i-CONVEXITY OF MANIFOLDS WITH REAL PROJECTIVE STRUCTURES

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Abstract. We compare the notion of higher-dimensional convexity, as defined by Carrière, for real projective manifolds with the existence of hemispheres. We show that if an i-convex real projective manifold $M$ of dimension $n$ for an integer $i$ with $0 < i < n$ has an $i$-dimensional hemisphere, then $M$ is projectively homeomorphic to $S^n/\Gamma$ where $\Gamma$ is a finite subgroup of $O(n + 1, \mathbb{R})$ acting freely on $S^n$.

A real projective structure (RP$^n$-structure) on a smooth manifold of dimension $n$, $n > 0$, is given by an atlas of charts to the sphere $S^n$ where transition functions of the charts are restrictions of projective automorphisms of $S^n$. An RP$^n$-manifold is a manifold with an RP$^n$-structure, and a projective map is an immersion-preserving real projective structures locally. The Klein model of hyperbolic geometry implies that $n$-dimensional hyperbolic manifolds provide examples of RP$^n$-manifolds. (See [2] and [3].)

Throughout this paper, let $M$ be an RP$^n$-manifold; let $\widetilde{M}$ denote the universal cover of $M$ with the induced RP$^n$-structure. $M$ has a developing map $\text{dev}: \widetilde{M} \to S^n$, a projective map. The sphere $S^n$ has a standard Riemannian metric $\mu$ of curvature 1 and the associated distance metric $d$. Note that dev induces a Riemannian metric $\mu$ on $\widetilde{M}$ from $S^n$. Associated with $\mu$ is the induced distance metric $d$ on $\widetilde{M}$. The completion $\hat{M}$ of $\widetilde{M}$ is obtained by completing $d$. Let $\sigma$ be the frontier set $\hat{M} - \widetilde{M}$. The sets $\hat{M}$ and $\sigma$ are topologically independent of the choice of dev, and dev extends uniquely to a distance decreasing map on $\hat{M}$. The extended map is also called a developing map and is denoted by the same symbol dev.

Let $i$ be an integer such that $0 < i \leq n$ holds. A great $i$-sphere is a totally geodesic $i$-dimensional sphere imbedded in $S^n$; a subset of $\hat{M}$ for which the restriction of dev is an imbedding onto a great $i$-sphere in $S^n$ is also called a great $i$-sphere. A great $i$-ball is a hemisphere of a great $i$-sphere in $S^n$; a
subset of \( \tilde{M} \) for which the restriction of \( \text{dev} \) is an imbedding onto a great \( i \)-ball is also called a great \( i \)-ball. An \( n \)-dimensional open hemisphere of \( S^n \) has a natural affine structure. An \( i \)-simplex is the convex hull of \( i + 1 \) independent points in the hemisphere (under affine geometry). An \( i \)-simplex in \( \tilde{M} \) is a subset for which the restriction of \( \text{dev} \) is an imbedding onto an \( i \)-simplex in an \( n \)-dimensional open hemisphere of \( S^n \).

We introduce the definition given by Carrière [1]. We say that \( M \) is \( i \)-convex for an integer \( i \) with \( 0 < i < n \) if the following holds: Given an \((i+1)\)-simplex \( T \) in \( \tilde{M} \), if \( F_1 \) is a face of \( T \) such that \( T \cap \sigma = F_1 \cap \sigma \), then \( T \subset \tilde{M} \). (Note that the \( i \)-convexity of \( M \) implies the \( j \)-convexity of \( M \) where \( i \leq j < n \)).

A subset \( A \) of \( \tilde{M} \) is called \textit{convex} if every two points of \( A \) are connected by an imbedded arc \( \alpha \) such that \( \text{dev}\alpha \) is an imbedding onto a segment of \( \text{d-length} \leq \pi \) (see [2, §1] for more detail). It is easy to see that given a convex open subset \( A \) of \( \tilde{M} \), the map \( \text{dev}|A \) is isometric with respect to the metrics on \( \tilde{M} \) and \( S^n \). Thus, \( \text{dev}|\text{Cl}(A) \) is an imbedding onto \( \text{Cl}(\text{dev}(A)) \) where \( \text{Cl}(A) \) is the closure of \( A \) in \( \tilde{M} \).

We prove the following theorem in this paper.

**Theorem.** Suppose that \( M \) is \( i \)-convex where \( 0 < i < n \) holds. Suppose that \( \tilde{M} \) includes a great \( j \)-ball or a great \( j \)-sphere for \( i \leq j < n \). Then \( \tilde{M} \) is projectively homeomorphic to \( S^n \).

