COMPACT PERTURBATIONS OF FREDHOLM \( n \)-TUPLES

RĂZVAN GELCA

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Abstract. Let \( T \) be an operator on a Hilbert space. We show that the pair \((T, T)\) can be perturbed to an invertible pair if and only if \( T \) is Fredholm of index zero. We also exhibit a large class of Fredholm \( n \)-tuples acting on a Banach space which cannot be perturbed by finite rank operators to invertible ones.

Introduction

It is well known that a Fredholm operator of index zero on a Banach space can be perturbed by a finite rank operator to an invertible one. In [2] it is asked if the same property remains true for commuting pairs of operators, or at least if one can perturb a pair of index zero with compact operators to get an invertible one.

There are several properties of the index that are preserved when we pass from one operator to commuting \( n \)-tuples of operators, for example, the index is invariant under small-norm perturbations or under perturbations with operators in the norm-closure of finite rank ones; in the Hilbert space case these are exactly the compact operators [1, 4]. It has been proved in [3] that the Koszul complex of a Fredholm \( n \)-tuple of index zero has a finite-dimensional perturbation to an exact complex, but the new complex is usually not a Koszul complex of a commuting \( n \)-tuple.

In the first part of the present paper we shall prove that on an infinite-dimensional Banach space there exists a large class of Fredholm \( n \)-tuples of index zero that cannot be perturbed with finite rank operators to invertible operators. Using the same idea we shall prove in the second part that on an infinite-dimensional Hilbert space no pair of the form \((T, 0)\) with \( T \) Fredholm and \( \text{ind}(T) \neq 0 \) can be perturbed with compact operators to an invertible pair; this will give a negative answer to the question raised by Raúl Curto in [2].

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1. Finite rank perturbations

In this section we consider the case of (bounded) linear operators acting on an infinite-dimensional Banach space \( \mathscr{X} \).

To each commuting \( n \)-tuple \( T = (T_1, T_2, \ldots, T_n) \) of operators on \( \mathscr{X} \), we attach a complex of Banach spaces, called the Koszul complex [5], as follows. Let \( \Lambda^p = \Lambda^p[e_1, e_2, \ldots, e_n] \) be the \( p \)-forms on \( \mathbb{C}^n \). Define the operator \( D_T : \mathscr{X} \otimes \Lambda^p \to \mathscr{X} \otimes \Lambda^{p+1} \) by \( D_T := T_1 \otimes E_1 + T_2 \otimes E_2 + \cdots + T_n \otimes E_n \), where \( E_i := e_i \omega \), \( i = 1, \ldots, n \).

The Koszul complex is

\[
0 \to \mathscr{X} \otimes \Lambda^0 \xrightarrow{D_1} \mathscr{X} \otimes \Lambda^1 \xrightarrow{D_2} \cdots \xrightarrow{D_n} \mathscr{X} \otimes \Lambda^n \to 0.
\]

Let \( H^p(T) \) be its cohomology spaces. The \( n \)-tuple \( T \) is called invertible if \( H^p(T) = 0 \), \( 0 \leq p \leq n \), and Fredholm if \( \dim H^p(T) < \infty \), \( 0 \leq p \leq n \), in which case we define its index to be \( \text{ind} \, T := \sum_{p=0}^n (-1)^p \dim H^p(T) \).

The Taylor spectrum of \( T \), denoted by \( \sigma(T) \), is the set of all \( z = (z_1, z_2, \ldots, z_n) \) in \( \mathbb{C}^n \) such that \( z - T = (z_1 - T_1, z_2 - T_2, \ldots, z_n - T_n) \) is not invertible. It is known that \( \sigma(T) \) is a compact nonvoid set. For any holomorphic map \( f : U \to \mathbb{C}^m \) on a neighborhood \( U \) of \( \sigma(T) \) one can define \( f(T) \) [6]; this functional calculus extends the polynomial calculus. By the spectral mapping theorem [6], \( f(\sigma(T)) = \sigma(f(T)) \).

If \( T = (T_1, T_2, \ldots, T_n) \) and \( T' = (T_1, T_2, \ldots, T_n, S) \) are commuting tuples, then we have a long exact sequence in cohomology

\[
0 \to H^0(T') \to H^0(T) \xrightarrow{S} H^0(T) \to H^1(T') \to H^1(T) \to \cdots
\]

where \( S \) is the operator induced by \( S \otimes 1 : \mathscr{X} \otimes \Lambda^p \to \mathscr{X} \otimes \Lambda^p \), \( 0 \leq p \leq n \). If \( T' \) is invertible, then \( S \) is an isomorphism. As a consequence of the long exact sequence, if \( T \) is Fredholm, then \( T' \) is Fredholm of index zero. We shall prove that there exists a class of \( (n+1) \)-tuples of index zero of the form \((T_1, T_2, \ldots, T_n, S)\) with \( \text{ind}(T_1, T_2, \ldots, T_n) \neq 0 \) and \((T_1, T_2, \ldots, T_n, S)\) Fredholm that cannot be made invertible by finite rank perturbations.

