LIFTING VECTOR-VALUED MEROMORPHIC FUNCTIONS IN INFINITE DIMENSIONS

NGUYEN VAN KHUE AND NGUYEN THU NGA

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Abstract. It is shown that the lifting problem for Fréchet-valued meromorphic functions on open subsets of a (DFN)-space has a solution.

Lifting holomorphic functions in infinite dimensions has been investigated by some authors. The problem for vector-valued meromorphic functions on complex manifolds was studied in [6]. The aim of this paper is to prove that the lifting problem has a solution for Fréchet-valued meromorphic functions on open subsets of a (DFN)-space.

1. Preliminaries

We shall use the standard notation from the theory of locally convex spaces as presented in the books of Pietsch [7] and Schaefer [8]. All locally convex spaces are assumed to be complex vector spaces and Hausdorff.

For a locally convex space $E$, we denote by $\mathcal{U}(E)$ the set consisting of all balanced convex neighbourhoods of zero in $E$. Let $U \in \mathcal{U}(E)$ and $p_U$ denote the Minkowski functional on $E$ associated to $U$. Then $E_U$ denotes the completion of the canonical normed space $E/\text{Ker} p_U$. The canonical map from $E$ into $E_U$ is written by $\pi_U$.

2. Holomorphic and meromorphic functions

Let $E$ and $F$ be locally convex spaces and $D \subseteq E$ be open. A map $f: D \to F$ is called holomorphic if $f$ is continuous and $f|D \cap V$ is holomorphic for every finite dimensional subspace $V$ of $E$.

Now a holomorphic function $f: D_0 \to F$, where $D_0$ is a dense open subset of $D$, is said to be meromorphic on $D$ if for every $z \in D$ there exists a neighbourhood $U$ of $z$ in $D$ and holomorphic functions $g: U \to F$, $\sigma: U \to \mathbb{C}$ such that

$$f|U \cap D_0 = g/\sigma|U \cap D_0 \quad \text{with} \quad \sigma \neq 0.$$
Theorem 1. Let $S$ be a continuous linear map of a Fréchet space $E$ onto a Fréchet space $F$ and let $D$ be an open subset of a (DFN)-space $P$. Then for every $F$-valued meromorphic $f$ on $D$ there exists an $E$-valued meromorphic function $g$ on $D$ such that $Sg = f$.

To prove the theorem we need the following.

Lemma 2. Every holomorphic function on an open subset of a (DFN)-space with values in a Fréchet space $F$ can be factorized through a compact map from a Banach space into $F$.

Proof. Consider an open set $D$ in a (DFN)-space $P$ and a holomorphic function $f$ on $D$ with values in $F$.

(i) Let $z \in D$. Since $D$ is $\sigma$-compact [5], it can be exhausted by an increasing sequence of compact sets $\{K_n\}$, with $z \in K_1$. Let $\{V_n\}$ be a decreasing neighbourhood basis of zero in $F$. For each $n \geq 1$ there exists $U_n \in \mathcal{U}(P)$ and $d_n > 0$ such that

$$K_n + U_n \subseteq D \quad \text{and} \quad f(K_n + U_n) \subseteq d_nV_n.$$

Set

$$U = \bigcap_{n \geq 1} (K_n + U_n).$$

Since

$$K_n \cap U = \bigcap_{1 \leq k \leq n} (K_k + U_k) \cap K_n$$

and since $\bigcap_{1 \leq k \leq n} (K_k + U_k)$ is a neighbourhood of $z$ in $P$, it follows that $K_n \cap U$ is a neighbourhood of $z$ in $K_n$ for every $n \geq 1$. On the other hand, since $D$ is a $k$-space [5], $U$ is a neighbourhood of $z$ in $D$. From the inclusion

$$f(U) \subseteq f(K_n + U_n) \subseteq d_nV_n \quad \text{for every } n \geq 1$$

we obtain the boundedness of $f(U)$.

