A LARGE $\Pi_2^1$ SET, ABSOLUTE FOR SET FORCINGS

SY D. FRIEDMAN

(Communicated by Andreas R. Blass)

Abstract. We show how to obtain, by class-forcing over $L$, a set of reals $X$ which is large in $L(X)$ and has a $\Pi_2^1$ definition valid in all set-generic extensions of $L(X)$. As a consequence we show that it is consistent for the Perfect Set Property to hold for $\Sigma_2^1$ sets yet fail for some $\Pi_2^1$ set. Also it is consistent for the perfect set property to hold for $\Sigma_2^1$ sets and for there to be a long $\Pi_2^1$ well-ordering. These applications (necessarily) assume the consistency of an inaccessible cardinal.

The purpose of this note is to prove the following.

Theorem. Let $\kappa$ be an $L$-cardinal, definable in $L$. Then there is a set of reals $X$, class-generic over $L$, such that

(a) $L(X) \models \text{Card} = \text{Card}^L$ and $X$ has cardinality $\kappa$.
(b) Some fixed $\Pi_2^1$ formula defines $X$ in all set-generic extensions of $L(X)$.

By Lévy-Shoenfield Absoluteness, any $\Pi_2^1$ formula defining $X$ in $L(X)$ defines a superset of $X$ in each extension of $L(X)$. The point of (b) is that this superset is just $X$ in set-generic extensions of $L(X)$. If $\mathcal{O}^\#$ exists then $X$ as in the Theorem actually exists in $V$, though of course it will be only countable there.

The basic idea of the proof comes from David [2]. In his paper a real $R$ class-generic over $L$ is produced so that $\{R\}$ is $\Pi_2^1$, uniformly for set-generic extensions of $L(R)$. The added technique here is to use “diagonal supports” to take a large product of David-style forcings.

The following corollaries are further applications of the Theorem and its proof.

Corollary 1. Assume consistency of an inaccessible cardinal. Then it is consistent for the Perfect Set Property to hold for $\Sigma_2^1$ sets yet fail for some $\Pi_2^1$ set.

Proof. Use the Theorem to obtain a $\Pi_2^1$ set $X$ which has cardinality $\kappa$ in $L(X)$, $\kappa$ least $L$-inaccessible, and which has a $\Pi_2^1$-definition uniform for set-generic extensions. Then gently collapse $\kappa$ to $\omega_1$ and add $\omega_2$ Cohen reals. In this extension, $\omega_1 > \omega_1^{L(R)}$ for each real $R$ and $X$ is a $\Pi_2^1$ set of cardinality $\omega_1 < \omega_2 = 2^{\aleph_0}$. $\Box$

Received by the editors December 8, 1992.
1991 Mathematics Subject Classification. Primary 03E15, 03E35, 04A15.
Research supported by NSF contract #9205530-DMA.
Corollary 2. Assume consistency of an inaccessible. Then it is consistent that the Perfect Set Property holds for $\Sigma^1_3$ sets and there is a $\Pi^1_2$ well-ordering of some set of reals of length $\aleph_{1000}$.

The latter answers a question of Harrington [4].

The proof

We modify the construction of David [2] to suit our purposes. First we describe the $\alpha^+$-Souslin tree $T_\alpha$ in $L$, where $\alpha$ is a successor $L$-cardinal: $T_\alpha$ has a unique node on level 0 and exactly two immediate successors on level $\beta + 1$ to each node on level $\beta$, for $\beta < \alpha^+$. If $\beta < \alpha^+$ is a limit of cofinality $< \alpha$ then level $\beta$ assigns a top to each branch through the tree below level $\beta$. Now suppose $\beta < \alpha^+$ has cofinality $\alpha$. Let $\mathcal{P}$ be the forcing consisting of pairs $(\gamma, f)$ where $\gamma < \beta$ and $f$ is a function from $\gamma$ into the nodes at levels $\beta$, with extension defined by $(\gamma', f') \leq (\gamma, f)$ iff $\gamma' \geq \gamma$, $f'\delta$-tree-extends $f:\delta$ for each $\delta < \gamma$. Choose $G$ to be $\mathcal{P}$-generic over $L_\beta$, where $\beta^* = \text{largest p.r. closed } \beta^* > \beta$ such that $\beta^* = \beta$ or $L_{\beta^*}$ has cardinality greater than $\alpha$. Then the nodes on level $\beta$ are obtained by putting tops on the branches defined by $\{f(\delta)\} = G$ some $\gamma$ for $\delta < \beta$. This completes the definition of the $\alpha^+$-Souslin tree $T_\alpha$.

