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THE STRUCTURE OF MEASURABLE MAPPINGS ON METRIC SPACES

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ABSTRACT. The purpose of this paper is to investigate the conditions under which every measurable mapping on a metric space X with the measure μ is a limit of a sequence of continuous mappings, with respect to the convergence μ-almost everywhere.

1. INTRODUCTION

It is well known that if f is a function on a finite interval which is measurable with respect to the Lebesgue measure, then f is the limit almost everywhere of a sequence of continuous functions (see, e.g., [4, Theorem 5, p. 104]). This paper is devoted to the generalization of the above fact.

Let X and Y be metric spaces. Denote by $\mathfrak{B}(X)$ and $\mathfrak{B}(Y)$ respectively the Borel σ-algebras on these spaces. Let μ be a finite Borel measure on X. By $\mathfrak{B}_\mu(X)$ we shall denote the completion in the measure μ of the σ-algebra $\mathfrak{B}(X)$. A mapping f from X into Y is called μ-measurable if it is measurable with respect to $(\mathfrak{B}_\mu(X), \mathfrak{B}(Y))$. The simplest example of a μ-measurable mapping is a continuous mapping from X into Y.

In [3, p. 544] Gihman and Skorohod proved the theorem which shows the connection between continuous and μ-measurable mappings on Hilbert spaces. Namely, they proved that if X and Y are separable Hilbert spaces and μ is a probability Borel measure on X, then for every μ-measurable mapping f from X into Y there exists a sequence of $\{f_n\}$ of continuous mappings from X into Y such that $f_n \to f$ μ-a.e.

In the present paper we shall consider the following problem: is the above theorem of Gihman and Skorohod true for arbitrary metric spaces X and Y and an arbitrary finite Borel measure μ on X?

It is easy to see that in general such an extension of this theorem needs not always be true, even under the additional assumption that X and Y are separable and complete metric spaces.

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For example, if $X = [0, 1]$ with the usual metric and the Lebesgue measure and $Y = \{0, 1\}$ with the discrete metric, then only the constant functions 0 and 1 are continuous from X to Y, while the set of measurable functions consists of all functions χ_A, where A is a μ-measurable set.

In this paper we shall give one of the affirmative answers of the stated problem. First we show that this problem has a positive solution if X is an arbitrary metric space and $Y = R$ is the real line (Theorem 1). Then we extend this result to the case where Y is a separable Banach space with the approximation property (Theorem 2).

2. Main results

Theorem 1. *Let μ be a finite Borel measure on a metric space X. If f is a μ-measurable mapping from X into a real line R, then there exists a sequence $\{f_n\}$ of continuous mappings from X into R such that $f_n \to f$ μ-a.e.*

Proof. To prove the theorem it suffices to show that for any $\varepsilon > 0$ and $\rho > 0$ there exists a continuous mapping $g : X \to R$ such that

$$\mu\{x : |f(x) - g(x)| > \varepsilon\} < \rho. \tag{1}$$

Indeed, if this is true, then choosing the sequences $\varepsilon \to 0$ and $\rho \to 0$ we can construct a sequence of continuous mappings from X into R which is convergent in the measure μ to f, and from this sequence we may choose a subsequence which is convergent to f μ-almost everywhere.

Therefore, let $\varepsilon > 0$ and $\rho > 0$ be fixed. We must construct a continuous mapping $g : X \to R$ which satisfies (1). From Lusin's theorem [6, Theorem 21.4] there exists a closed subset D of X such that $\mu(X - D) < \rho$ and the restriction of f to D is continuous. Denote by $\bar{g}$ this restriction, i.e., $\bar{g} = f|_D$. Thus $\bar{g}$ is a continuous mapping from D into R. Since D is a closed subset of X, by virtue of Tietze's theorem [5, p. 115] we can extend the mapping $\bar{g}$ to a mapping which is continuous on the whole space X. This means that there is a continuous mapping $g : X \to R$ such that

$$\{x : |f(x) - g(x)| > \varepsilon\} \subset X - D. \tag{2}$$

In fact, if the inclusion is not true, then there exists $x \in D$ such that $|f(x) - g(x)| > \varepsilon$. But on the set D the mappings f and g are equal. Hence $f(x) = g(x)$, which contradicts the above inequality.

From the inclusion (2) and the fact that $\mu(X - D) < \rho$ we obtain (1). The theorem is thus proved.

Now we prove the main result of this paper which shows that Theorem 1 is true if the range of a mapping f is a separable Banach space with the approximation property.