Consider the Taylor expansion of $f$ at $z$:

$$f(z + h) = \sum_{n \geq 0} P_nf(z)(h),$$

where

$$P_nf(z)(h) = \frac{1}{2\pi i} \int_{|\lambda| = 2} f(z + \lambda h)/\lambda^{n+1} \, d\lambda$$

for $h \in V$, $V \in \mathcal{U}(P)$, $z + 2V \subseteq U$. Set

$$B = \overline{\text{conv}} \bigcup_{n \geq 0} P_nf(z)(V).$$

Then $B$ is a balanced convex closed set in $F$ and $f$ induces a holomorphic function on $\pi_V(z + V)$ with values in the canonical Banach space $F(B)$ spanned by $B$.

(ii) By (i) we can find a countable open cover of $D$, $\{\tilde{U}_i = z_i + U_i\}$, $U_i \in \mathcal{U}(P)$, a sequence of balanced convex bounded closed sets in $F$, $\{B_i\}$, and a sequence of holomorphic functions $f_i: \pi_{U_i}(\tilde{U}_i) \to F(B_i)$ such that

$$f_i\pi_{U_i} = f|\tilde{U}_i \quad \text{for every } i \geq 1.$$
Take two sequences \( \{\lambda_i\} \downarrow 0 \) and \( \{\mu_i\} \uparrow \infty \) such that
\[
B = \text{conv} \bigcup_{i \geq 1} \lambda_i B_i \quad \text{is bounded in } F
\]
and
\[
U = \bigcap_{i \geq 1} \mu_i U_i \in \mathcal{U}(P) .
\]
Such sequences exist by [4] for \( \{\lambda_i\} \) and [5] for \( \{\mu_i\} \). Since the canonical map from \( F(\lambda_i B_i) \) into \( F(B) \) is continuous for every \( i \geq 1 \), and \( \{\pi_U(\tilde{U}_i)\} \) is an open cover of \( \pi_U(D) \) in \( P/\text{Ker} p_U \), it follows that the sequence \( \{f_i\} \) defines an \( F(B) \)-valued holomorphic function \( g \) on \( \pi_U(D) \) with \( g \pi_U = f \).

(iii) Let \( \tilde{g} \) be a holomorphic extension of \( g \) to a neighbourhood \( \tilde{D}_U \) of \( \pi_U(D) \) in \( P_U \). Take \( V \in \mathcal{U}(P) \), \( V \subseteq U \) such that the canonical map \( \pi_{V,U}: P_V \to P_U \) is compact. Cover \( \tilde{D}_V = \pi_{V,U}^{-1}(\tilde{D}_U) \) by a sequence of bounded open sets in \( P_V \), \( \{W_i\} \), such that \( \pi_{V,U}(W_i) \) is relatively compact in \( \tilde{D}_U \) for every \( i \geq 1 \). Then \( A_i = g \pi_{V,U}(W_i) \) for every \( i \geq 1 \) is relatively compact in \( F(B) \). Take again a sequence \( \{\alpha_i\} \downarrow 0 \) such that
\[
A = \text{conv} \bigcup_{i \geq 1} \alpha_i A_i
\]
is compact in \( F(B) \). It is easy to see that \( \tilde{g} \pi_{V,U}: \tilde{D}_V \to F(A) \) is holomorphic. Hence \( f \) can be factorized through the compact map \( F(A) \to F \). The lemma is thus proved.

