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ON THE ESSENTIAL SELFADJOINTNESS OF DIRICHLET OPERATORS ON GROUP-VALUED PATH SPACES

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ABSTRACT. Let G be a compact Lie group with Lie algebra $\mathscr{G}$. Consider the Wiener measure P on the space

$$W_G = \{g\colon [0, 1] \to G,\ g \text{ continuous},\ g(0) = e\}$$

For each h in the Cameron-Martin space H over $\mathscr{G}$, let ∂_h be the associated right invariant vector field over W_G and let ∂_h^* be its adjoint with respect to P. We prove for a particular h that the space of functions on W_G generated by C^∞-cylindrical functions on W_G together with one Gaussian random variable is a core for the Dirichlet operator $\partial_h^*\partial_h$. This is the first step in proving the essential selfadjointness of the Number operator over group-valued path spaces in the natural presumed core.

1. INTRODUCTION

During the last few years a lot of work has been done in analysis on loop groups. These provide interesting examples of infinite dimensional manifolds. Particular interest has been paid to the study of properties of Dirichlet operators over these spaces: Airault [Air], Getzler [Get], Gross [Gr], Airault and Van Biesen [AirVan], and Airault and Malliavin [AirMal].

One of the basic properties one would like to have for a Dirichlet operator is that of essential selfadjointness in a nice domain. For the linear case, e.g., in an abstract Wiener space, properties of Dirichlet operators have been studied extensively. In particular, properties of the Number operator (Ornstein-Uhlenbeck operator) are well known and have been used especially for the development of Malliavin calculus. Closability and essential selfadjointness have been recently obtained in a quite general setting by Takeda [Tak1Tak2], Röckner and Wielens [RöcWie], Röckner and Sheng [RöcShe], and Albeverio and Röckner [AlRöc].

There are two points of view in studying loop groups depending on the domain one chooses for the differential operators. Let G be a compact Lie group and let $\mathscr{G}$ its Lie algebra. One can consider functions defined on

$$W_{\mathscr{G}} = \{b\colon [0, 1] \to \mathscr{G};\ b \text{ continuous},\ b(0) = 0\},$$

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or functions defined on

$$W_G = \{g\colon [0,1] \to G;\ g \text{ continuous},\ g(0) = e\}.$$

Getzler [Get] and Malliavin [Mal1, Mal2] have formulated their work on analysis over loop spaces in terms of functions on $W_{\mathscr{G}}$. We will work with spaces of functions on W_G since they reflect topological properties of G. We make progress towards the solution of the problem of essential selfadjointness of the Number operator by proving the essential selfadjointness for the simplest Dirichlet operator over W_G in the natural presumed core.

For simplicity, since all compact Lie groups are locally isomorphic to a matrix group, we think of G as a matrix group. We fix an $\operatorname{Ad} G$-invariant inner product $(\ ,\ )$ in $\mathscr{G}$, which in our case is given by the trace, and an orthonormal basis $\{\xi_i\}_{i=1}^n$ for $\mathscr{G}$. Let $b(t)$, $t \in [0,1]$, be a $\mathscr{G}$-valued Wiener process with $b(0) = 0$ such that

$$E((\xi, b(s))(\eta, b(t))) = (\xi, \eta)(s \wedge t).$$

The solution of the stochastic differential equation

$$dg = dbg + \tfrac{1}{2} C g\, dt, \qquad g(0) = e, \tag{1}$$

where $C = \sum_1^n \xi_i^2$, lies in the space W_G [IkeWat, Chapter 5]. The map $I\colon b \mapsto g$ induces a probability measure P on W_G. In fact P is the diffusion process measure with generator $\frac{1}{2}\sum_1^n \hat{\xi}_i^2$ and initial value $g(0) = e$, where $\hat{\xi}_i$ is the left invariant vector field on G corresponding to ξ_i. I is called the *Itô* mapping and it turns out to be an isomorphism of the measure spaces $(W_{\mathscr{G}}, \mu)$ and (W_G, P) [McK], where μ is the Wiener measure on $W_{\mathscr{G}}$ corresponding to the process b. We remark again that, even though I is a measure theoretical isomorphism, it has poor topological properties when G is not commutative.

