MATRIX TRANSFORMATIONS OF POWER SERIES

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ABSTRACT. We consider the sequence of transforms \( \{g_n\} \) of a power series \( \sum_{n=0}^{\infty} a_n z^n \) given by \( g_n(z) := \sum_{k=0}^{\infty} b_{nk} a_k z^k \). We establish necessary and sufficient conditions on the matrix \( (b_{nk}) \) for the sequence \( \{g_n\} \) to converge uniformly on compact subsets of the disk \( D_P := \{z : |z| < P\} \) to a function holomorphic on \( D_P \).

1. INTRODUCTION

Suppose throughout that \( 0 < P \leq \infty \), \( 0 < R < \infty \), and that all sequences and matrices are complex with indices running through \( 0, 1, 2, \ldots \). We make the following definitions:

- \( D_P \) is the disk \( \{z : |z| < P\} \);
- \( \mathcal{E} \) is the set of all sequences \( a \equiv (a_n) \) such that \( \lim \left| a_n \right|^{1/n} = 0 \);
- \( \mathcal{E}^\beta \) is the set of all sequences \( a \equiv (a_n) \) such that \( \lim \sup \left| a_n \right|^{1/n} < \infty \);
- \( \mathcal{E}_R \) is the set of all sequences \( a \equiv (a_n) \) such that \( \sum_{n=0}^{\infty} |a_n| R^n < \infty \);
- \( A_R \) is the set of all sequences \( a \equiv (a_n) \) such that \( \lim \sup |a_n|^{1/n} = \frac{1}{R} \).

It will follow from the lemma (below) that \( \mathcal{E}^\beta \) is the \( \beta \)-dual of \( \mathcal{E} \).

The following are the first three of eight theorems we shall prove concerning matrix transformations of power series.

Theorem 1. A matrix \( B \equiv (b_{nk}) \) has the property that whenever the sequence \( a \equiv (a_n) \in \mathcal{E}_R \) the sequence of functions \( \{g_n\} \) given by

\[
g_n(z) := \sum_{k=0}^{\infty} b_{nk} a_k z^k, \quad n = 0, 1, \ldots,
\]

converges uniformly on every compact subset of \( D_P \), each power series \( \sum_{k=0}^{\infty} b_{nk} a_k z^k \) being convergent on \( D_P \), if and only if

- (i) \( \lim_{n \to \infty} b_{nk} = b_k \) for \( k = 0, 1, \ldots \);
- (ii) \( \sup_{n \geq 0, k \geq 0} |b_{nk}| \left( \frac{p}{R} \right)^k < \infty \) for each positive \( p < P \).

And then \( \lim_{n \to \infty} g_n(z) = \sum_{k=0}^{\infty} b_k a_k z^k \) on \( D_P \).
Theorem 2. A matrix $B \equiv (b_{nk})$ has the property that whenever the sequence $a \equiv (a_n) \in A_R$ the sequence of functions $(g_n)$ given by

$$g_n(z) := \sum_{k=0}^{\infty} b_{nk} a_k z^k, \quad n = 0, 1, \ldots,$$

converges uniformly on every compact subset of $D_P$, each power series $\sum_{k=0}^{\infty} b_{nk} a_k z^k$ being convergent on $D_P$, if and only if

(i) $\lim_{n \to \infty} b_{nk} = b_k$ for $k = 0, 1, \ldots$;

(ii) $\sup_{n \geq 0, k \geq 0} |b_{nk}| \left(\frac{R}{p}\right)^k < \infty$ for each positive $p < P$.

And then $\lim_{n \to \infty} g_n(z) = \sum_{k=0}^{\infty} b_k a_k z^k$ on $D_P$.

Theorem 3. A matrix $B \equiv (b_{nk})$ has the property that whenever the sequence $a \equiv (a_n) \in A_R$ the sequence of functions $(g_n)$ given by

$$g_n(z) := \sum_{k=0}^{\infty} b_{nk} a_k z^k, \quad n = 0, 1, \ldots,$$

converges uniformly on every compact subset of $D_{\infty}$, each power series $\sum_{k=0}^{\infty} b_{nk} a_k z^k$ being convergent on $D_{\infty}$, if and only if

(i) $\lim_{n \to \infty} b_{nk} = b_k$ for $k = 0, 1, \ldots$;

(ii) $\sup_{n \geq 0, k \geq 0} |b_{nk}| \left(\frac{1}{p}\right)^k < \infty$.

And then $\lim_{n \to \infty} g_n(z) = \sum_{k=0}^{\infty} b_k a_k z^k$ on $D_{\infty}$.

These theorems show that if the series-to-sequence transform given by $B$ is regular, then it is necessary in each case that $\lim_{n \to \infty} b_{nk} = b_k = 1$ for $k = 0, 1, \ldots$, and this in turn implies that $P \leq R$ in Theorems 1 and 2 (i.e., the sequence $(g_n)$ cannot converge uniformly in any disk $D_P$ with $P > R$). Regular sequence-to-sequence transforms of power series have been considered by Peyerimhoff [5] and Luh [4] among others. One of the novel features of our approach is that we deal with series-to-sequence transforms rather than sequence-to-sequence transforms.

Let $(B_n)$ be a sequence of nonzero complex numbers. The associated Nörlund series-to-sequence matrix $N_B$ is the triangular matrix $(b_{nk})$ with

$$b_{nk} := \begin{cases} \frac{B_{n-k}}{B_n} & \text{if } 0 \leq k \leq n, \\ 0 & \text{otherwise}. \end{cases}$$

The following theorem is an immediate consequence of Theorem 2. The case $R = 1$ of Theorem KS is due to Karin Stadtmüller [6, Theorem 5]. Her method of proof is different from and more complicated than the one developed below.

