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ARITHMETIC GROUPS OF HIGHER $\mathbb{Q}$-RANK CANNOT ACT ON 1-MANIFOLDS

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ABSTRACT. Let Γ be a subgroup of finite index in $\mathrm{SL}_n(\mathbb{Z})$ with $n \geq 3$. We show that every continuous action of Γ on the circle S^1 or on the real line $\mathbb{R}$ factors through an action of a finite quotient of Γ. This follows from the algebraic fact that central extensions of Γ are not right orderable. (In particular, Γ is not right orderable.) More generally, the same results hold if Γ is any arithmetic subgroup of any simple algebraic group G over $\mathbb{Q}$, with $\mathbb{Q}$-rank$(G) \geq 2$.

1. INTRODUCTION

Let Γ be a subgroup of finite index in $\mathrm{SL}_n(\mathbb{Z})$ with $n \geq 3$, or, more generally, let Γ be an arithmetic subgroup of a simple algebraic $\mathbb{Q}$-group of $\mathbb{Q}$-rank at least 2. (For background on arithmetic subgroups, see [B, §7.11, p. 49].) We show that any continuous action of Γ on a connected 1-manifold (that is, on the circle S^1 or on the real line $\mathbb{R}$) factors through an action of a finite quotient of Γ. To do this, we prove that no central extension of Γ is right orderable. Every group is trivially a central extension of itself, so, in particular, we prove that Γ is not right orderable.

Definition [MR, Chapter VII]. Let Γ be a group that is equipped with a total order $<$. We say Γ is *right ordered* if $a < b \Rightarrow ac < bc$ for all $a, b, c \in \Gamma$. A group is *right orderable* if there exists an order relation under which the group is right ordered.

Definition. A subgroup Z of a group $\tilde{\Gamma}$ is *central* if every element of Z commutes with every element of $\tilde{\Gamma}$. A group $\tilde{\Gamma}$ is a *central extension* of a group Γ if there is a central subgroup Z of $\tilde{\Gamma}$ such that $\tilde{\Gamma}/Z \cong \Gamma$.

Main Theorem 4.3′. *If Γ is a subgroup of finite index in $\mathrm{SL}_n(\mathbb{Z})$ with $n \geq 3$, then no central extension of Γ is right orderable.*

Corollary 2.4′. *If Γ is a subgroup of finite index in $\mathrm{SL}_n(\mathbb{Z})$ with $n \geq 3$, then Γ has no continuous, faithful actions on S^1 or $\mathbb{R}$. In fact, every action of Γ on S^1 or $\mathbb{R}$ factors through an action of a finite quotient of Γ.*

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It is not difficult to see that the group of orientation-preserving homeomorphisms of $\mathbb{R}$ has no finite subgroups and that every finite subgroup of the group of orientation-preserving homeomorphisms of S^1 is cyclic. Thus, the corollary implies that every continuous, orientation-preserving action of Γ on $\mathbb{R}$ is trivial and that every continuous, orientation-preserving action of Γ on S^1 factors through an action of the finite group $\Gamma/[\Gamma, \Gamma]$.

The implications for general actions, which need not be orientation preserving, are slightly weaker. Namely, every continuous action of Γ on $\mathbb{R}$ factors through an action of the finite group $\Gamma/[\Gamma, \Gamma]$, and every continuous action of Γ on S^1 factors through an action of the finite group Γ/Γ'', where Γ'' denotes the second derived group of Γ (that is, the commutator subgroup of the commutator subgroup of Γ).

The same conclusions hold if Γ is any arithmetic subgroup of any simple algebraic group G over $\mathbb{Q}$, with $\mathbb{Q}\text{-rank}(G) \geq 2$ (see Theorem 4.3 and Corollary 2.4). On the other hand, the conclusions sometimes fail to hold when $\mathbb{Q}\text{-rank}(G) < 2$. For example, the group $\mathrm{PSL}_2(\mathbb{R})$ acts on S^1 (by linear-fractional transformations), so every subgroup (including every arithmetic subgroup) of $\mathrm{PSL}_2(\mathbb{R})$ has a faithful action on S^1. Furthermore, every torsion-free, cocompact, arithmetic subgroup of $\mathrm{PSL}_2(\mathbb{R})$ has a faithful action on $\mathbb{R}$. For the general arithmetic group of $\mathbb{Q}$-rank less than two, it is not yet known how to determine whether actions on $\mathbb{R}$ or S^1 exist.