Denote by H the Cameron-Martin space over $\mathscr{G}$, i.e., the Hilbert space of $\mathscr{G}$-valued absolutely continuous functions h on $[0,1]$ with

$$\int_0^1 \|\dot{h}\|_{\mathscr{G}}^2\, ds < \infty, \qquad h(0) = 0.$$

Denote by K the space of finite energy functions on W_G, i.e., the space of functions $k\colon [0,1] \to G$ such that

$$|k^{-1}\dot{k}|^2 = \int_0^1 \|k(t)^{-1}\dot{k}(t)\|_{\mathscr{G}}^2\, dt < \infty,$$

and consider the action of K on W_G

$$\begin{aligned} K \times W_G &\to W_G \\ (k, \gamma) &\mapsto k \cdot \gamma. \end{aligned}$$

Define the action of h in H on W_G by $\gamma \mapsto e^{th}\gamma$, where $(e^{th}\gamma)(s) = e^{th(s)}\gamma(s)$, and the derivative of a function $f\colon W_G \to R$ in the direction of h by

$$(\partial_h f)(\gamma) = \frac{d}{dt}\bigg|_{t=0} f(e^{th}\gamma),$$

if it exists. Let $C^\infty(G^m)$ be the space of C^∞-functions on G^m and write

$$\begin{aligned} C_{cyl}^\infty(W_G) = \{f\colon W_G \to R;\ f(\gamma) &= u(\gamma(T_1), \dots, \gamma(T_m)), \\ &u \in C^\infty(G^m),\ 0 < T_1 < \cdots < T_m \le 1\}, \end{aligned}$$

as domain for the operator ∂_h. The adjoint operator ∂_h^* is given by $-\partial_h + j_h$, where j_h is the derivative with respect to t at $t = 0$ of the Jacobian

$$J_{e^{-th}}(\gamma) = \frac{dP(e^{-th}\gamma)}{dP(\gamma)}.$$

The main result of this work is the essential selfadjointness of $\partial_h^*\partial_h$ restricted to certain invariant subspaces of $L^2(W_G, P)$ that we shall describe below. From this the essential selfadjointness of the same in a nice domain in $L^2(W_G, P)$ follows easily. More precisely, for each partition $0 < T_1 < \cdots < T_m \leq 1$ of $[0, 1]$, define the space

$$\mathscr{F}_h^m = \{f\colon W_G \to R;\ f(\gamma) = u(j_h(\gamma), \gamma(T_1), \ldots, \gamma(T_m)),\ u \in C_c^\infty(R \times G^m)\}.$$

This family is defined by the pullback of $C_c^\infty(R \times G^m)$ with respect to the map $\vartheta_m\colon W_G \to R \times G^m$ given by

$$\vartheta_m(\gamma) = (j_h(\gamma), \gamma(T_1), \ldots, \gamma(T_m)).$$

The completion $\overline{\mathscr{F}}_h^m$ of $\mathscr{F}_h^m$ in the space $L^2(W_G, P)$ is isomorphic to $L^2(R \times G^m, \rho_m d\sigma\, dx)$ via ϑ_m, where $\rho_m d\sigma\, dx$ is the distribution law of ϑ_m. The restriction of $\partial_h^*\partial_h$ to $\overline{\mathscr{F}}_h^m$ leaves invariant $\mathscr{F}_h^m$. We apply Wielens' technique [Wie] to prove that $\partial_h^*\partial_h|_{\mathscr{F}_n^m}$ as an operator on $L^2(R \times G^m, \rho_m d\sigma\, dx)$ is essentially self-adjoint in $\mathscr{F}_h^m$. From this, it follows that the essential selfadjointness of $\partial_h^*\partial_h$ in $C_{cyl}^\infty(R, W_G) = \bigcup \mathscr{F}_h^m$.