Theorem KS. The Nörlund matrix $N_B$ has the property that whenever the sequence $a \equiv (a_n) \in A_R$ the sequence of functions $(g_n)$ given by

$$g_n(z) := \frac{1}{B_n} \sum_{k=0}^{n} B_{n-k} a_k z^k, \quad n = 0, 1, \ldots,$$

converges uniformly on every compact subset of $D_P$, if and only if

$$\lim_{n \to \infty} \frac{B_{n-1}}{B_n} = b \quad \text{with} \quad |b| = \frac{R}{P}.$$

And then $\lim_{n \to \infty} g_n(z) = \sum_{k=0}^{\infty} a_k (bz)^k$ on $D_P$. 


Note. In view of Theorem 1, Theorem KS remains true if $A_R$ is replaced by $\mathcal{E}_R$.

2. A PRELIMINARY RESULT

**Lemma.** A sequence $b$ has the property that $\sum_{n=0}^{\infty} b_n a_n$ is convergent for each $a \in \mathcal{E}$ if and only if $b \in \mathcal{E}^\beta$.

**Proof.** Sufficiency. If $b \in \mathcal{E}^\beta$, then there exists a positive number $M$ such that $|b_n| \leq M^{n+1}$ for $n = 0, 1, \ldots$. Hence, if $a \in \mathcal{E}$, then $\sum_{k=0}^{\infty} |b_k a_k| \leq M \sum_{k=0}^{\infty} |a_k| M^k < \infty$.

Necessity. Assume $b \notin \mathcal{E}^\beta$, i.e., $\limsup |b_n|^{1/n} = \infty$. Then there exists a strictly increasing sequence of positive integers $(n_j)$ such that $0 < |b_{n_j}|^{1/n_j} \to \infty$. Choose

$$a_n := \begin{cases} \frac{1}{\sqrt{|b_n|}} & \text{if } n = n_j, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$|a_n|^{1/n} = \begin{cases} \left( \frac{1}{|b_n|^{1/n}} \right)^{1/2} & \text{if } n = n_j, \\ 0 & \text{otherwise.} \end{cases}$$

Hence $\lim |a_n|^{1/n} = 0$, so $a \in \mathcal{E}$. But

$$|b_{n_j} a_{n_j}| = \sqrt{|b_{n_j}|} = \left( |b_{n_j}|^{1/n_j} \right)^{n_j} \to \infty \quad \text{as } j \to \infty,$$

and therefore $\sum_{n=0}^{\infty} b_n a_n$ is not convergent. ☐

3. PROOFS OF THEOREMS 1, 2, AND 3

**Proof of Theorem 1.** Sufficiency. We assume that

$$\lim_{n \to \infty} b_{nk} := b_k$$

for $k = 0, 1, \ldots$;

$$M(p) := \sup_{n \geq 0, k \geq 0} |b_{nk}| \left( \frac{p}{R} \right)^k < \infty$$

for $0 < p < P$.

Let $a \in \mathcal{E}_R$. We have, for $n = 0, 1, \ldots$ and $|z| \leq p < P$,

$$\left| \sum_{k=0}^{\infty} b_{nk} a_k z^k \right| \leq \sum_{k=0}^{\infty} |b_{nk}| |a_k| p^k \leq M(p) \sum_{k=0}^{\infty} |a_k| R^k < \infty.$$

Hence the functions $g_n(z) := \sum_{k=0}^{\infty} b_{nk} a_k z^k$ are holomorphic and uniformly bounded on $D_p$. Also $g_n^{(k)}(0) = k! b_{nk} a_k \to k! b_k a_k$ as $n \to \infty$ for $k = 0, 1, \ldots$. Further, from Cauchy’s inequalities for the coefficients of power series we get that, for $|z| \leq p_1 < p < P$, $n = 0, 1, \ldots$, and $k = 0, 1, \ldots$,

$$|b_{nk} a_k z^k| \leq M(p, a) (p_1/p)^k,$$

where $M(p, a) := \sup_{n \geq 0, |z| = p} |g_n(z)| < \infty$.

Therefore, by the Weierstrass M-test, $\lim_{n \to \infty} g_n(z) = \sum_{k=0}^{\infty} b_k a_k z^k$ on $D_p$, and the sequence $(g_n)$ is uniformly convergent on compact subsets of $D_p$. 

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Necessity. Let \( a_k := \frac{1}{((k + 1)^2 R^k)} \) for \( k = 0, 1, 2, \ldots \). Since \( a \in \mathcal{A}_R \), our assumption is that the series \( g_n(z) := \sum_{k=0}^{\infty} b_{nk} a_k z^k \) converges on \( D_P \) and that the sequence \( (g_n) \) is uniformly convergent on \( D_p \) for \( 0 < p < P \). By the Weierstrass double-series theorem, \( \lim_{n \to \infty} b_{nk} a_k \) exists for \( k = 0, 1, \ldots \). Since \( a_k \neq 0 \) for \( k = 0, 1, \ldots \), it follows that the condition

\[
\lim_{n \to \infty} b_{nk} =: b_k \quad \text{for } k = 0, 1, \ldots
\]

must necessarily hold. Suppose now that \( p \) and \( \hat{p} \) are fixed and \( 0 < p < \hat{p} < P \). Since the sequence \( (g_n) \) is uniformly convergent on \( D_{\hat{p}} \), the closure of \( D_{\hat{p}} \), we have, for \( |z| \leq \hat{p} \) and \( n = 0, 1, \ldots \), that \( |g_n(z)| \leq M(\hat{p}, a) < \infty \). From Cauchy's inequalities for the coefficients of power series we get that

\[
|b_{nk} a_k \hat{p}^k| \leq M(\hat{p}, a) \quad \text{for } n = 0, 1, \ldots \text{ and } k = 0, 1, \ldots,
\]

and hence that

\[
\sup_{n \geq 0, k \geq 0} |b_{nk}| \left( \frac{\hat{p}}{R} \right)^k \leq M(\hat{p}, a) \sup_{k \geq 0} \left( \frac{\hat{p}}{R} \right)^k (k + 1)^2 < \infty.
\]