3. Proof of Theorem 1

Cover \( D \) by a sequence of open subsets \( \{\tilde{U}_i\} \) of \( D \) such that \( f|\tilde{U}_i \) can be written in the form \( h_i/\sigma_i \), where \( h_i \) and \( \sigma_i \) are holomorphic functions on \( \tilde{U}_i \) with values in \( F \) and \( \mathbb{C} \), respectively. By Lemma 2, for each \( i \geq 1 \) we can find \( U_i \in \mathcal{U}(P) \) and \( B_i \), a balanced convex compact set in \( F \) such that \( h_i \) and \( \sigma_i \) are factorized through \( \pi_{U_i}: \tilde{U}_i \to \pi_{U_i}(\tilde{U}_i) \) and \( F(B_i) \to F \). Take two sequences \( \{\lambda_i\} \downarrow 0 \) and \( \{\mu_i\} \uparrow \infty \) such that
\[
B = \text{conv} \bigcup_{i \geq 1} \lambda_i B_i \quad \text{is compact in } F
\]
and
\[
U = \bigcap_{i \geq 1} \mu_i U_i \in \mathcal{U}(P) .
\]
This implies that the two sequences \( \{h_i\} \) and \( \{\sigma_i\} \) define a meromorphic function \( g \) on a neighbourhood \( \tilde{D}_U \) of \( \pi_U(D) \) in \( P_U \) with \( f = g \pi_U \). By [1] there exists a balanced convex compact set \( A \) in \( E \) such that \( S(A) = B \).

Cover \( \tilde{D}_U \) by a sequence of open sets \( \{W_j\} \) in \( \tilde{D}_U \) such that for each \( j \geq 1 \) there exist bounded holomorphic functions \( g_j \) and \( \sigma_j \) on \( W_j \) with values in \( F(B) \) and \( \mathbb{C} \), respectively, with \( g_j W_j = g_j/\sigma_j \), \( \sigma_j \neq 0 \). Let \( V \in \mathcal{U}(P) \), \( V \subseteq U \) such that \( T = \pi_{V,U} \) is nuclear. Thus \( T \) can be written in the form
\[
T(x) = \sum_{k \geq 1} \lambda_k(x)e_k
\]
with \( a = \sum_{k \geq 1} \| \lambda_k \| e_k \| < \infty \). Fix an index \( j \). For each \( x \in T^{-1}(W_j) \), set 
\( 2r_{j,x} = p_U(Tx, \partial W_j) > 0 \). Consider the Taylor expansion of \( g_j \) at \( T(x) \):
\[
g_j(Tx + z) = \sum_{n \geq 0} P_n g_j(Tx)(z)
\]
for \( \| z \| < 2r_{j,x} \), \( z \in P_U \), where
\[
P_n g_j(Tx)(z) = \frac{1}{2\pi i} \int_{|\lambda| = r_{j,x}} g_j(Tx + \lambda z)/\lambda^{n+1} d\lambda
\]
for \( \| z \| \leq 1 \), \( z \in P_U \). We have
\[
\| P_n g_j(Tx) \| \leq M_j/(r_{j,x})^n \quad \text{with} \quad M_j = \sup\{ \| g_j(z) \| : z \in W_j \}.
\]
Therefore
\[
\sum_{n \geq 0} \sum_{k_1, \ldots, k_n \geq 1} (\delta_{j,x})^n \| \lambda_{k_1} \| \| e_{k_1} \| \cdots \| \lambda_{k_n} \| \| e_{k_n} \| \\
\times \| P_n g_j(Tx)(e_{k_1}/\| e_{k_1} \|, \ldots, e_{k_n}/\| e_{k_n} \|) \|
\leq M_j \sum_{n \geq 0} (1/n!)(\delta_{j,x} n/r_{j,x})^n \left( \sum_{k \geq 1} \| \lambda_k \| \| e_k \| \right)^n
\]
\[
= M_j \sum_{n \geq 0} (1/n!)(\delta_{j,x} nna/r_{j,x})^n < \infty \quad \text{with} \quad \delta_{j,x} = r_{j,x}/2ae.
\]
It follows that \( g_jT|x + \delta_{j,x}B_V \), where \( B_V \) is the unit ball in \( P_V \), can be written in the form
\[
g_jT(z) = \sum_{\alpha \in \mathcal{A}} \lambda_\alpha (z - x)a_\alpha^{j,x}
\]
for \( z \in P_V \), \( \| z - x \| < \delta_{j,x} \), where
\[
\mathcal{A} = \{ \alpha \in (\mathbb{Z}^+)^N : \alpha_j \neq 0 \ \text{for only finitely many} \ j \in \mathbb{N} \},
\]
\[
\lambda_\alpha = \lambda_{\alpha_1} \cdots \lambda_{\alpha_n}, \ a_\alpha^{j,x} = P_n g_j(Tx)(e_{\alpha_1}, \ldots, e_{\alpha_n}),
\]
\[
\alpha = (\alpha_1, \ldots, \alpha_n, 0, \ldots),
\]
and
\[
\sum_{n \geq 0} (\delta_{j,x})^n \| \lambda_\alpha \| \| a_\alpha^{j,x} \| < \infty.
\]
Thus there exist \( x_{j,k} \in T^{-1}(W_j) \), \( \delta_{j,k} > 0 \), \( j, k = 1, 2, \ldots, \) such that \( \{ x_{j,k} + \frac{1}{2} \delta_{j,k} B_V \} \) is an open cover of \( T^{-1}(W_j) \) and
\[
\sum_{n \geq 0} (\delta_{j,k})^n \| \lambda_\alpha \| \| a_\alpha^{j,k} \| < \infty, \quad a_\alpha^{j,k} = a_\alpha^{j,x_{j,k}}
\]
with \( (g_jT)(x) = \sum_{\alpha \in \mathcal{A}} \lambda_\alpha (x - x_{j,k})a_\alpha^{j,k} \) for \( x \in x_{j,k} + \delta_{j,k} B_V \).