In §2 we state the main theorems and a nice property of ρ_m. In §3 we give the proofs of the main theorems and state sufficient conditions on h for ρ_m to be smooth.

2. Main theorems and preliminaries

We begin stating a theorem by Albeverio and Hoegh-Krohn [AlHoe] which gives an expression for the Radon-Nikodim derivative of P under the action of a finite energy function k on W_G.

Theorem 2.0. *For each* $k \in K$

$$\frac{dP(k\gamma)}{dP(\gamma)} = \exp\left\{-\frac{1}{2}|k^{-1}\dot{k}|^2 + \int_0^1 (k(s)^{-1}\dot{k}(s),\ db(s))_{\mathscr{G}}\right\}. \tag{2}$$

For a proof of Theorem 2.0 see [Mal1]. Let $k \in K$ and $\gamma(s) = k(s)g(s)$. Then by (1) we have that

$$\begin{aligned} d\gamma &= dkg + kdg = \dot{k}gdt + k(dbg + \tfrac{1}{2}Cgdt) \\ &= Adk(db + k^{-1}\dot{k}dt)\gamma + \tfrac{1}{2}C\gamma dt. \end{aligned} \tag{3}$$

Let us write

$$J_k(\gamma) = \frac{dP(k\gamma)}{dP(\gamma)}.$$

The expression for J_k given by (2) allows us to compute explicitly ∂_h^*, the adjoint operator of ∂_h with $h \in H$. A straightforward computation shows that

$$\partial_h^* = -\partial_h + j_h,$$

where $j_h(\gamma) = (d/dt)|_{t=0} J_{e^{-th}}(\gamma) = \int_0^1 (\dot{h}, db)$. By (3) we have that

$$j_h(e^{th}\gamma) = \int_0^1 (\dot{h}(s), Ad\, e^{th(s)}(db(s) + e^{-th(s)} e^{t\dot{h}(s)} ds))_{\mathscr{G}},$$

where $e^{t\dot{h}(s)} = (d/ds)e^{th(s)}$. Then we have that

$$\partial_h j_h(\gamma) = -\int_0^1 (Ad\, h(\dot{h}), db) + \int_0^1 (\dot{h}, \dot{h})\, ds.$$

From now on we work with h pointing always in the same direction, i.e.,

$$h(s) = \varphi(s)\xi, \tag{4}$$

with $\varphi(0) = 0$, $\xi \in \mathscr{G}$. For this h we get

$$j_h(e^{th}\gamma) = j_h(\gamma) + t\|h\|_H^2$$

and

$$\partial_h j_h(\gamma) = \int_0^1 (\dot{h}, \dot{h})\, ds = \|h\|_H^2. \tag{5}$$

Given a partition $0 < T_1 < \cdots < T_m \leq 1$ of $[0, 1]$ define the space $\mathscr{F}_h^m$ of functions on W_G by

$$\mathscr{F}_h^m = \{f\colon W_G \to R;\ f(\gamma) = u(j_h(\gamma), \gamma(T_1), \ldots, \gamma(T_m)),\ u \in C_c^\infty(R \times G^m)\},$$

and define the $(R \times G^m)$-valued random variable ϑ_m on W_G by

$$\vartheta_m(\gamma) = (j_h(\gamma), \gamma(T_1), \ldots, \gamma(T_m)).$$

Let $\rho_m d\sigma\, dx$ be the distribution law of ϑ_m, i.e., the measure on $R \times G^m$ such that

$$\int_{W_G} f(\gamma)\, dP(\gamma) = \int_{R \times G^m} u(\sigma, x)\rho_m(\sigma, x)\, d\sigma\, dx,$$

where $d\sigma$ is the Lebesgue measure on R and dx is the Haar measure on G^m. We will continue to write this measure as if it has a density, which is not always the case for all h as we will see later. The reason to do so is that the computations will be more perspicuous this way. We see easily by (5) that $\mathscr{F}_h^m$ is an invariant subspace of $L^2(W_G, P)$ under $\partial_h^*\partial_h$ and that ϑ_m defines an isomorphism of $L^2(R \times G^m, \rho_m d\sigma\, dx)$ and $\overline{\mathscr{F}}_h^m$, the completion of $\mathscr{F}_h^m$ in $L^2(W_G, P)$.