Therefore the condition

\[
\sup_{n \geq 0, k \geq 0} |b_{nk}| \left( \frac{p}{R} \right)^k < \infty \quad \text{for all positive } p < P
\]

is also necessary. \( \Box \)

Proof of Theorem 2. Sufficiency. We assume that

\[
\begin{align*}
\lim_{n \to \infty} b_{nk} &=: b_k \quad \text{for } k = 0, 1, \ldots; \\
M(p) := \sup_{n \geq 0, k \geq 0} |b_{nk}| \left( \frac{p}{R} \right)^k &< \infty \quad \text{for } 0 < p < P.
\end{align*}
\]

Let \( a \in \mathcal{A}_R \). For \( 0 < p < P \) choose \( r \) so that \( 0 < r < R \) and \( \frac{p}{r} < \frac{R}{r} \). Now choose \( p_1 \) such that \( 0 < p_1 < P \) and \( \frac{p}{r} = \frac{p_1}{R} \). We have, for \( |z| \leq p \), that

\[
\begin{align*}
\left| \sum_{k=0}^{\infty} b_{nk} a_k z^k \right| &\leq \sum_{k=0}^{\infty} |b_{nk}| |a_k|r^k = \sum_{k=0}^{\infty} |b_{nk}| \left( \frac{p_1}{R} \right)^k |a_k|r^k \\
&= \sum_{k=0}^{\infty} |b_{nk}| \left( \frac{p_1}{R} \right)^k |a_k|r^k \leq M(p_1) \sum_{k=0}^{\infty} |a_k|r^k < \infty.
\end{align*}
\]

Hence the functions \( g_n(z) := \sum_{k=0}^{\infty} b_{nk} a_k z^k \) are uniformly bounded on \( D_p \) for \( 0 < p < P \). Also \( g_n^{(k)}(0) = k! b_{nk} a_k \to k! b_k a_k \) as \( n \to \infty \) for \( k = 0, 1, \ldots \). Further, from Cauchy’s inequalities for the coefficients of power series we get that, for \( |z| \leq p_1 < p < P \), \( n = 0, 1, \ldots \) and \( k = 0, 1, \ldots \),

\[
|b_{nk} a_k z^k| \leq M(p, a)(p_1/p)^k, \quad \text{where } M(p, a) := \sup_{n \geq 0, |z| = p} |g_n(z)| < \infty.
\]

Therefore, by the Weierstrass M-test, \( \lim_{n \to \infty} g_n(z) = \sum_{k=0}^{\infty} b_{nk} a_k z^k \) on \( D_P \), and the sequence \( (g_n) \) is uniformly convergent on compact subsets of \( D_P \).

Necessity. Let \( a_k := 1/R^k \) for \( k = 0, 1, 2, \ldots \). Since \( a \in \mathcal{A}_R \), our assumption is that the series \( g_n(z) := \sum_{k=0}^{\infty} b_{nk} a_k z^k \) converges on \( D_P \) and that the
sequence \((g_n)\) is uniformly convergent on \(D_p\) for \(0 < p < P\). By the Weierstrass double-series theorem, \(\lim_{n \to \infty} b_{nk}a_k\) exists for \(k = 0, 1, \ldots\). Since \(a_k \neq 0\) for \(k = 0, 1, \ldots\), it follows that the condition

\[
\lim_{n \to \infty} b_{nk} =: b_k \quad \text{for } k = 0, 1, \ldots
\]

must necessarily hold. Suppose now that \(p\) is fixed and \(0 < p < P\). Since the sequence \((g_n)\) is uniformly convergent on \(D_p\), we have, for \(|z| \leq p\) and \(n = 0, 1, \ldots\), that \(|g_n(z)| \leq M(p, a) < \infty\). From Cauchy’s inequalities for the coefficients of power series we get that

\[
|b_{nk}| \left(\frac{p}{R}\right)^k = |b_{nk}a_k p^k| \leq M(p, a) \quad \text{for } n = 0, 1, \ldots \text{ and } k = 0, 1, \ldots.
\]

Therefore, the condition

\[
\sup_{n \geq 0, k \geq 0} |b_{nk}| \left(\frac{p}{R}\right)^k < \infty \quad \text{for all positive } p < P
\]

is also necessary.