The conclusion implies that \( M \) is projectively homeomorphic to \( S^n/\Gamma' \) where \( \Gamma' \) is a subgroup of the projective automorphism group \( \text{Aut}(S^n) \) acting freely and properly discontinuously on \( S^n \). It is a standard fact that \( \Gamma' \) is conjugate to a finite subgroup of \( O(n+1, \mathbb{R}) \), the group of isometries of \( S^n \). Thus, \( M \) is projectively homeomorphic to \( S^n/\Gamma \) where \( \Gamma \) is a finite subgroup of \( O(n+1, \mathbb{R}) \) acting freely on \( S^n \).

**Proof of the Theorem.** We give the proof of the theorem assuming that Lemma 1, which follows, holds. We can prove the theorem by using induction on \( j \). Suppose that \( j = n - 1 \). Then by Lemma 1 \( \tilde{M} \) includes a great sphere of dimension \( n \). Therefore, \( \tilde{M} \) is projectively homeomorphic to \( S^n \).

Suppose that the conclusion is true for the case where \( j = k \geq i \) holds. We verify the conclusion for the case where \( j = k - 1 > i \) holds. By Lemma 1, \( \tilde{M} \) includes a great \( k \)-sphere. By the induction hypothesis, \( \tilde{M} \) is projectively homeomorphic to \( S^n \). This completes the proof.

Let us discuss Lemma 1. For an integer \( j \) with \( 0 < j < n \), a \((j+1)\)-bihedron in \( S^n \) is a closed domain in a great \((j+1)\)-sphere \( S^{j+1} \) in \( S^n \) bounded by two great \( j \)-balls with common boundary equal to a great \((j - 1)\)-sphere or the set of two points antipodal to each other; a \((j+1)\)-bihedron in \( M \) is a subset for which the restriction of \( \text{dev} \) is an imbedding onto a \((j+1)\)-bihedron in \( S^n \). The bounding great \( j \)-balls of a bihedron are called \textit{faces}. A bihedron in \( S^n \) or \( \tilde{M} \) is convex if and only if the interior angle between two faces of the bihedron is less than or equal to \( \pi \).

**Lemma 1.** Assume that \( M \) is \( i \)-convex for an integer \( i \) with \( 0 < i < n \). Suppose that \( \tilde{M} \) includes a great \( j \)-ball \( B_0 \) where \( i \leq j < n \). Then \( \tilde{M} \) includes a great \((j+1)\)-sphere.
Proof. We choose a convex \((j+1)\)-bihedron \(T_0\) including \(B_0\) in a neighborhood of \(B_0\) with faces \(B_a\) and \(B_b\) such that \(\delta B_a = \delta B_b = \delta B_0\) holds. (Assume that \(B_0\) is not a face of \(T_0\).) Let us agree that bihedra in this proof are \((j+1)\)-dimensional always. Let \(A^+\) be the set of convex bihedra with a face \(B_0\) and including \(B_a\), and let \(A^-\) be the set of convex bihedra with a face \(B_0\) and including \(B_b\). Then there is a unique great \((j+1)\)-sphere \(S^{j+1}\) in \(S^n\) including the images of elements of \(A^+\) and \(A^-\) under \(\text{dev}\).

We may parameterize \(A^+\) and \(A^-\) by positive intervals. Given an element \(T\) of \(A^+\) or \(A^-\), let \(\theta(T)\) be the interior angle between \(B_0\) and the other face of \(T\). This defines a function \(\theta\) from the set of elements of \(A^+\) and \(A^-\) to \(\mathbb{R}\). Let \(T_a\) be the bihedron in \(T_0\) bounded by \(B_0\) and \(B_a\); let \(T_b\) be the bihedron in \(T_0\) bounded by \(B_0\) and \(B_b\). Suppose that \(T'\) and \(T''\) are two bihedra in \(A^+\) with \(\theta(T') = \theta(T'')\). Then Lemma 2 implies that \(T' = T''\). Thus \(\theta|A^+\) is an injective map into \([\theta(T_a), \pi]\). Similarly, \(\theta|A^-\) is an injective map into \([\theta(T_b), \pi]\).

If we have \(t < t'\) where \(t' \in \theta(A^+\) and \(t \in [\theta(T_a), \pi]\) hold, then \(t\) is realized as the angle of a bihedron in \(A^+\) which is included in the bihedron corresponding to \(t'\). It follows that \(\theta(A^+)\) is connected. Similarly, \(\theta(A^-)\) is connected.