Theorem 2.1. *$\partial_h^*\partial_h|_{\overline{\mathscr{F}}_h^m}$ is essentially selfadjoint in $\mathscr{F}_h^m$.*

Theorem 2.2. *$\partial_h^*\partial_h$ is essentially selfadjoint in*

$$C_{cyl}^\infty(R, W_G) = \{f\colon W_G \to R;\ f(\gamma) = u(j_h(\gamma), \gamma(T_1), \ldots, \gamma(T_m)),\ u \in C_c^\infty(R \times G^m),\ 0 < T_1 < \cdots < T_m \leq 1\}.$$

We shall prove Theorem 2.1 by proving that $\partial_h^*\partial_h|_{\overline{\mathscr{F}}_h^m}$ as an operator on $L^2(R \times G^m, \rho_m d\sigma\, dx)$ is essentially selfadjoint in $C_c^\infty(R \times G^m)$, and Theorem 2.2 by using the invariance of $\partial_h^*\partial_h$ in $\mathscr{F}_h^m$. The criterion of essential

selfadjointness to be used is the following [ReSim, Theorem X.26]. Let A be a nonnegative symmetric operator with domain $\mathscr{D}$ and $\alpha > 0$. Then

$$(6)\qquad \begin{aligned} &A \text{ is essentially selfadjoint in } \mathscr{D} \text{ if and only if} \\ &((A+\alpha)\varphi, u) = 0 \text{ for all } \varphi \in \mathscr{D} \text{ implies } u = 0. \end{aligned}$$

We spend the rest of this paragraph stating a nice property of ρ_m which has very useful consequences.

Lemma 2.3. *Let* $v = (h(T_1), \dots, h(T_m)) = (v_1, \dots, v_m)$. *Then*

$$\rho_m(sa, e^{sv}x) = \rho_m(0, x)e^{-as^2/2},$$

where $a = \|h\|_H^2$ *and* $e^{sv}x = (e^{sv_1}x_1, \dots, e^{sv_m}x_m)$.

Proof. By (5) $j_h(e^{th}\gamma) = j_h(\gamma) + ta$. On the other hand by Theorem 2.0

$$\int_{W_G} f(e^{th}\gamma)\,dP(\gamma) = \int_{W_G} f(\gamma)e^{-t^2\alpha/2+tj_h(\gamma)}\,dP(\gamma).$$

Then, if $f \in \mathscr{F}_h^m$ and $u \in C_c^\infty(R \times G^m)$ defines f

$$\begin{aligned} &\int_{R\times G^m} u(\sigma, x)\rho_m(\sigma - ta, e^{-tv}x)\,d\sigma\,dx \\ &\quad = \int_{R\times G^m} u(\sigma + ta, e^{tv}x)\rho_m(\sigma, x)\,d\sigma\,dx \\ &\quad = \int_{R\times G^m} u(\sigma, x)e^{-t^2a/2+t\sigma}\rho_m(\sigma, x)\,d\sigma\,dx. \end{aligned}$$

It follows that

$$e^{-t^2a/2+t\sigma}\rho_m(\sigma, x) = \rho_m(\sigma - ta, e^{-tv}x).$$

If we put $\sigma = 0$ and $s = -t$ we have

$$\rho_m(sa, e^{sv}x) = e^{-s^2a/2}\rho_m(0, x). \quad \square$$

Lemma 2.3 says that the density ρ_m is nothing but a Gaussian function along each spiral $\rho_m(sa, e^{sv}x)$ in $R \times G^m$. Now, let $\beta\colon R \times G^m \to R \times G^m$ be defined by $\beta(\sigma, x) = (s, y)$, where $s = a^{-1}\sigma$ and $y = e^{-sv}x$. Then if $\tilde{\rho}_m = \rho_m \circ \beta^{-1}$ we have

$$(7)\qquad \tilde{\rho}_m(s, y) = \rho_m(sa, e^{sv}y) = e^{-s^2a/2}\rho_m(0, y),$$

by Lemma 2.3.