**Proof of Theorem 3. Sufficiency.** We assume that

\[
\left\{ \begin{array}{l}
\lim_{n \to \infty} b_{nk} = b_k \quad \text{for } k = 0, 1, \ldots, \\
M := \sup_{n \geq 0, k \geq 0} |b_{nk}|^{\frac{1}{k+1}} < \infty.
\end{array} \right.
\]

Let \(a \in \mathbb{G}\). We have, for \(|z| \leq R < \infty\), that

\[
\left| \sum_{k=0}^{\infty} b_{nk}a_k z^k \right| \leq \sum_{k=0}^{\infty} |b_{nk}| |a_k| M^k \leq M \sum_{k=0}^{\infty} |a_k| (MR)^k < \infty.
\]

Hence the functions \(g_n(z) := \sum_{k=0}^{\infty} b_{nk}a_k z^k\) are entire and are uniformly bounded on each closed disk \(D_R\). Also \(g_n^{(k)}(0) = k! b_{nk}a_k \to k! b_k a_k\) as \(n \to \infty\) for \(k = 0, 1, \ldots\). Further, from Cauchy’s inequalities for the coefficients of power series we get that, for \(|z| < p < R\), \(n = 0, 1, \ldots\) and \(k = 0, 1, \ldots\),

\[
|b_{nk}a_k z^k| \leq M(R, a) \left(\frac{p}{R}\right)^k, \quad \text{where } M(R, a) := \sup_{n \geq 0} \max_{|z| = R} |g_n(z)| < \infty.
\]

Therefore, by the Weierstrass M-test, \(\lim_{n \to \infty} g_n(z) = \sum_{k=0}^{\infty} b_{nk}a_k z^k\) on \(D_\infty\), and the sequence \((g_n)\) is uniformly convergent on compact subsets of \(D_\infty\).

**Necessity.** We assume that for each \(a \in \mathbb{G}\) the series \(g_n(z) := \sum_{k=0}^{\infty} b_{nk}a_k z^k\) is convergent on \(D_\infty\) and that the sequence \((g_n)\) is uniformly convergent on compact subsets of \(D_\infty\). By the Weierstrass double-series theorem, \(\lim_{n \to \infty} b_{nk}a_k\) exists for \(k = 0, 1, \ldots\). Since there is an \(a \in \mathbb{G}\) such that \(a_k \neq 0\) for \(k = 0, 1, \ldots\), it follows that the condition

\[
\lim_{n \to \infty} b_{nk} =: b_k \quad \text{for } k = 0, 1, \ldots
\]

must necessarily hold.

Suppose that \(a \in \mathbb{G}\). Since the sequence \((g_n)\) is uniformly convergent on \(D_R\), we have, for \(|z| \leq R\) and \(n = 0, 1, \ldots\), that \(|g_n(z)| \leq M(R, a) < \infty\). From Cauchy’s inequalities for the coefficients of power series we get that

\[
|b_{nk}a_k R^k| \leq M(R, a) \quad \text{for } n = 0, 1, \ldots \text{ and } k = 0, 1, \ldots.
\]
Also, since \( \sum_{k=0}^{\infty} b_{nk} a_k \) is convergent whenever \( a \in \mathcal{E} \), we have, by the lemma, that
\[
M_n := \sup_{k \geq 0} |b_{nk}|^{\frac{1}{k+1}} < \infty \quad \text{for } n = 0, 1, \ldots.
\]
Assume now that
\[
\sup_{n \geq 0} \sup_{k \geq 0} |b_{nk}|^{\frac{1}{k+1}} = \sup_{n \geq 0} M_n = \infty.
\]
This implies that there exists a strictly increasing sequence of positive integers \( (n_j) \) such that \( M_{n_j} \to \infty \). This in turn implies that there exists a sequence of nonnegative integers \( (k_j) \) such that
\[
(*) \quad |b_{n_j,k_j}|^{\frac{1}{k_j+1}} > \frac{1}{2} M_{n_j} \to \infty \quad \text{as } j \to \infty.
\]
We show now that the sequence \( (k_j) \) is not bounded. Assume that it is bounded. Then there is a positive integer \( k^* \) such that \( 0 \leq k_j \leq k^* \). Since \( \lim_{n \to \infty} b_{nk} = b_k \) for \( k = 0, 1, \ldots, k^* \), it follows that the set of numbers \( (b_{nk})_{n \geq 0, 0 \leq k \leq k^*} \) is bounded and hence that the set of numbers \( (|b_{nk}|^{\frac{1}{k+1}})_{n \geq 0, 0 \leq k \leq k^*} \) is bounded. But this contradicts \( (*) \). Therefore, the sequence \( (k_j) \) is not bounded. We can suppose (by considering a subsequence if necessary) that the sequence is strictly increasing. Choose
\[
ak_j := \begin{cases} 1/(|b_{n_j,k_j}|)^{\frac{k_j+1}{k_j}} & \text{if } k_j = k_j, \\ 0 & \text{otherwise} \end{cases}
\]
We then have
\[
|a_{k_j}|^{\frac{1}{k_j+1}} = \frac{1}{\sqrt{|b_{n_j,k_j}|}} \left( \frac{1}{\frac{1}{2} M_{n_j}} \right)^{\frac{1}{k_j+1}} \to 0 \quad \text{as } j \to \infty.
\]
Therefore \( a \in \mathcal{E} \), but
\[
|b_{n_j,k_j}|a_{k_j} = \sqrt{|b_{n_j,k_j}|} \to \infty \quad \text{as } j \to \infty,
\]
which contradicts \( (1) \). Thus the condition
\[
\sup_{n \geq 0, k \geq 0} |b_{nk}|^{\frac{1}{k+1}} < \infty
\]
is also necessary. \( \square \)

4. Additional theorems

In this section we prove some theorems showing that the disk of convergence \( D_P \) specified in Theorem 2 cannot be enlarged when the matrix \( B \) satisfies conditions (i) and (ii) of that theorem together with certain other conditions.