For each \( n \geq 1 \), set \( A_n = \{ \alpha \in \mathcal{A} : \max \alpha_j \leq n \} \) and
\[
S_n^{j,k}(x) = \sum_{\alpha \in A_n} \lambda_\alpha (x - x_{j,k})a_\alpha^{j,k}
\]
for \( x \in x_{j,k} + \delta_{j,k} B_V \).
Let $\varepsilon > 0$ be given. Take $m \in \mathbb{N}$ such that

$$\sum_{i > m} \|\lambda_i\|\|e_i\| < \varepsilon.$$  

We have

$$\left\| \sum_{\alpha \in I} \lambda_{\alpha}(x - x_{j,k})a_{\alpha,j,k} - S_{m}^{j,k}(x) \right\|$$

$$\leq \max_{\alpha_j > m, \alpha > 0} (\delta_{j,k})^n \|\lambda_{\alpha_1}\| \cdots \|\lambda_{\alpha_n}\| \|a_{\alpha,j,k}\|$$

$$\leq (M\varepsilon/a) \sum_{n \geq 0} (\delta_{j,k}(n/r_{j,k}, x_{j,k})^n (1/n!) \left( \sum_{i \geq 1} \|\lambda_i\|\|e_i\| \right)^n$$

$$= (M\varepsilon/a) \sum_{n \geq 0} 1/n!(n/2a)^n.$$  

Thus $S_{n}^{j,k} \to g_{jT}$ uniformly on $x_{j,k} + \delta_{j,k}B_{V}$ for $j, k \geq 1$ as $n \to \infty$. Applying the method of Bishop [2] to the sequence $\{S_{n}^{j,k}\}, j, k \geq 1$, we can find a sequence of disjoint 1-dimensional projections $P_{q}^{n}$ in $F(B)$ such that

$$S_{n}^{j,k} = \sum_{q \geq 1} P_{q}^{n} S_{n}^{j,k} = \sum_{q \geq 1} h_{q,j,k}^{n,j,k} u_{q}^{n}$$

and

$$\|u_{q}^{n}\| = 1, \quad \|P_{q}^{n}\| \leq 2^{\log 2} q, \quad P_{q}^{n}(u_{q}^{n}) = u_{q}^{n}$$

$$\sup_{n \geq 1} \sum_{q \geq 1} \|h_{q,j,k}^{n,j,k}\|_{x_{j,k} + \delta_{j,k}B_{V}} < \infty$$

for $j, k \geq 1$, where $\tilde{\delta}_{j,k} = \frac{1}{2}\delta_{j,k}$ and $\|h_{q,j,k}^{n,j,k}\|_{x_{j,k} + \delta_{j,k}B_{V}}$ denotes the sup-norm of $h_{q,j,k}^{n,j,k}$ on $x_{j,k} + \tilde{\delta}_{j,k}B_{V}$.