Let \(T \in A^+\). Then \(\text{dev}|T\) is an imbedding onto a convex bihedron \(\text{dev}(T)\). Choose an open neighborhood \(N\) of \(T\) in \(M\) such that \(\text{dev}|N\) is an imbedding onto an open subset of \(S^n\). Then for every convex bihedron \(T'\) in \(\text{dev}(N)\) the open subset \(N\) includes a convex bihedron \(T''\) such that \(\text{dev}(T'') = T'\). This implies that \([\theta(T_a), \pi]\) includes an open neighborhood of \(\theta(T)\) whose elements are realized by bihedra in \(A^+\). Hence, \(\theta(A^+)\) is an open subset of \([\theta(T_a), \pi]\). Similarly, \(\theta(A^-)\) is an open subset of \([\theta(T_b), \pi]\).

We claim that \(\theta(A^+)\) is closed. (We use \(i\)-convexity now.) Suppose that it is not closed. Then \(\theta(A^+)\) is the half-open interval \([\theta(T_a), t^+\) for a real number \(t^+\) less than or equal to \(\pi\). Let \(T_i = \bigcup_{T \in A^+} T_i\). Since by Lemma 2, \(\text{dev}|T_i\) is injective, \(\text{dev}|T_i\) is an imbedding onto the interior of a convex bihedron of angle \(t^+\). For the closure \(\text{Cl}(T_1)\) of \(T_1\) in \(\overline{M}\), the map \(\text{dev}|\text{Cl}(T_1)\) is an imbedding onto \(\text{Cl}(\text{dev}(T_1))\). Since \(\text{Cl}(\text{dev}(T_1))\) is a convex bihedron, so is \(\text{Cl}(T_1)\). The bihedron \(\text{Cl}(T_1)\) has two faces \(B_0\) and \(B_{\text{Cl}(T_1)}\) where \(\sigma \cap \text{Cl}(T_1) \subset B_{\text{Cl}(T_1)}^\circ\). Since \(\sigma \cap B_{\text{Cl}(T_1)}^\circ\) is compact, \(\sigma \cap B_{\text{Cl}(T_1)}^\circ\) is a bounded subset of the open great ball \(B_{\text{Cl}(T_1)}^\circ\). Thus, the bihedron \(\text{Cl}(T_1)\) includes a \((j+1)\)-simplex \(K\) such that \(\sigma \cap \text{Cl}(T_1) \subset K^\circ\), where \(K^\circ\) is a \(j\)-dimensional face of \(K\) and is included in \(B_{\text{Cl}(T_1)}^\circ\). The definition of \(i\)-convexity implies that \(K \cap \sigma = \emptyset\) and \(\text{Cl}(T_1) \subset \overline{M}\) hold. Since \(\text{Cl}(T_1) \supset T_a\), we have \(\text{Cl}(T_1) \in A^+\) and hence \(t^+ \in \theta(A^+)\). This is absurd. Therefore, \(\theta(A^+)\) and, similarly, \(\theta(A^-)\) are closed.

We therefore have \(\theta(A^+) = [\theta(T_a), \pi]\) and \(\theta(A^-) = [\theta(T_b), \pi]\). Let \(T^+\) be the convex bihedron corresponding to \(\pi\) belonging to \(A^+\), and let \(T^-\) be that belonging to \(A^-\). We have \(T^+, T^- \subset \overline{M}\). The maps \(\text{dev}|T^+\) and \(\text{dev}|T^-\) are imbeddings onto great \((j+1)\)-balls in the great \((j+1)\)-sphere \(S^{j+1}\). Since the intersection \(\text{dev}(T^+) \cap \text{dev}(T^-)\) is a great \(j\)-sphere and, hence, is path-connected, Lemma 2 implies that \(\text{dev}|T^+ \cup T^-\) is an imbedding onto \(S^{j+1}\). Therefore, \(T^+ \cup T^-\) is a great \((j+1)\)-sphere in \(\overline{M}\). This completes the proof of Lemma 1.
Lemma 2. Suppose that $A$ and $B$ are path-connected compact subsets of $\tilde{M}$ such that $\text{dev}|A$ and $\text{dev}|B$ are imbeddings. Suppose that $A \cap B \neq \emptyset$ and that $\text{dev}(A) \cap \text{dev}(B)$ is a path-connected subset of $S^n$.

Then $\text{dev}|A \cup B$ is an imbedding onto $\text{dev}(A) \cup \text{dev}(B)$.

Proof. We only need to deduce the injectivity of $\text{dev}|A \cup B$ from the fact that given a path in $S^n$ and an initial point in $\tilde{M}$ there is at most one lift of the path to $\tilde{M}$ (see [1, Proposition 1.3.1]).

Let us end this paper with the following remark: Carrière conjectured in 1988 that the homotopy groups in dimensions greater than or equal to $i$, $i > 1$, for an $i$-convex affine manifold are trivial. This conjecture is still open. What we did here may aid us in understanding similar questions for projective manifolds.

References


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