**Theorem 4.** Suppose that \( P \) and \( R \) are positive numbers, and that \( B \equiv (b_{nk}) \) is a normal infinite matrix (i.e., \( b_{nk} = 0 \) for \( k > n \) and \( b_{nn} \neq 0 \)) satisfying
\[
M(p) := \sup_{n \geq 0, k \geq 0} |b_{nk}| \left( \frac{p}{R} \right)^{k} < \infty \quad \text{for } 0 < p < P.
\]
Then, for each \( a \in A_R \) and each \( R_1 \geq P \),

\[
\limsup_{n \to \infty} \max_{|z| = R_1} \left| \sum_{k=0}^{n} b_{nk} a_k z^k \right|^{\frac{1}{2}} \leq \frac{R_1}{P}.
\]

**Proof.** Choose \( R_1 \geq P \), and suppose \( a \in A_R \). Let \( 0 < \lambda < 1 \), and take \( p := \lambda P \). Then \( 0 < p < P \). Since \( \limsup |a_k|^{\frac{1}{1-k}} = \frac{1}{k} \), there is a positive constant \( c(\lambda) \) such that

\[
|a_k| \leq \frac{c(\lambda)}{(\lambda R)^k} \quad \text{for } k \geq 0.
\]

Now for \( |z| = R_1 \) we have

\[
\left| \sum_{k=0}^{n} b_{nk} a_k z^k \right| \leq \sum_{k=0}^{n} |b_{nk}| \left( \frac{P}{R} \right)^k |a_k| R^k \left( \frac{R_1}{p} \right)^k 
\]

\[
\leq M(p) c(\lambda) \sum_{k=0}^{n} \left( \frac{R}{\lambda R} \right)^k \left( \frac{R_1}{\lambda P} \right)^k = M(p) c(\lambda) \sum_{k=0}^{n} \left( \frac{R_1}{\lambda^2 P} \right)^k.
\]

Since \( R_1/(\lambda^2 P) > R_1/P \geq 1 \), it follows that

\[
\limsup_{n \to \infty} \max_{|z| = R_1} \left| \sum_{k=0}^{n} b_{nk} a_k z^k \right|^{\frac{1}{2}} \leq \lim_{n \to \infty} \left( \sum_{k=0}^{n} \left( \frac{R_1}{\lambda^2 P} \right)^k \right)^{\frac{1}{2}} = \frac{R_1}{\lambda^2 P}.
\]

Letting \( \lambda \to 1 \) we get

\[
\limsup_{n \to \infty} \max_{|z| = R_1} \left| \sum_{k=0}^{n} b_{nk} a_k z^k \right|^{\frac{1}{2}} \leq \frac{R_1}{P}. \quad \square
\]

**Remark.** Assume that a normal matrix \( B \) satisfies

\[
M(p) := \sup_{n \geq 0, \ k \geq 0} |b_{nk}| \left( \frac{P}{R} \right)^k < \infty \quad \text{for } 0 < p < P.
\]

Then

\[
|b_{nn}|^{\frac{1}{2}} \frac{P}{R} \leq M(p)^{\frac{1}{2}} \to 1 \quad \text{as } n \to \infty,
\]

and hence

\[
\limsup_{n \to \infty} |b_{nn}|^{\frac{1}{2}} \leq \frac{R}{p} \quad \text{for each positive } p < P.
\]

Letting \( p \to P \) we get

\[
\limsup_{n \to \infty} |b_{nn}|^{\frac{1}{2}} \leq \frac{R}{P}.
\]

This suggests that it is not inappropriate to impose the condition

\[
\lim_{n \to \infty} |b_{nn}|^{\frac{1}{2}} = \frac{R}{P},
\]

as we do in the following theorem.
**Theorem 5.** Let $B$ be a normal matrix. Suppose that
\[
\lim_{n \to \infty} \left| b_{nn} \right|^\frac{1}{n} = \frac{R}{P},
\]
where $P$ and $R$ are positive numbers. Then for each $a \in A_R$ and each $R_1 \geq P$ we have
\[
\limsup_{n \to \infty} \max_{|z| = R_1} \left| \sum_{k=0}^{n} b_{nk} a_k z^k \right|^\frac{1}{n} \geq \frac{R_1}{P}.
\]

**Proof.** Assume that the conclusion of the theorem is not true. Then there is an $a^* \in A_R$ and an $R_1 \geq P$ such that
\[
\limsup_{n \to \infty} \max_{|z| = R_1} \left| \sum_{k=0}^{n} b_{nk} a_k^* z^k \right|^\frac{1}{n} < \frac{R_1}{P}.
\]

Therefore, there exists a positive $\tilde{R} < R_1$ such that, for all $n$ sufficiently large,
\[
\max_{|z| = \tilde{R}} \left| \sum_{k=0}^{n} b_{nk} a_k^* z^k \right|^\frac{1}{n} \leq \frac{\tilde{R}}{P},
\]
and hence
\[
\max_{|z| = \tilde{R}} \left| \sum_{k=0}^{n} b_{nk} a_k^* z^k \right| \leq \left( \frac{\tilde{R}}{P} \right)^n.
\]

Applying the Cauchy inequalities to the function $g_n(z) := \sum_{k=0}^{n} b_{nk} a_k^* z^k$ we get in particular that, for all large $n$,
\[
|b_{nn}| |a_n^*| R_1^n \leq \left( \frac{\tilde{R}}{P} \right)^n, \quad \text{and therefore} \quad |b_{nn}|^{\frac{1}{n}} |a_n^*|^\frac{1}{n} R_1 \leq \frac{\tilde{R}}{P}.
\]

From the last inequality we get that
\[
\frac{\tilde{R}}{P} \geq \limsup_{n \to \infty} \left( |b_{nn}|^{\frac{1}{n}} |a_n^*|^\frac{1}{n} R_1 \right) = R_1 \lim_{n \to \infty} |b_{nn}|^{\frac{1}{n}} \cdot \limsup_{n \to \infty} |a_n^*|^\frac{1}{n} = \frac{R_1}{P}.
\]
But this is a contradiction since $0 < \tilde{R} < R_1$. Hence the conclusion of the theorem must hold. \(\Box\)

The next two theorems generalize results about regular and nonregular Nörlund matrices due respectively to Luh [3] and K. Stadtmüller [6, Theorems 6 and 7]. The first of these theorems, which follows immediately from Theorems 4 and 5, shows, inter alia, that the sequence $(g_n)$ specified in Theorem 2 cannot converge uniformly in any disk $D_{P_1}$ with $P_1 > P$ when $B$ is a normal matrix satisfying condition (ii) of Theorem 2 together with the diagonal condition of Theorem 5.