Lemma 2.4. *ϑ_m carries $\partial_h|_{\mathscr{F}_h^m}$ to a vector field $\mathscr{X}_m$ on $R \times G^m$ which in the β-coordinates is given by $\partial/\partial s$.*

Proof. Let $f \in \mathscr{F}_h^m$ and let $u \in C_c^\infty(R \times G^m)$ define f. Then

$$(\partial_h f)(\gamma) = \frac{d}{dt}\bigg|_{t=0} u(\sigma + ta, e^{tv}x) = \frac{d}{dt}\bigg|_{t=0} \tilde{u}\left(\frac{\sigma}{a} + t, e^{-\sigma v/a}x\right) = \frac{\partial \tilde{u}}{\partial s}(s, y). \quad \square$$

Corollary 2.5. $\mathscr{X}_m^*\mathscr{X}_m \circ \beta^{-1} = -\partial^2/\partial s^2 + sa\partial/\partial s$.

Remark. Lemma 2.3 says only that the Radon-Nykodim derivative of the distribution law $d\nu_m = \rho_m\,ds\,dx$ under the action $(\sigma, x) \mapsto (\sigma - ta, e^{-tv}x)$ is given by

$$\frac{d\nu_m(\sigma - ta, e^{-tv}x)}{d\nu_m(\sigma, x)} = e^{-at^2/2+t\sigma},$$

or, what is the same, that the Radon-Nykodim derivative of $d\tilde{\nu}_m = \tilde{\rho}_m\, ds\, dy$ under the action $(s, y) \mapsto (s - \tau, y)$ is given by

$$\frac{d\tilde{\nu}_m(s-\tau, y)}{d\tilde{\nu}_m(s, y)} = e^{-at^2/2+tsa}. \tag{8}$$

But identity (7) says that $\tilde{\nu}_m$ is a product measure

$$d\tilde{\nu}_m(s, y) = e^{-as^2/2} ds\, d\mu_m(y).$$

This fact can be proved in a rigorous way as follows. Let μ_m be the measure on G^m defined by

$$\int_{G^m} f(y)\, d\mu_m(y) = \int_{R\times G^m} f(y)\, d\tilde{\nu}_m(s, y),$$

where $f \in C(G^m)$. For $f \in C(G^m)$ let Λ be the linear functional on $C_c(R)$ defined by

$$\Lambda(g) = \int_{R\times G^m} f(y) g(s) e^{as^2/2} d\tilde{\nu}_m(s, y).$$

Using (8) we can show easily that

$$\int_{R\times G^m} f(y) g(s+t) e^{as^2/2} d\tilde{\nu}_m(s, y) = \int_{R\times G^m} f(y) g(s) e^{as^2/2}\, d\tilde{\nu}_m(s, y),$$

which means that Λ is invariant under translations. Therefore

$$\Lambda(g) = c(f) \int_R g(s)\, ds.$$

It is easy to see that

$$c(f) = \int_{R\times G^m} f(y)\, d\tilde{\nu}_m(s, y) = \int_{G^m} f(y)\, d\mu_m(y)$$

and thus

$$\int_{R\times G^m} f(y) g(s) e^{as^2/2} d\tilde{\nu}_m(s, y) = \int_{G^m} f(y)\, d\mu_m(y) \int_R g(s) e^{-as^2/2}\, ds,$$

which means $d\tilde{\nu}_m = \exp\{-as^2/2\} ds d\mu_m$ on products $f \cdot g$, $f \in C(G^m)$, $g \in C_c(R)$. Finally, any function $\varphi \in C_c(R \times G^m)$ is a uniform limit of finite sums of products $f \cdot g$, and then $d\tilde{\nu}_m = \exp\{-s^2 a/2\}\, ds\, d\mu_m(y)$.