Lemma 1.1. Let \( (S_1, S_2, \ldots, S_n) \) be an invertible commuting \( n \)-tuple, and let \( f : \mathbb{C}^n \to \mathbb{C}^m \) be a holomorphic function with \( f^{-1}(0) = 0 \). Then the \( m \)-tuple \( f(S_1, S_2, \ldots, S_n) \) is invertible.

**Proof.** Straightforward from the spectral mapping theorem by noticing the fact that \( f^{-1}(0) \cap \sigma(S_1, S_2, \ldots, S_n) = \emptyset \).

Lemma 1.2. Let \( (S_1, S_2, \ldots, S_n) \) be a Fredholm commuting \( n \)-tuple with the property that \( \text{ind}(S_1, S_2, \ldots, S_n) \neq 0 \). Then there exist a sequence of positive integers \( \{m_k\}_k \) and \( 0 \leq p_0 \leq n \) such that \( \dim H^{p_0}(S_1^{m_k}, S_2, \ldots, S_n) \to \infty \) for \( k \to \infty \).

**Proof.** By Corollary 3.8 in [4] \( \dim(S_1^{m_k}, S_2, \ldots, S_n) = m \cdot \text{ind}(S_1, S_2, \ldots, S_n) \), so \( \dim H^{p_0}(S_1^{m_k}, S_2, \ldots, S_n) \) cannot all remain bounded.

**Theorem 1.1.** Let \( (T_1, T_2, \ldots, T_n) \) be a Fredholm commuting \( n \)-tuple with \( \text{ind}(T_1, T_2, \ldots, T_n) \neq 0 \) and \( p \in \mathbb{C}[z_1, z_2, \ldots, z_n] \) with \( p(0) = 0 \). Define
the operator $T_{n+1} = p(T_1, T_2, \ldots, T_n)$. Then there do not exist finite rank operators $R_1, R_2, \ldots, R_n, R_{n+1}$ such that $(T_1 + R_1, T_2 + R_2, \ldots, T_{n+1} + R_{n+1})$ is an invertible commuting $(n+1)$-tuple.

**Proof.** Suppose that such finite rank operators exist, and let $S_i = T_i + R_i$, $1 \leq i \leq n+1$. Applying Lemma 1.1 to the function $f: C^{n+1} \rightarrow C^{n+1}$, $f(z_1, z_2, \ldots, z_n, z_{n+1}) = (z_1, z_2, \ldots, z_n, z_{n+1} - p(z_1, z_2, \ldots, z_n))$, we get that $(S_1, S_2, \ldots, S_n, R)$ must be invertible, where

$$R = S_{n+1} - p(S_1, S_2, \ldots, S_n).$$

Clearly, $R$ is a finite rank operator. By applying Lemma 1.1 to the function $\phi: C^{n+1} \rightarrow C^{n+1}$, $\phi(z_1, z_2, \ldots, z_n, z_{n+1}) = (z_1^n, z_2, \ldots, z_n, z_{n+1})$, we get that $(S_1^n, S_2, \ldots, S_n, R)$ is also invertible for every positive integer $m$.

Let $\{m_k\}_k$ and $p_0$ be the numbers obtained by applying Lemma 1.2 to the $n$-tuple $(S_1, S_2, \ldots, S_n)$; let $\tilde{R} = \tilde{R}(m_k, p_0)$ be the operator induced by $R$ on $H^{p_0}(S_1^{m_k}, S_2, \ldots, S_n)$. Because $(S_1^{m_k}, S_2, \ldots, S_n)$ is invertible, $\tilde{R}$ must be an isomorphism for every $m_k$. But this is impossible since

$$\dim H^{p_0}(S_1^{m_k}, S_2, \ldots, S_n) \to \infty \quad \text{and} \quad \text{rank}(\tilde{R}) \leq \binom{n}{p_0} \cdot \text{rank}(R).$$

This proves the theorem.

2. The main example

In what follows we shall restrict ourselves to bounded linear operators on an infinite-dimensional Hilbert space $H$. We shall start with a result about the structure of a Fredholm operator of positive index.

**Lemma 2.1.** Let $T$ be a Fredholm operator with $\text{ind } T > 0$. Define $\mathcal{H}_n = \ker T^n \oplus \ker T^{n-1}$. Then $\mathcal{H}_n \neq (0)$, $n \geq 2$. Let $T_n = T|\ker T^n$.

(3) $T_n: \mathcal{H}_n \oplus \ker T^{n-1} \to \mathcal{H}_{n-1} \oplus \ker T^{n-2}$, $T_n = \begin{bmatrix} A_n & 0 \\ B_n & C_n \end{bmatrix}$.

Then there exists $n_0$ such that, for $n > n_0$, $A_n$ is an isomorphism.