**Theorem 6.** Suppose that $P$ and $R$ are positive numbers and that $B$ is a normal matrix satisfying
\[
M(p) := \sup_{n \geq 0, \: k \geq 0} |b_{nk}| \left( \frac{p}{R} \right)^k < \infty \quad \text{for} \quad 0 < p < P \quad \text{and} \quad \lim_{n \to \infty} |b_{nn}|^{\frac{1}{n}} = \frac{R}{P}.
\]

Then, for each $a \in A_R$ and each $R_1 \geq P$,
\[
\limsup_{n \to \infty} \max_{|z| = R_1} \left| \sum_{k=0}^{n} b_{nk} a_k z^k \right|^\frac{1}{n} = \frac{R_1}{P}.
\]

The next theorem shows that the circle $|z| = R_1$ in the conclusion of Theorem 6 can be replaced by any arc of that circle when condition (i) of Theorem 2 is also satisfied.
Theorem 7. Suppose that $P$ and $R$ are positive numbers and that $B$ is a normal matrix such that
\[
\lim_{n \to \infty} b_{nk} =: b_k \quad \text{for} \quad k = 0, 1, \ldots, \quad \text{where} \quad b_k \neq 0 \quad \text{for} \quad k > k^*;
\]
\[
M(p) := \sup_{n \geq 0, \ k \geq 0} |b_{nk}| \left( \frac{p}{R} \right)^k < \infty \quad \text{for} \quad 0 < p < P, \quad \text{and} \quad \lim_{n \to \infty} |b_{nn}|^{1/n} = \frac{R}{P}.
\]
Then, for each $a \in A_R$ and each $R_1 \geq P$,
\[
\lim_{n \to \infty} \sup_{z \in \Gamma} \max_{k=0}^n |a_k z_k|^{1/n} = \frac{R_1}{P},
\]
where $\Gamma$ is any closed non-trivial arc of $|z| = R_1$.

Proof. By Theorem 6 we know that
\[
\lim_{n \to \infty} \sup_{z \in \Gamma} \max_{k=0}^n |a_k z_k|^{1/n} \leq \frac{R_1}{P}.
\]
Hence it is enough to prove that, for every $a \in A_R$,
\[
\lim_{n \to \infty} \sup_{z \in \Gamma} \max_{k=0}^n |a_k z_k|^{1/n} \geq \frac{R_1}{P},
\]
which we now proceed to do.

Case 1. $R_1 = P$: Suppose (2) is not true. Then for some $a^* \in A_R$ we have
\[
\lim_{n \to \infty} \sup_{z \in \Gamma} \max_{k=0}^n |a_k z_k|^{1/n} < \frac{R_1}{P} = 1.
\]
It follows that there exists a positive number $q < 1$ such that, for all $n$ sufficiently large,
\[
\sup_{z \in \Gamma} \left| \sum_{k=0}^n b_{nk} a_k^* z_k^k \right| < q^n.
\]
Given $\epsilon > 0$ we get from Theorem 6 that, for all $n$ sufficiently large,
\[
\max_{|z|=P} \left| \sum_{k=0}^n b_{nk} a_k^* z_k^k \right| \leq 2\epsilon^n.
\]
For $0 < r < P$ we have, by Nevanlinna's $N$-constants theorem (see [1, Theorem 18.3.3]), that there exists a positive number $\theta < 1$ (depending on $r$ but not on $\epsilon$) such that, for all large $n$,
\[
\max_{|z|=r} \left| \sum_{k=0}^n b_{nk} a_k^* z_k^k \right| \leq (q^\theta 2^{(1-\theta)\epsilon})^n.
\]
Since we can choose $\epsilon > 0$ so small that $q^\theta 2^{(1-\theta)\epsilon} < 1$, it follows that
\[
\max_{|z|=r} \left| \sum_{k=0}^n b_{nk} a_k^* z_k^k \right| \to 0 \quad \text{as} \quad n \to \infty.
\]
By the Weierstrass double-series theorem we get that
\[ 0 = \lim_{n \to \infty} b_{n k} a_k^* = b_k a_k^* \quad \text{for } k = 0, 1, \ldots. \]

Since \( a^* \in A_R \), we have that \( a_k^* \neq 0 \) for some \( k > k^* \). Hence \( b_k = 0 \) for such a \( k \). But this contradicts the assumption that \( b_k \neq 0 \) for \( k > k^* \). Therefore (2) must hold when \( R_1 = P \).