3. Proofs of Theorem 2.1 and Theorem 2.2

We prove first Theorem 2.2 using Theorem 2.1.

Proof of Theorem 2.2. Let P_m be the orthogonal projection in $L^2(W_G, P)$ onto $L^2(R \times G^m, \rho_m\, d\sigma\, dx)$ and u in $L^2(W_G, P)$ such that

$$((\partial_h^* \partial_h + \alpha)\varphi, u)_{L^2(P)} = 0,$$

for all $\varphi \in C^\infty_{cyl}(R, W_G)$, $\alpha > 0$. Then

$$\begin{aligned}((\mathscr{Z}_m^* \mathscr{Z}_m + \alpha)\varphi, P_m u)_{L^2(\rho_m)} &= (P_m^*(\mathscr{Z}_m^* \mathscr{Z}_m + \alpha)\varphi, u)_{L^2(P)} \\ &= ((\partial_h^* \partial_h + \alpha)\varphi, u)_{L^2(P)} = 0,\end{aligned}$$

for all $\varphi \in \mathscr{F}_h^m$. By Theorem 2.1 and (6), $P_m u = 0$, and since P_m converges strongly to the identity operator in $L^2(W_G, P)$, $u = 0$ in $L^2(W_G, P)$. Again by (6), $\partial_h^* \partial_h$ is essentially selfadjoint in $C_{cyl}^\infty(R, W_G)$.

Proof of Theorem 2.1. Let $u \in L^2(R \times G^m, \rho_m \, d\sigma \, dx)$ be such that

$$((\mathscr{X}_m^* \mathscr{X}_m + \alpha)\varphi, u)_{\rho_m} = 0, \qquad \alpha > 0, \tag{9}$$

for all $\varphi \in C_c^\infty(R \times G^m)$. Since $C_c^\infty(R \times G^m)$ is dense in $C_c^2(R \times G^m)$, equation (9) is still true for $\varphi \in C_c^2(R \times G^m)$. Consider functions $\varphi \in C_c^2(R \times G^m)$ of the form $\varphi(s, y) = \varphi_1(s)\varphi_2(y)$, where $\varphi_1 \in C_c^2(R)$ and $\varphi_2 \in C^\infty(G^m)$. Then (9) becomes by (7)

$$\int_{G_m} \varphi_2(y) \int_R \tilde{u}(s, y)\{(e^{-as^2/2}\varphi_1'(s))' + \alpha\varphi_1(s)e^{-\alpha s^2/2}\} \, ds \, \rho_m(0, y) \, dy = 0. \tag{10}$$

Let $\{\varphi_1^{(k)}\}_{k \geq 1}$ be a countable dense subset of $C_c^2(R)$. For each k let $N_k := \{y \in G^m; F(\varphi_1^{(k)})(y) \neq 0\}$ and $N_0 := \{y \in G^m; u(\cdot, y) \notin L^2(R, \rho(\cdot, y) \, ds)\}$, where

$$F(\varphi_1)(y) = \int_R \tilde{u}(s, y)\{(e^{-as^2/2}\varphi_1'(s))' + \alpha\varphi_1(s)e^{-\alpha s^2/2}\} \, ds \, \rho_m(0, y). \tag{11}$$