The conclusions of Corollary 2.4 are related to work of R. J. Zimmer. Specifically, Zimmer conjectures that if $n \geq 3$, then the natural action of $\mathrm{SL}_n(\mathbb{Z})$ on the n-torus $\mathbb{T}^n$ is the smallest volume-preserving action of $\mathrm{SL}_n(\mathbb{Z})$. More precisely, for $n \geq 3$, Zimmer conjectures that every volume-preserving, C^∞ action of $\mathrm{SL}_n(\mathbb{Z})$ on a compact manifold of dimension less than n factors through an action of a finite quotient of $\mathrm{SL}_n(\mathbb{Z})$ (see [Z2, Conjecture II]). Corollary 2.4 can be viewed as a first step toward establishing analogous results for actions that are not assumed to preserve volume.

The paper is organized as follows. Section 2 describes the connection between right orderability and the existence of actions on 1-manifolds; in particular, Corollary 2.4 is derived from Theorem 4.3. Section 3 presents a simple proof of the nonexistence of right orderings of finite-index subgroups of $\mathrm{SL}_n(\mathbb{Z})$; this illustrates all the main ideas of §4, in a simple setting. Section 4 presents a proof of Theorem 4.3, the nonexistence of right orderings of any arithmetic group of higher $\mathbb{Q}$-rank.

2. Proof of Corollary 2.4

In this section, we derive Corollary 2.4 from Theorem 4.3. To do this, we apply Lemmas 2.2 and 2.3, which explain the relationship between right orderability and the existence of continuous actions. We also appeal to the following special case of an important theorem of Margulis on quotients of arithmetic groups.

Theorem 2.1 (Margulis [M] or [Z1, Theorem 8.1.2, p. 149]). *Let Γ be an arithmetic subgroup of a $\mathbb{Q}$-simple algebraic $\mathbb{Q}$-group G, with $\mathbb{R}\text{-rank}(G) \geq 2$. If N is a normal subgroup of Γ, then either Γ/N is finite or N is a finite, central subgroup of G.*

Lemma 2.2. *If a group Γ acts faithfully on $\mathbb{R}$, by orientation-preserving homeomorphisms, then it is right orderable. Conversely, if a countable group is right orderable, then it has faithful actions on $\mathbb{R}$ and S^1 by orientation-preserving homeomorphisms.*

Proof. Any orientation-preserving homeomorphism of $\mathbb{R}$ is order preserving, and it is not difficult to show that every group of order-preserving permutations of any ordered set is a right-orderable group [C, §5, III]. (Briefly, let $\{x_i\}$ be a well-ordered list of the elements of the set. Define $\gamma < \lambda$ if $x_i^\gamma < x_i^\lambda$, where i is minimal with $x_i^\gamma \neq x_i^\lambda$.) Thus, Γ is right orderable.

Conversely, suppose a countable group Γ is right orderable. If the order type of $(\Gamma, <)$ is $\mathbb{Q}$, then the Dedekind completion of $(\Gamma, <)$ is $\mathbb{R}$. Because Γ acts by order-preserving homeomorphisms on this completion, we have an action of Γ on $\mathbb{R}$. Any action on $\mathbb{R}$ extends to an action on the one-point compactification S^1.

If the order type of $(\Gamma, <)$ is not $\mathbb{Q}$, that is, if the order is not dense, then extend the right order on Γ to a dense right order on $\Gamma \times \mathbb{Q}$. From the argument of the preceding paragraph, we know that $\Gamma \times \mathbb{Q}$ acts faithfully on $\mathbb{R}$ and S^1. By restriction, then Γ also acts. □

Lemma 2.3. *Any faithful, orientation-preserving action of a group Γ on S^1 lifts to a faithful, orientation-preserving action of some central extension of Γ on $\mathbb{R}$.*

Proof. Each $\gamma \in \Gamma$ lifts (in several ways) to a homeomorphism of $\mathbb{R}$. Let $\widetilde{\Gamma}$ be the group consisting of all the lifts of all the elements of Γ. Then $\pi_1(S^1)$, the fundamental group of the circle, is a normal subgroup of $\widetilde{\Gamma}$, with quotient Γ. Because the action of Γ is orientation preserving, it is not difficult to see that Γ acts trivially on $\pi_1(S^1)$, so $\pi_1(S^1)$ is a central subgroup of $\widetilde{\Gamma}$. □