Case 2. \( R_1 > P \): Assume that (2) is not true. Then there exists a sequence \( a^* \in A_R \) and a number \( \tilde{R} \) such that \( P < \tilde{R} < R_1 \) and
\[ \limsup_{n \to \infty} \max_{z \in \Gamma} \left| \sum_{k=0}^{n} b_{n k} a_k^* z^k \right|^{1/2} \leq \frac{\tilde{R}}{P}. \]

Hence given \( \varepsilon > 0 \) we have, for all sufficiently large \( n \),
\[ \max_{z \in \Gamma} \left| \sum_{k=0}^{n} b_{n k} a_k^* z^k \right| \leq \left( \frac{\tilde{R}}{P} \cdot \frac{1}{R_1} \right)^n 2^n \varepsilon^n = \left( \frac{\tilde{R}}{R_1} \right)^n \left( \frac{2\varepsilon}{P} \right)^n. \]

Further, from Theorem 6 we get that, for all large \( n \),
\[ \max_{|z|=P} \left| \sum_{k=0}^{n} b_{n k} a_k^* z^k \right| \leq \left( \frac{2\varepsilon}{P} \right)^n \]
\[ \text{and} \]
\[ \max_{|z|=R_1} \left| \sum_{k=0}^{n} b_{n k} a_k^* z^k \right| \leq \left( \frac{2\varepsilon}{P} \right)^n. \]

Let \( g_n(z) := \sum_{k=0}^{n} b_{n k} a_k^* z^k \), and let \( P < r < R_1 \). Then, by Nevanlinna's \( N \)-constants theorem, there exist positive constants \( \theta_1, \theta_2, \theta_3 \) (depending on \( r \) but not on \( \varepsilon \) ) such that \( \theta_1 + \theta_2 + \theta_3 = 1 \) and
\[ \max_{|z|=r} \left| \frac{g_n(z)}{z^n} \right| \leq \left( \frac{\tilde{R}}{R_1} \right)^{n\theta_1} \left( \frac{2\varepsilon}{P} \right)^{n\theta_2} \left( \frac{2\varepsilon}{P} \right)^{n\theta_3} = \left( \frac{\tilde{R}}{R_1} \right)^{n\theta_1} \left( \frac{2\varepsilon}{P} \right)^n \]
for all sufficiently large \( n \). Hence, choosing \( \varepsilon > 0 \) so small that \( (\tilde{R}/R_1)^{\theta_1} 2^n \varepsilon < 1 \), we get
\[ \limsup_{n \to \infty} \max_{|z|=r} \left| g_n(z) \right|^\frac{1}{n} \leq \left( \frac{\tilde{R}}{R_1} \right)^{\theta_1} \frac{r}{P} < \frac{r}{P}. \]
Since \( r > P \), the last inequality contradicts the conclusion of Theorem 5. Hence (2) must hold when \( R_1 > P \). \( \square \)

The next theorem deals with the possibility of pointwise convergence of the sequence \( (g_n(z)) \) specified in Theorem 2 outside the convergence disk \( D_P \). It generalizes results due to Lejá [2] and Stadtmüller [6, Theorem 8] about regular and nonregular Nörlund matrices respectively. Both authors mistakenly assumed that their proofs were valid when, in the notation of the following theorem, \( R = 1 \) and sequence \( (a_n) \) is bounded. The example \( a_n := 1/(n+1) \) shows that their method of proof cannot be used in this case. The difficulty is avoided in our Theorem 8 by the imposition of the limsup condition.
Theorem 8. Suppose that $P$ and $R$ are positive numbers and that $B$ is a normal matrix such that

$$\lim_{n \to \infty} b_{nk} =: b_k \quad \text{for } k = 0, 1, \ldots, \text{ where } b_k \neq 0 \text{ for } k > k^*;$$

$$M(p) := \sup_{n \geq 0, k \geq 0} |b_{nk}| \left( \frac{P}{R} \right)^k < \infty \quad \text{for } 0 < p < P; \quad \lim_{n \to \infty} |b_{nn}|^{\frac{1}{n}} = \frac{R}{P},$$

and

$$|b_{nk}| \leq c(\hat{R}) |b_{nn}| \left( \frac{P}{\hat{R}} \right)^{n-k} \quad \text{for } 0 < \hat{R} < R \text{ and } 0 \leq k \leq n.$$

Suppose that $a \in L_R$ and that $\limsup_{n \to \infty} |a_n| R^n > 0$. Let

$$g_n(z) := \sum_{k=0}^{n} b_{nk} a_k z^k.$$

Then $\limsup_{n \to \infty} |g_n(z)|^{\frac{1}{n}} \leq 1$ for at most a finite number of points $z$ satisfying $|z| > P_1 > P$, and hence, in particular, the sequence $(g_n)$ can converge at most at a finite number of points $z$ satisfying $|z| > P_1 > P$.

Proof. Let $c_n := a_n R^n$ where $a \in L_R$, and let $\limsup_{n \to \infty} |c_n| > c > 0$. Define

$$M := \begin{cases} 1 & \text{if } \sup_{n \geq 0} |c_n| = \infty, \\ c^{-1} \sup_{n \geq 0} |c_n| & \text{otherwise.} \end{cases}$$

By considering the unbounded monotonic sequence $(d_n)$ where $d_n := \max_{0 \leq k \leq n} |c_k|$ when $\max_{n \geq 0} |c_n| = \infty$, we see that there is a strictly increasing sequence of positive integers $(n_k)$ integers such that

$$|c_{n_k}| \leq M |c_{n_k}| \text{ for } 0 \leq n < n_k, \quad \text{and } |c_{n_k}| > c.$$