It follows from (10) that each N_k has measure zero and thus $N := \bigcup_{k=0}^\infty N_k$ does too. Now, let $\varphi_1 \in C_c^2(R)$ and $\varphi_1^{(k_i)}$ be a sequence in $\{\varphi_1^{(k)}\}_{k \geq 1}$ which converges to φ_1 in $C_c^2(R)$. We have that $F(\varphi_1^{(k_i)})$ converges pointwisely to $F(\varphi_1)$ in $G^m - N$. Therefore $F(\varphi_1)(y) = 0$ for all $y \in G^m - N$. Using integration by parts in (4) it is easy to show that for each $y \in G^m - N$, $\tilde{u}(\cdot, y)$ is continuously differentiable and that $\partial\tilde{u}/\partial s(\cdot, y)$ is absolutely continuous (cf. [ReSim, Example VIII 1.3]). From this it is easy to see that if $\varphi \in C_c^\infty(R \times G^m)$ then $\varphi u \in \mathscr{D}(\overline{\mathscr{X}_m^* \mathscr{X}_m + \alpha})$, where $\overline{\mathscr{X}_m^* \mathscr{X}_m + \alpha}$ is the closure of $\mathscr{X}_m^* \mathscr{X}_m + \alpha$. Let us now compute $(\varphi u, (\overline{\mathscr{X}_m^* \mathscr{X}_m + \alpha})\varphi u)_{\rho_m}$. Since $(\mathscr{X}_m^* \mathscr{X}_m + \alpha)^* u = 0$,

$$\begin{aligned}
(\varphi u, (\overline{\mathscr{X}_m^* \mathscr{X}_m + \alpha})\varphi u)_{\rho_m} &= (\varphi u, u\mathscr{X}_m^* \mathscr{X}_m \varphi - 2\mathscr{X}_m \varphi \mathscr{X}_m u)_{\rho_m} \\
&= (\varphi, u^2 \mathscr{X}_m^* \mathscr{X}_m \varphi - \mathscr{X}_m \varphi \mathscr{X}_m(u^2))_{\rho_m} \\
&= -(\varphi, \mathscr{X}_m(\rho_m u^2 \mathscr{X}_m \varphi)) \\
&= (u\mathscr{X}_m \varphi, u\mathscr{X}_m \varphi)_{\rho_m}.
\end{aligned} \tag{12}$$

On the other hand since $\mathscr{X}_m^* \mathscr{X}_m$ is nonnegative

$$(\varphi u, (\mathscr{X}_m^* \mathscr{X}_m + \alpha)\varphi u)_{\rho_m} \geq \alpha(\varphi u, \varphi u)_{\rho_m}. \tag{13}$$

Combining (12) and (13) we get

$$\alpha(\varphi u, \varphi u)_{\rho_m} \leq (u\mathscr{X}_m \varphi, u\mathscr{X}_m \varphi)_{\rho_m}. \tag{14}$$

Now consider the following sequence of functions in $C_0^\infty(R)$:

$$\begin{gathered}
f_r(\sigma) = 1 \quad \text{for } |\sigma| \leq r, \\
f_r(\sigma) = 0 \quad \text{for } |\sigma| \geq r + 1, \\
0 \leq f_r \leq 1, \quad \text{for all } r \in N, \\
\sup_\sigma |f_r'(\sigma)| < M \quad \text{for all } r \in N \text{ and } M > 0.
\end{gathered}$$

Set $\varphi_r(\sigma, x) = f_r(\sigma)$, then $\varphi_r \in C_c^\infty(R \times G^m)$. By (14)

$$0 \le \alpha \int_{|\sigma|<r} u^2 \rho_m \, d\sigma \, dx + \alpha \int_{|\sigma|>r} (\varphi_r u)^2 \rho_m \, d\sigma \, dx$$
$$\le \int_{|\sigma|>r} (\mathscr{Z}_m \varphi_r)^2 u^2 \rho_m \, d\sigma \, dx = \int_{|\sigma|>r} (f_r'(\sigma))^2 u^2 \rho_m \, d\sigma \, dx.$$

Then

$$0 \le \alpha \int_{|\sigma|\le r} u^2 \rho_m \, d\sigma \, dx \le M \int_{|\sigma|>r} u^2 \rho_m \, d\sigma \, dx.$$

Letting $r \to \infty$ we get $u = 0$ in $L^2(R \times G^m, \rho_m \, d\sigma \, dx)$. By (6) $\mathscr{Z}_m^* \mathscr{Z}_m$ is essentially selfadjoint in $C_c^\infty(R \times G^m)$. □