A Banach space Y is said to be a space with the approximation property if for every compact subset K of Y and every $\varepsilon > 0$ there exists a finite-dimensional continuous linear operator T on Y such that $\|Ty - y\| < \varepsilon$ for any $y \in K$. It is easy to see that each Banach space with the Schauder basis has the approximation property (see [1, p. 514]).

Theorem 2. *Let μ be a finite Borel measure on a metric space X, and let Y be a separable Banach space with the approximation property. If f is a μ-*

measurable mapping from X into Y, then there exists a sequence $\{f_n\}$ of continuous mappings from X into Y such that $f_n \to f$ μ-a.e.

Proof. Similarly as in the proof of Theorem 1 it is enough to show that for any $\varepsilon > 0$ and $\rho > 0$ there exists a continuous mapping $g : X \to Y$ such that

$$\mu\{x : \|f(x) - g(x)\| > \varepsilon\} < \rho. \tag{3}$$

Let ν denote a finite Borel measure on Y given by the formula $\nu(B) = \mu(f^{-1}(B))$ for every Borel subset B of Y. Since each finite Borel measure on Y is tight (see [2, Theorem 1.4]), there exists a compact subset K of Y such that $\nu(Y - K) < \rho/2$. Put $K' = f^{-1}(K)$. Then

$$\mu(X - K') < \rho/2. \tag{4}$$

Now, since Y is a Banach space with the approximation property, there exists a finite-dimensional operator T on Y such that $\|y - Ty\| < \varepsilon/2$ for any $y \in K$. Hence

$$\|f(x) - T(f(x))\| < \varepsilon/2 \quad \text{for any } x \in K'. \tag{5}$$

It is well known that each finite-dimensional operator on T can be represented in the form $Ty = \sum_{k=1}^{m} f_k(y)y_k$, where $f_1, \dots, f_m \in Y^*$ (Y^* denotes the dual space of Y) and $\{y_1, \dots, y_m\}$ is a basis of $T(y)$ with $\|y_k\| = 1$ for $k = 1, \dots, m$ (see [1, p. 492]).

Taking into account this representation, we can write inequality (5) in the form

$$\left\| f(x) - \sum_{k=1}^{m} f_k(f(x))y_k \right\| < \varepsilon/2 \quad \text{for any } x \in K'.$$

Hence, using (4), we infer that

$$\mu\left\{x : \left\| f(x) - \sum_{k=1}^{m} f_k(f(x))y_k \right\| > \varepsilon/2 \right\} \le \mu(X - K') \le \rho/2. \tag{6}$$

For every $k = 1, \dots, m$ the function $g_k : X \to R$ defined by the formula $g_k(x) = f_k(f(x))$ is a μ-measurable mapping from X into R. Then, in view of Theorem 1, we see that for every $k = 1, \dots, m$ there exists a sequence $\{g_n^{(k)}\}$ of continuous mappings from X into R such that $g_n^{(k)} \to g_k$ (as $n \to \infty$) μ-a.e.

Hence for every $k = 1, \dots, m$ there exists $n_k > 0$ such that

$$\mu\{x : |g_k(x) - g_{n_k}^{(k)}(x)| > \varepsilon/2m\} < \rho/2m. \tag{7}$$

Let $g(x) = \sum_{k=1}^{m} g_{n_k}^{(k)}(x)y_k$. Then g is a continuous mapping from X into

Y. Moreover, from (6) and (7) we have

$$\begin{aligned}
\mu\{x\colon \|f(x)-g(x)\|>\varepsilon\} &= \mu\left\{x\colon \left\|f(x)-\sum_{k=1}^{m} g_{n_k}^{(k)}(x)y_k\right\|>\varepsilon\right\}\\
&\le \mu\left\{x\colon \left\|f(x)-\sum_{k=1}^{m} f_k(f(x))y_k\right\|>\varepsilon/2\right\}\\
&\quad+\mu\left\{x\colon \left\|\sum_{k=1}^{m} f_k(f(x))y_k-\sum_{k=1}^{m} g_{n_k}^{(k)}(x)y_k\right\|>\varepsilon/2\right\}\\
&\le \rho/2+\sum_{k=1}^{m}\mu\left\{x\colon |g_k(x)-g_{n_k}^{(k)}(x)|>\varepsilon/2m\right\}\\
&\le \rho/2+m\cdot\rho/2m=\rho.
\end{aligned}$$

This completes the proof of the theorem.

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