative measure, and, by the reasoning at the beginning of §3, $X$ is isometric to a subspace of $L_q$. 

Now we are able to prove the main result of this paper. Let us only note that for every function $f$ of the class $C^2$ on the sphere $\Omega_n$ and for a small enough number $\lambda$ the function $N(x) = 1 + \lambda f(x), x \in \Omega_n$, is the restriction to the sphere of some norm in $\mathbb{R}^n$. This is an easy consequence of the following one-dimensional fact: If $a, b \in \mathbb{R}$, $g$ is a convex function on $[a, b]$ with $g'' > \delta > 0$ on $[a, b]$ for some $\delta$ and $h \in C^2[a, b]$, then functions $g + \lambda h$ have positive second derivatives on $[a, b]$ for sufficiently small $\lambda$'s and, hence, are convex on $[a, b]$.

**Theorem 2.** Let $Q$ be a compact subset of $(0, \infty) \setminus \{2k, k \in \mathbb{N}\}$. Then there exists a Banach space different from Hilbert spaces which is isometric to a subspace of $L_q$ for every $q \in Q$.

**Proof.** Let $f$ be any infinitely differentiable function on $\Omega_n$, and fix a number $r \in \mathbb{N}$ so that $2r > n + q$ for every $q \in Q$. Choose a sufficiently small number $\lambda$ such that the function $N(x) = 1 + \lambda f(x), x \in \Omega_n$, is the restriction to the sphere of some norm in $\mathbb{R}^n$ (see the remark before Theorem 2) and such that for every $q \in Q$, the function $H(x) = (N(x))^q$ satisfies the condition of Lemma 2. The possibility of such a choice of $\lambda$ follows from the facts that $K(q), L(q)$, and $c(q)$ are continuous functions of $q$ on the set $Q$ and that $\|H - 1\|_{L^2(\Omega_n)}$ and $\|\Delta^r H\|_{L^2(\Omega_n)}$ tend to zero uniformly with respect to $q \in Q$ as $\lambda$ tends to zero. Now we can apply Lemma 2 to complete the proof. 

Finally, let us consider the case where $q$ is an even integer. It is easy to see that for any fixed number $2k$, $k \in \mathbb{N}, k > 1$, we can make the space $X$ constructed in Theorem 2 isometric to a subspace of $L_{2k}$. In fact, let $N(x) = (1 + \lambda (x_1^{2k} + \cdots + x_n^{2k}))^{1/2k}$. For sufficiently small numbers $\lambda$, $N$ is the restriction to the sphere of some norm in $\mathbb{R}^n$ and the corresponding space $X$ is isometric to a subspace of $L_{2k}$ for every $q \in Q$. On the other hand, $X$ is isometric to a subspace of $L_{2k}$ because the norm admits the Levy representation with a measure on the sphere
\[
1 + \lambda (x_1^{2k} + \cdots + x_n^{2k}) = \int_{\Omega_n} |\langle x, \xi \rangle|^{2k} (c(2k) \, d\xi + \lambda \, d\delta_1(\xi) + \cdots + \lambda \, d\delta_n(\xi)),
\]
where $\delta_i$ is a unit mass at the point $\xi \in \mathbb{R}_n$ with $\xi_i = 1, \xi_j = 0, j \neq i$. 


Let us show that one cannot make the space $X$ isometric to subspaces of $L_{2^p}$ and $L_{2^q}$ if $p, q \in N$ and do not have common factors. In fact, if $(X, \| \cdot \|)$ is such a space, then for every $x \in R_n$

$$\|x\|^{4pq} = \left( \int_{\Omega_n} |\langle x, \xi \rangle|^{2p} d\mu(\xi) \right)^{2q} = \left( \int_{\Omega_n} |\langle x, \xi \rangle|^{2q} d\nu(\xi) \right)^{2p}$$

for some measures $\mu, \nu$ on $\Omega_n$. The functions in the latter equality are polynomials, and since the polynomial ring has the unique factorization property, we conclude that $\|x\|^2$ is a homogeneous polynomial of the second order and $X$ is a Hilbert space.

The situation is not clear if $p$ and $q$ have common factors. One can find some interesting results on Banach spaces with polynomial norms and on the structure of subspaces of $L_{2^k}$, $k \in N$ [7].

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