Now fix an $L$-definable cardinal $\kappa$ and also fix an $L$-definable 1-1 function $F : \kappa \times \omega \times \text{ORD} \rightarrow \text{Successor } L$-cardinals greater than $\kappa$. The forcing $\mathcal{P}(\gamma, n)$, $\gamma < \kappa$ and $n < \omega$, is designed to produce a real $R(\gamma, n)$ coding branches through $T_\alpha$ whenever $\alpha$ is of the form $F(\gamma, n, \delta)$ for some $\delta$. This forcing is obtained by modifying the Jensen coding of the empty class (see Beller, Jensen, and Welch [1]) as follows: In defining the strings $s : [\alpha, \omega] \rightarrow 2$ in $S_\alpha$, require that $\text{Even}(s)$ code a branch through $T_\alpha$ if $\alpha \in \text{Card}(\gamma, n) = \{F(\gamma, n, \delta)\} \delta \in \text{ORD}$). Also use David's trick to create a $\Pi^1_2$ condition implying that branches through the appropriate trees are coded: for any $\alpha$, for $s$ to belong to $S_\alpha$ require that for $\xi \leq |s|$ and $\eta > \xi$, if $L_\eta(s \upharpoonright \xi) \equiv \xi = \alpha^+ + ZF^+ + \text{Card} = \text{Card}^\xi$ then $L_\eta(s \upharpoonright \xi) \equiv s$ for some $\gamma^* < \kappa^*$, $\text{Even}(s \upharpoonright \xi)$ codes a branch through $T_{\gamma^*}$ whenever $\alpha^* \in \text{Card}^\xi(\gamma^*, n)$, where $\kappa^*$, $T_{\alpha^*}$, $\text{Card}^\xi(\gamma^*, n)$ are defined in $L_\eta$ as were $\kappa$, $T_\alpha$, $\text{Card}(\gamma, n)$ in $L$. The $\leq \alpha$-distributivity of $\mathcal{P}(\gamma, n)$ is established in David [2], with one added observation: if $\alpha' \in \text{Card}(\gamma, n)$ then we have to be sure that $\text{Even}(p_{\alpha'})$ codes a branch through $T_{\alpha'}$, where $p$ arises as the greatest lower bound to an $\alpha$-sequence constructed to meet $\alpha$-many open dense sets. There is no problem if $\alpha' > \alpha$ since then $T_{\alpha'}$ is $\leq \alpha$-closed. If $\alpha' = \alpha$ then the property follows from the definition of level $|p_{\alpha}|$ of $T_\alpha$, since we can arrange that $\text{Even}(p_{\alpha})$ is sufficiently generic for $T_\alpha \upharpoonright (\text{levels } < |p_{\alpha}|)$. (In fact the latter genericity is a consequence of the usual construction of the $\alpha$-sequence leading to $p$.)

The forcing $\mathcal{P}(\gamma)$, $\gamma < \kappa$, is designed to produce a real $R(\gamma)$ such that $n \in R(\gamma)$ iff $R(\gamma)$ codes a branch through $T_\alpha$ for each $\alpha$ in $\text{Card}(\gamma, n)$. A condition is $p \in \prod_n \mathcal{P}(\gamma, n)$ where $p(n)(0)$ (a finite object) is $(\varnothing, \varnothing)$ for all but finitely many $n$. Extension is defined by $q \leq p$ iff $q(n) \leq p(n)$ in $\mathcal{P}(\gamma, n)$ unless $n$ is not of the form $2^n 3^{n_1}$ or $n = 2^n 3^{n_1}$ where $q(n_0)q(n_1) = 0$, in which case there is no requirement on $q(n)$. A generic $G$ can be identified with the real $\{2^n 3^m | p(n)_0(m) = 1 \text{ for some } p \in G\} = R(\gamma)$. The forcing at or above
Our desired forcing $\mathcal{P}$ is the "diagonally supported" product of the $\mathcal{P}(\gamma)$, $\gamma < \kappa$. Specifically, a condition is $p \in \prod_{\gamma < \kappa} \mathcal{P}(\gamma)$ where for infinite cardinals $\alpha < \kappa$, $\{\gamma \mid p(\gamma)(\alpha) \neq (\emptyset, \emptyset)\}$ has cardinality $\leq \alpha$ and in addition $\{\gamma \mid p(\gamma)(0) \neq (\emptyset, \emptyset)\}$ is finite. Quasi-distributivity for $\mathcal{P}_\alpha =$ forcing at or above $\alpha$ follows just as for $\mathcal{P}(\gamma)$. The point of the diagonal supports is that for infinite successor cardinals $\alpha$, $\mathcal{P}$ factors as $\mathcal{P} \times \mathcal{P}_\alpha$ where $\mathcal{P}_\alpha$ denotes the $\mathcal{P}_\alpha$-generic and $\mathcal{P}_\alpha$ is $\alpha^+ - \text{CC}$. Thus we get cardinal-preservation.