Since $S(A) = B$, the map $S$ induces a continuous linear map $\tilde{S}$ from $E(A)$ onto $F(B)$. Thus the open mapping theorem gives a constant $C > 0$ such that for each $(n, q)$ there exists $u_{q}^{n} \in E(A)$ for which $\tilde{S}(u_{q}^{n}) = u_{q}^{n}$ with $\|u_{q}^{n}\| \leq C\|v_{q}^{n}\|$ for $n, q \geq 1$.

Set

$$\tilde{S}_{n}^{j,k}(x) = \sum_{q \geq 1} h_{q,j,k}^{n,j,k} u_{q}^{n}$$

for $x \in x_{j,k} + \tilde{\delta}_{j,k}B_{V}$. Then

$$\sup_{n \geq 1} \{||\tilde{S}_{n}^{j,k}(x)||: x \in x_{j,k} + \tilde{\delta}_{j,k}B_{V}\}$$

$$\leq C \sup_{n \geq 1} \sum_{q \geq 1} \|h_{q,j,k}^{n,j,k}\|_{x_{j,k} + \delta_{j,k}B_{V}} < \infty.$$  

Thus the sequence $\{\tilde{S}_{n}^{j,k}\}_{n \geq 1}$ is bounded in $\mathcal{O}(x_{j,k} + \tilde{\delta}_{j,k}B_{V}, E(A))$, the space of holomorphic functions on $x_{j,k} + \tilde{\delta}_{j,k}B_{V}$ with values in $E(A)$ equipped with
the compact-open topology. From the compactness of the canonical map \( E(A) \to E \), we can assume that \( \{ \tilde{S}^{i,k}_n \}_{n \geq 1} \) converges to \( \tilde{S}^{i,k} \) in \( \mathcal{O}(x_j,k + \delta_j,k B_V, E) \) as \( n \to \infty \) for all \( j, k \geq 1 \). Moreover we assume also that the sequences \( \{ v^n_q \}, \{ p^n_q \}, \) and \( \{ h^{n,j,k}_q \} \) converge to \( v_q, p_q \) and \( h^{i,j,k}_q \) in \( F, \text{Hom}(F(B), F) \), the space of continuous linear maps from \( F(B) \) into \( F \), and \( \mathcal{O}(x_j,k + \delta_j,k B_V) \), respectively, as \( n \to \infty \) for all \( j, k, q \geq 1 \). On the other hand, from the relations

\[ P^n_s v^n_q = 0 \quad \text{if} \quad s \neq q \quad \text{and} \quad P^n_s v^n_s = v^n_s \]

we have

\[
\lim_{n \to \infty} P^n_s \sigma_j g_j T = \lim_{n \to \infty} P^n_s \sigma_j s^{i,k}_n \\
= \lim_{n \to \infty} \sum_{q \geq 1} P^n_s \sigma_j h^{n,i,k}_q v^n_q = \lim_{n \to \infty} \sigma_j h^{n,i,k}_n v^n_s = \sigma_j h^{i,j,k}_s v_s.
\]

Similarly

\[
\lim_{n \to \infty} P^n_s \sigma_i g_j T = \lim_{n \to \infty} P^n_s \sigma_i s^{i,l}_n = \sigma_i h^{i,l}_s v_s.
\]

Hence

\[ \sigma_j h^{i,j,k}_s = \sigma_i h^{i,l}_s \quad \text{for all} \quad i, j, k, l \geq 1. \]

This yields

\[ \sigma_i \tilde{S}^{i,k} = \sigma_j \tilde{S}^{i,l} \]

on \( (x_j,k + \delta_j,k B_V) \cap (x_i,l + \delta_i,l B_V) \) for all \( i, j, k, l \geq 1 \).

Thus the system \( \{ \tilde{S}^{i,j,k}/\sigma_j \} \) defines an \( E \)-valued meromorphic function \( g \) on \( D \) such that \( Sg = f \). The theorem is proved.

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