**Proof.** Suppose that, for some $n$, $\mathcal{H}_n = (0)$. Then $\ker T^n = \ker T^{n-1}$. Hence $\ker T^{n+k} = \ker T^n \forall k \geq 0$, but this contradicts the fact that $\lim_{n \to \infty} \text{ind } T^{n+k} = \infty$.

Since $\mathcal{H}_n \perp \ker T_{n-1}$, $T|\mathcal{H}_n$ is injective and $T\mathcal{H}_n \cap \ker T_{n-2} = (0)$. This shows that $A_n$ is injective. But then the sequence $\{\dim \mathcal{H}_n\}$ is decreasing, so it becomes stationary. Let $n_0$ be such that, for $n > n_0$, $\dim \mathcal{H}_n = \dim \mathcal{H}_{n-1}$. Then, for $n > n_0$, $A_n$ is an injective operator between finite-dimensional spaces of the same dimension, so it is an isomorphism.

**Lemma 2.2.** Let $T$ and $\mathcal{H}_n$, $n \geq 2$, be as in the statement of the previous lemma. If $S$ is an operator that commutes with $T$, then, for all $n \geq 1$, ker $T^n$ is an invariant subspace for $S$. Let $S_n = S|\ker T^n$.

(4) $S_n: \mathcal{H}_n \oplus \ker T^{n-1} \to \mathcal{H}_{n-1} \oplus \ker T^{n-1}$, $S_n = \begin{bmatrix} X_n & 0 \\ Y_n & Z_n \end{bmatrix}$.

Then there is $n_0$ such that, for $n \geq n_0$, $X_n$ is similar to $X_{n_0}$.

**Proof.** The fact that ker $T^n$ is invariant for $S$ follows from the commutativity. Let $A_n$ and $n_0$ be as in Lemma 2.1. Then $ST = TS$ implies $S_{n-1}T_n = \begin{bmatrix} X_{n-1} & 0 \\ Y_{n-1} & Z_{n-1} \end{bmatrix} = \begin{bmatrix} X_{n_0} & 0 \\ Y_{n_0} & Z_{n_0} \end{bmatrix}$.
Lemma 2.3. Let \((T, S)\) be an invertible commuting pair. Then, for any \(n\), \(S|\ker T^n\) is an isomorphism for \(\ker T^n\).

Proof. Applying Lemma 1.1 to \((T, S)\) and \(f: \mathbb{C}^2 \to \mathbb{C}^2, f(z_1, z_2) = (z_1^n, z_2)\), we get that \((T^n, S)\) is invertible for any \(n\). By the remarks made at the beginning of the first section, \(\hat{S} : H^0(T^n) \to H^0(T^n)\) is an isomorphism. But \(H^0(T^n) = \ker T^n\), and the lemma is proved.

Theorem 2.1. Let \(T\) be a Fredholm operator with \(\text{ind} T \neq 0\). Then there do not exist compact operators \(K_1\) and \(K_2\) such that \((T + K_1, K_2)\) is an invertible commuting pair.

Proof. Suppose such \(K_1\) and \(K_2\) exist. Without loss of generality we may assume \(\text{ind} T > 0\), otherwise we take \(T^*\) instead of \(T\). We can also assume that \(K_1 = 0\), otherwise we can denote \(T + K_1\) by \(T\) and let \(K_2 = K\).

Consider the spaces \(\mathcal{H}_n, n \geq 2\), obtained by applying Lemma 2.1 to \(T\), and let \(K_n = K|\ker T^n\). By Lemma 2.2,

\[
K_n : \mathcal{H}_n \oplus \ker T^n \to \mathcal{H}_n \oplus \ker T^{n-1}, \quad K_n = \begin{bmatrix} X_n & 0 \\ Y_n & Z_n \end{bmatrix}
\]

have the property that \(X_n\) is similar to \(X_{n_0}\) for some \(n_0\) and \(n \geq n_0\). Applying Lemma 2.3 we get that the operators \(X_n, n \geq 2\), are isomorphisms. If we denote by \(r\) the spectral radius of \(X_{n_0}\), then \(r > 0\). From the fact that \(X_n\) is similar to \(X_{n_0}\) for \(n \geq n_0\) (so all \(X_n\)'s have the same spectral radius), it follows that \(||X_n|| \geq r\).

But \(||K|\mathcal{H}_n|| = ||K_n|\mathcal{H}_n|| \geq ||X_n|| \geq r\) for \(n \geq n_0\). Because \(\mathcal{H}_n \perp \mathcal{H}_m, n \neq m\), and \(\mathcal{H}_n \neq (0)\) for any \(n\), it follows that \(K\) is not compact, a contradiction. Therefore, such \(K_1\) and \(K_2\) cannot exist.

Corollary 2.1. The pair \((T, T)\) can be perturbed by compacts to an invertible commuting pair if and only if \(T\) can be perturbed by a compact to an invertible operator.

References