Now note that if $\langle R(\gamma) \mid \gamma < \kappa \rangle$ comes from (and therefore determines) a $\mathcal{P}$-generic then $n \in R(\gamma) \rightarrow R(\gamma)$ codes a branch through $T_\alpha$ for $\alpha \in \text{Card}(\gamma, n)$. Conversely, if $n \notin R(\gamma)$ then there is no condition on extension of conditions in $\mathcal{P}(\gamma)$ to cause $R(\gamma)$ to code a branch through such $T_\alpha$. In fact, by the quasi-distributivity argument for $\mathcal{P}_\alpha$, given any term $\tau$ for a subset of $\alpha^+$ and any condition $p$, we can find $\beta < \alpha^+$ of cofinality $\alpha$ and $q \leq p$ such that $q$ forces $\tau \land \beta$ to be one of $\alpha$-many possibilities, each constructed before $\beta^*$, where $\beta = |q_\alpha|$. Thus $q$ forces that $\tau$ is not a branch through $T_\alpha$, so we get: $n \in R(\gamma)$ iff $R(\gamma)$ codes a branch through each $T_\alpha$, $\alpha \in \text{Card}(\gamma, n)$, iff $R(\gamma)$ codes a branch through some $T_\alpha$, $\alpha \in \text{Card}(\gamma, n)$. The coding is localized in the sense that if $n \in R(\gamma)$ then whenever $L_\eta(R(\gamma)) \models ZF^- + \text{Card} = \text{Card}^\kappa$, there is $\gamma^* < \kappa^*$ such that $L_\eta(R(\gamma)) \models R(\gamma)$ codes a branch through $T_{\gamma^*}$ whenever $\alpha^* \in \text{Card}^*(\gamma^*, n)$, where $\kappa^*$, $T_{\alpha^*}$, $\text{Card}^*(\gamma^*, n)$ are defined in $L_\eta$ just as $\kappa$, $T_\alpha$, $\text{Card}(\gamma^*, n)$ are defined in $L$. The latter condition on $R(\gamma)$ is sufficient to know that $R(\gamma)$ is equal to one of the intended $R(\gamma)$, $\gamma < \kappa$, even if we restrict ourselves to countable $\eta$. With that restriction we get a $\Pi_1^1$ condition equivalent to membership in $X = \{R(\gamma) \mid \gamma < \kappa\}$. Since set-forcing preserves the Souslinness of trees at sufficiently large cardinals, the above $\Pi_1^1$ definition of $X$ works in any set-generic extension of $L(X)$. This completes the proof of the Theorem.

Proof of Corollary 2. As in the proof of Corollary 1 we can obtain $X = \{R(\gamma) \mid \gamma < \kappa\}$, $\kappa = 999$th cardinal after the least $L$-inaccessible, which has a $\Pi_1^1$ definition uniform for set-generic extensions of $L(X)$, where $\text{Card}^{L(X)} = \text{Card}^\kappa$. We can guarantee that $Y = \{\langle R(0), R(\gamma_1), R(\gamma_2) \rangle \mid 0 < \gamma_1 \leq \gamma_2 < \kappa\}$ also has such a uniform $\Pi_1^1$ definition, using the following trick: Design $R(0)$ so that $u \in R(0) \Leftrightarrow \text{Even}(R(0))$ codes a branch through $T_\alpha$ for each $\alpha$ in $\text{Card}(0, n)$, and so that $\text{Odd}(R_0)$ almost disjointly codes $\{\langle R(\gamma_1), R(\gamma_2) \rangle \mid 0 < \gamma_1 \leq \gamma_2 < \kappa\}$. Thus, for $R \in L(X)$, $R^*$ is almost disjoint from $\text{Odd}(R_0)$ iff $R = \langle R(\gamma_1), R(\gamma_2) \rangle$ for some $0 < \gamma_1 \leq \gamma_2 < \kappa$, where $R^* = \{n \mid n \text{ codes a finite initial segment of } R\}$. The former requires only a very small modification to the definition of the $\mathcal{P}(0)$ forcings. The latter requires only a small modification to the definition of $\mathcal{P}$: take the diagonally-supported product as before, but restrain $p(0)$ for $p \in \mathcal{P}$ so as to affect the desired almost disjoint coding. These finite restraints do not interfere with the quasi-distributivity argument for $\mathcal{P}$.

Now we have the desired $\Pi_1^1$ definition for $Y = \{\langle R(0), R(\gamma_1), R(\gamma_2) \rangle \mid 0 < \gamma_1 \leq \gamma_2 < \kappa\}$: $R$ belongs to $Y$ iff $R = \langle R_0, R_1, R_2 \rangle$ where $R_0 = R(0)$ and $\langle R_1, R_2 \rangle^*$ is almost disjoint from $R_0$ and $R_1$, $R_2$ belong to $X$. Since $R(0)$
is uniformly definable as a $\Pi^1_2$-singleton in set-generic extensions of $L(X)$, this is the desired definition. Of course, using $Y$ we obtain a $\Pi^1_2$ well-ordering of length $\kappa$. Finally, as in the proof of Corollary 1, gently collapse $\kappa$ to $\omega_1$ and we have $\omega_1 > \omega^L_1(R)$ for each real $R$ with a $\Pi^1_2$ well-ordering of length $\aleph_{1000}$.

Remarks. The same proof gives length $\aleph_\alpha$ for any $L$-definable $\alpha$. We can also add Cohen reals so that the continuum is as large as desired, without changing the maximum length of a $\Pi^1_2$ well-ordering.

It is possible to show that if $O^*$ exists then there is a $\Pi^1_2$ set $X$ such that $X$ has large cardinality in $L(X)$. But this requires the more difficult technique of Friedman [3].

REFERENCES

3. S. Friedman, The $\Pi^1_2$-singleton conjecture, J. Amer. Math. Soc. 3 (1990), 771–791.