Corollary 2.4. *If Γ is an arithmetic subgroup of a $\mathbb{Q}$-simple algebraic $\mathbb{Q}$-group G, with $\mathbb{Q}\text{-rank}(G) \geq 2$, then Γ has no continuous, faithful actions on S^1 or $\mathbb{R}$. In fact, every action of Γ on S^1 or $\mathbb{R}$ factors through an action of a finite quotient of Γ.*

Proof. Suppose, for a contradiction, that Γ acts faithfully on S^1 or $\mathbb{R}$. By passing to a subgroup of index 2 if necessary, we may assume that the action is orientation preserving. In this case, Lemma 2.3 asserts that a central extension $\widetilde{\Gamma}$ acts faithfully on $\mathbb{R}$. Then Lemma 2.2 asserts that $\widetilde{\Gamma}$ is right orderable. This contradicts the conclusion of Theorem 4.3.

Given any action of Γ on S^1 or $\mathbb{R}$, let N be the kernel of the action. If N is infinite, then Margulis's Theorem (2.1) on normal subgroups of arithmetic groups asserts that N is of finite index in Γ; thus the action factors through an action of the finite quotient Γ/N, as desired. If N is finite, then N is a normal subgroup of G, so, by passing to the arithmetic subgroup Γ/N of G/N, we obtain a faithful action. This is impossible. □

3. A simple proof of a special case

In this section, we present an essentially self-contained proof that finite-index subgroups of $\mathrm{SL}_n(\mathbb{Z})$ are not right orderable. (For $\mathrm{SL}_n(\mathbb{Z})$ itself, there is an easy proof based on the existence of elements of finite order, but this argument does not apply in general, because there are subgroups of finite index that have no elements of finite order [Rg, Corollary 6.13, p. 95].) This proof illustrates the main ideas of a proof of Main Theorem 4.3, without reference to central extensions, or the theory of algebraic groups, or other algebraic machinery. The basic group theory that we use can be found in standard texts such as Hall [H] and Gorenstein [G].

The proof is based on Lemma 3.2, which can be viewed as a complete description of the right orderings of a certain nilpotent group, the discrete Heisenberg group. Analogously, the general proof presented in §4 is based on an understanding of the right orderings of general nilpotent groups.

Definition. For elements a and b of a right-ordered group Γ, we say a is *infinitely larger* than b (denoted $a \gg b$) if either $a > b^i$ for all $i \in \mathbb{Z}$ or $a^{-1} > b^i$ for all $i \in \mathbb{Z}$. Notice that the relation $\gg$ is transitive.

Notation. The commutator $a^{-1}b^{-1}ab$ of elements a and b of a group Γ is denoted $[a, b]$. We use e to denote the identity element of a group. It is straightforward to check that a commutes with b if and only if $[a, b] = e$.

Lemma 3.1 [G, Lemmas 2.2.2(i) and 2.2.4(iii), pp. 19 and 20]. *Let a and b be elements of a group G. If $[a, b]$ commutes with both a and b, then $[b^n, a^m] = [a, b]^{-mn}$ for all $m, n \in \mathbb{Z}$.*

Lemma 3.2 (Ault [A, Lemma 1]). *Let a, b, c be nonidentity elements of a right-ordered group H, with $[a, b] = c^k$ for some nonzero $k \in \mathbb{Z}$ and $[a, c] = [b, c] = e$. Then either $c \ll a$ or $c \ll b$.*

Proof. Replacing some of a, b, c by their inverses if necessary and perhaps interchanging a and b, we may assume $a, b, c^k \geq e$. Replacing c with c^k, we may assume $k = 1$. We may assume $c \not\ll a$, so $a < c^i$ for some $i \in \mathbb{Z}^+$. Replacing b and c by b^i and c^i, we may assume $a < c$. Then e is less than each of ca^{-1}, b, and ca, so, for all $r \in \mathbb{Z}^+$, we have

$$e < (ca^{-1})^{3r}b^3(ca)^{3r} = c^{3r}a^{-3r}b^3c^{3r}a^{3r} = b^3[b^3, a^{3r}]c^{6r} = b^3c^{-9r}c^{6r} = (bc^{-r})^3.$$