Since $\limsup_{n \to \infty} |c_n|^{\frac{1}{n}} = 1$, we have

$$1 \geq \limsup_{k \to \infty} |c_{n_k}|^{\frac{1}{n_k}} \geq \liminf_{k \to \infty} |c_{n_k}|^{\frac{1}{n_k}} \geq \lim_{k \to \infty} c^{\frac{1}{n_k}} = 1,$$

so $\lim_{k \to \infty} |c_{n_k}|^{\frac{1}{n_k}} = 1$. Whenever $c_n \neq 0$, let

$$\tilde{g}_n(z) := \sum_{j=0}^{n} \frac{b_{nj}}{b_{nn}} c_j \left( \frac{z}{R} \right)^{j-n} = \frac{g_n(z)}{b_{nn} c_n(z/R)^n};$$

and let

$$h_k(w) := \tilde{g}_{n_k} \left( \frac{1}{w} \right).$$

Assume that $z^*$ is a point such that $|z^*| > P_1$ and $\limsup_{n \to \infty} |g_n(z^*)|^{\frac{1}{n}} \leq 1$. Since

$$\lim_{k \to \infty} \left| b_{n_k, n_k} c_{n_k} \left( \frac{z^*}{R} \right)^{\frac{1}{n_k}} \right|^{\frac{1}{n_k}} = \lim_{k \to \infty} |b_{n_k, n_k}|^{\frac{1}{n_k}} \cdot \lim_{k \to \infty} |c_{n_k}|^{\frac{1}{n_k}} \cdot \frac{|z^*|}{R} \geq \frac{R P_1}{P R} = \frac{P_1}{P} > 1,$$
it follows from (3) and (4) that \( \limsup_{k \to \infty} |\tilde{g}_{nk}(z^*)|^{1/k} < 1 \) and hence that

\[
(5) \quad \lim_{k \to \infty} h_k(w^*) = 0 \quad \text{where} \quad w^* := 1/z^*.
\]

Suppose \( |w| \leq 1/P^* \) where \( P_1 > P^* > P \). Then we have, for \( 0 < \tilde{R} < R \),

\[
|h_k(w)| \leq \sum_{j=0}^{n_k} c(\tilde{R}) \left( \frac{P}{\tilde{R}} \right)^{n_k-j} M \left( \frac{R}{P^*} \right)^{n_k-j} \sum_{j=0}^{n_k} \left( \frac{P}{P^*} \right)^{n_k-j}.
\]

Choose \( \tilde{R} < R \) so close to \( R \) that \( 0 < \frac{P}{P^*} \tilde{R} < 1 \). Then

\[
|h_k(w)| \leq \frac{c(\tilde{R}) M}{1 - \frac{P}{P^*} \tilde{R}} < \infty \quad \text{for} \quad |w| \leq \frac{1}{P^*} < \frac{1}{P} \quad \text{and} \quad k \geq 0.
\]

This means that the sequence \((h_k(w))\) is uniformly bounded for \( |w| \leq 1/P^* \). Suppose now that there are infinitely many points \( z_r \) with \( |z_r| > P_1 > P^* \) such that \( \limsup_{n \to \infty} |g_n(z_r)|^{1/n} \leq 1 \). Then by (5)

\[
\lim_{k \to \infty} h_k(w_r) = 0 \quad \text{for} \quad w_r := 1/z_r.
\]

By Vitali's theorem (see [7, Theorem 5.2.1]) the sequence \((h_k(w))\) converges uniformly to 0 on compact subsets of \( D_{P^*} \). In particular,

\[
0 = \lim_{k \to \infty} h_{nk}(0) = 1,
\]

which is a contradiction. Hence there are at most finitely many points \( z \) such that \( |z| > P_1 \) and \( \limsup_{n \to \infty} |g_n(z)|^{1/n} \leq 1 \). \( \Box \)

5. Construction

In this section we construct a Nörlund matrix \( N_B \) satisfying the hypotheses of Theorem 8 with \( P = 1 \) such that the corresponding sequence of transforms \((g_n)\) of the power series \( \sum_{k=0}^{\infty} (z/R)^k \) converges at \( N \) points outside the convergence disk \( D_1 \).

Let \( p(z) \) be a polynomial of degree \( N \) defined by

\[
p(z) := \sum_{k=0}^{\infty} p_k z^k := (z + \alpha_1)(z + \alpha_2) \cdots (z + \alpha_N),
\]

where \( 0 < \alpha_1 < \alpha_2 < \cdots < \alpha_N < 1 \). Define the Nörlund matrix \( N_B \equiv (b_{nk}) \) by setting

\[
b_{nk} := \frac{B_{n-k}}{B_n} \quad \text{for} \quad 0 \leq k \leq n, \quad \text{where} \quad B_n := \frac{1}{R^n} \sum_{k=0}^{n} p_k.
\]
Then, for $a_k := 1/R^k$, $w = 1/z$, and $n \geq N$,

$$g_n(z) := \sum_{k=0}^{n} b_{nk}a_kz^k = \frac{1}{B_n} \sum_{k=0}^{n} B_{n-k} \left(\frac{z}{R}\right)^k$$

$$= \frac{z^n}{B_n R^n} \sum_{k=0}^{n} B_k (Rw)^k = \frac{z^n}{B_n R^n} \sum_{k=0}^{n} w^k \sum_{j=0}^{k} p_j$$

$$= \frac{z^n}{B_n R^n} \sum_{j=0}^{n} p_j \sum_{k=j}^{n} w^k = \frac{z^n}{B_n R^n} \sum_{j=0}^{n} p_j \frac{w^j - w^{n+1}}{1 - w}$$

$$= \frac{z^n}{B_n R^n} \frac{p(w)}{1 - w} - \frac{w}{1 - w}.$$

Hence, for every $n \geq N$, we have $g_n(z) = z/(1 - z)$ whenever $p(w) = 0$, and this occurs when $z = -1/a_k$, $k = 1, 2, \ldots, N$.

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