Remark. As we said at the beginning, the distribution ρ_m may not have a density as the following example shows. Consider the case $m = 1$, $T_1 = 1$, $G = S^1$, and $h(t) = ti$. In this case the marginal distribution $\int_R \rho_m(\sigma, x) \, d\sigma$ is given by

$$\int_R \rho_1(\sigma, x) \, ds = P_1(x) = \sum_{-\infty}^{\infty} \exp\left(-\frac{m^2}{2}\right) e^{-m\theta i}, \tag{15}$$

where P_1 is the heat kernel on the Lie group S^1. Then

$$P_1(e^{it}x) = \int_R e^{-\sigma^2/2} \rho_1(0, e^{-(\sigma-t)}x^i) \, d\sigma = e^{-t^2/2} \int_R e^{-u^2/2} e^{-iut} \rho_1(0, e^u x^i) \, du.$$

By analytic continuation

$$P_1(e^{-t}x) = e^{t^2/2} \int_R e^{-u^2/2} e^{-iut} \rho_1(0, e^{-ui}x) \, du,$$

and taking the inverse Fourier transform

$$e^{-\sigma^2/2} \rho_1(0, e^{-\sigma i}x) = \frac{1}{2\pi} \int_R e^{-t^2/2} P_1(e^{-t}x) e^{it\sigma} \, dt.$$

So using (15) we have an explicit expression for $\rho_1(\sigma, x)$

$$\rho_1(\sigma, x) = \frac{1}{2\pi} \int_R e^{-t^2/2} \sum_{-\infty}^{\infty} \exp\left(-\frac{m^2}{2}\right) e^{im(\theta+it)} e^{it\sigma} \, dt$$
$$= \frac{1}{\sqrt{2\pi}} e^{-\sigma^2/2} \sum_{-\infty}^{\infty} e^{im(\theta-\sigma)} = \frac{1}{\sqrt{2\pi}} e^{-\sigma^2/2} \delta_\sigma(\theta).$$

Appealing to Malliavin calculus we can find a sufficient condition on h for ρ_m to be smooth. In fact, we will say that φ is piecewise linear on $[0, 1]$ if $\ddot{\varphi} \equiv 0$ in each open interval (T_{i-1}, T_i). Then,

Theorem 3.1. *Let $h(t) = \varphi(t)\xi$, where φ is nonpiecewise linear on $[0, 1]$. Then there exists $\alpha > 0$ such that*

$$D = \det(\theta'_{m\gamma} \circ \theta'^*_{m\gamma}) \ge \alpha,$$

for all $\gamma \in W_G$.

Proof. $D = (\Delta_1, \dots, \Delta_m)^n \{\|\eta\|_H^2 - \sum_{r=0}^{m-1} \|\eta_r - \eta_{r+1}\|_{\mathscr{F}}^2 / \Delta_{r+1}$, where

$$\eta(t) = h(t) = \phi(t) = h(t) + \int_0^t \int_s^1 a d\dot{h} \, db \, ds,$$

where $\eta_r = \eta(T_r)$, $\Delta_r = T_r - T_{r+1}$, and $T_0 = 0$. The sum in the expression for D is nothing but a Riemann sum for the integral $\int_0^1 \|\dot{\eta}(t)\|^2\,dt = \|\eta\|_H^2$. Now by the Schwartz inequality it is easy to show that

$$\sum \frac{1}{\Delta_{r+1}} \|h_r - h_{r+1}\|^2 \leq \|\eta\|_H^2.$$

By hypothesis the inequality is strict and then $\alpha = \|\eta\|_H^2 - \sum \frac{1}{\Delta_{r+1}} \|h_r - h_{r+1}\|^2 > 0$. The theorem follows form the fact that h and ϕ are orthogonal at each t. □

Corollary 3.2. *Under the hypothesis of Theorem* 3.1, ρ_m *is* C^∞.

Proof. It follows from Theorem 3.1 and Malliavin's theorem [Ike, Wat, Theorem 8.1]. □

For more details on the proofs of Theorem 2.1 and Corollary 3.2 see [Aco].

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