Therefore, $bc^{-r} > e$, so $b > c^r$ for all $r \in \mathbb{Z}^+$. Therefore, $b \gg c$. □

Proposition 3.3. *For $n \geq 3$, no subgroup of finite index in $\mathrm{SL}_n(\mathbb{Z})$ is right orderable.*

Proof. Because subgroups of right-orderable groups are right orderable, we may assume $n = 3$. Let Γ be a subgroup of finite index in $\mathrm{SL}_3(\mathbb{Z})$, and suppose, for a contradiction, that Γ is right orderable.

Because Γ has finite index, there is some $k \in \mathbb{Z}^+$ such that the six matrices

$$a_1 = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix},$$

$$a_4 = \begin{pmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{pmatrix}, \quad a_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{pmatrix}$$

all belong to Γ. A straightforward check shows that $[a_i, a_{i+1}] = e$ and $[a_{i-1}, a_{i+1}] = (a_i)^{\pm k}$ for $i = 1, \dots, 6$ (with subscripts read modulo 6). Thus, Lemma 3.2 asserts either $a_i \ll a_{i-1}$ or $a_i \ll a_{i+1}$. In particular, we must have either $a_1 \ll a_6$ or $a_1 \ll a_2$. Assume for definiteness that $a_1 \ll a_2$. For each i, Lemma 3.2 implies that if $a_{i-1} \ll a_i$, then $a_i \ll a_{i+1}$. Since $a_1 \ll a_2$, we conclude by induction that

$$a_1 \ll a_2 \ll a_3 \ll a_4 \ll a_5 \ll a_6 \ll a_1.$$

Thus, $a_1 \ll a_1$, a contradiction. □

4. Proof of the Main Theorem

Our proof of Main Theorem 4.3 is based on an important theorem proved independently by Ault [A] and Rhemtulla [Rh]. This result provides an understanding of all the right orderings of nilpotent groups. The information we need is stated in Theorem 4.1 below.

Definition. Let Γ be a right-ordered group, let $x \in \Gamma$, and let Y be a subgroup of Γ. We say x is an *upper bound* for Y (denoted $x > Y$) if $x \geq y$ for all $y \in Y$. (If $x > Y$ and $x = y$ for some $y \in Y$, then $x = e$ and $Y = \{e\}$. It is convenient to allow this trivial case.)

Definition. Let s be an element of a right-ordered group Γ. We define the absolute value $|s|$ of s to be s if $s \geq e$ and to be s^{-1} if $s < e$.

Definition. A subgroup C of a right-ordered group Γ is *convex* if, for all $a, b, c \in \Gamma$ with $a < b < c$ and $a, c \in C$, we have $b \in C$. Equivalently, C is convex if, for all $a \in \Gamma \setminus C$, we have $|a| > C$, where $\Gamma \setminus C$ denotes the set-theoretic difference of Γ and C.

Theorem 4.1 (Ault, Rhemtulla). *Let S be a nonempty, finite generating set for a right-ordered, nilpotent group G. Then there is some $s \in S$ with $|s| > [G, G]$.*

Proof. It is well known (and easy to see) that the class of convex subgroups is linearly ordered by inclusion and closed under arbitrary unions. Since G is finitely generated, Zorn's Lemma implies that G has a maximal convex subgroup C (except in the trivial case $G = \{e\}$). Since S generates G, but C is a proper subgroup of G, there is some $s \in S$ with $s \notin C$. The convexity of C implies $|s| > C$, so it suffices to show $[G, G] \subset C$.

The maximality of C implies that no convex subgroup of G lies strictly between C and G, which is exactly what it means to say that the inclusion $C \subset G$ is a "convex jump." Therefore, the fundamental theorem of Ault [A] and Rhemtulla [Rh] (or see [MR, Theorem 7.5.1, p. 141]) asserts that C is a normal subgroup of G and that G/C is abelian; in other words, we have $[G, G] \subset C$, as desired. □

For technical reasons, we will use the following strengthened form of the theorem.

Corollary 4.2. *Let S be a nonempty, finite subset of a right-ordered, nilpotent group G, and assume $\langle S, Z(G)\rangle = G$. Then there is some $s \in S$ such that, for all nonzero $k \in \mathbb{Z}$ and all $z \in Z(G)$, we have $|s^k z| > [G, G]$.*

Proof. We argue by contradiction. Suppose for each $s \in S$ that there is some nonzero $k_s \in \mathbb{Z}$ and some $z_s \in Z(G)$ such that $|s^{k_s} z_s| \not> [G, G]$. The theorem asserts that, for some $t \in S$, the element $|t^{k_t} z_t|$ is an upper bound for the commutator subgroup of $\langle s^{k_s} z_s \mid s \in S\rangle$. Because each z_s belongs to $Z(G)$, the groups $\langle s^{k_s} z_s \mid s \in S\rangle$ and $\langle s^{k_s} \mid s \in S\rangle$ have the same commutator subgroup.

Because $\langle S\rangle$ is a finitely generated, nilpotent group, the commutator subgroup of $\langle s^{k_s} \mid s \in S\rangle$ is of finite index in $[\langle S\rangle, \langle S\rangle] = [G, G]$. Therefore, $|t^{k_t} z_t| > [G, G]$, which contradicts the choice of k_t and z_t. $\square$

Main Theorem 4.3. *If Γ is an arithmetic subgroup of a $\mathbb{Q}$-simple algebraic $\mathbb{Q}$-group G, with $\mathbb{Q}$-rank$(G) \geq 2$, then no central extension of Γ is right orderable.*

Proof. From the structure theory of reductive algebraic groups, we know that G contains a $\mathbb{Q}$-split simple subgroup whose root system is the reduced subsystem of the $\mathbb{Q}$-root system of G (see [BT, Theorem 7.2, p. 117]). Replacing G with this $\mathbb{Q}$-subgroup, we may assume G is $\mathbb{Q}$-split (and the root system of G is reduced). Because G is simple and $\mathbb{Q}$-rank$(G) \geq 2$, we know that the $\mathbb{Q}$-root system of G is irreducible and has rank at least two. Therefore, the $\mathbb{Q}$-root system of G contains an irreducible subsystem of rank two, that is, a root system of type A_2, B_2, or G_2. By inspection, we see that the long roots of G_2 form a subsystem of type A_2, so we conclude that the system of $\mathbb{Q}$-roots of G contains an irreducible subsystem Φ of type A_2 or B_2.

Let $\widetilde{\Gamma}$ be a central extension of Γ, and fix a central subgroup Z of $\widetilde{\Gamma}$ with $\widetilde{\Gamma}/Z \cong \Gamma$. For each $\alpha \in \Phi$, the root subgroup G_α is a 1-dimensional, unipotent $\mathbb{Q}$-subgroup of G, so the group $\Gamma_\alpha = \Gamma \cap G_\alpha$, being an arithmetic subgroup of G_α, is infinite cyclic. We let $\widetilde{\Gamma}_a$ be the inverse image of Γ_α in $\widetilde{\Gamma}$ under the quotient map $\widetilde{\Gamma} \to \Gamma$.

Suppose, for a contradiction, that $\widetilde{\Gamma}$ is right ordered.

Definition. For $\alpha, \beta \in \Phi$, we write $\alpha < \beta$ if $\exists x \in \widetilde{\Gamma}_\alpha \setminus Z$, $\forall y \in \widetilde{\Gamma}_\beta \setminus Z$, $x \ll y$.

Claim A. *If $\alpha, \beta \in \Phi$ with $\alpha+\beta \in \Phi$, then either $\alpha+\beta < \alpha$ or $\alpha+\beta < \beta$.* Let $[\alpha, \beta] = \Phi \cap \{m\alpha + n\beta \mid m, n \in \mathbb{Z}^+\}$, and let $G_{[\alpha,\beta]} = \langle G_\phi \mid \phi \in [\alpha, \beta]\rangle$. The subgroup generated by G_α and G_β is unipotent, and its commutator subgroup is $G_{[\alpha,\beta]}$ (see [BT, Proposition 2.5, p. 66] or [Ch, p. 27]). Similarly, one can show, by induction on the cardinality of $[\alpha, \beta]$, that the commutator subgroup of $\langle \Gamma_\alpha, \Gamma_\beta\rangle$ contains a finite-index subgroup of Γ_ϕ, for each $\phi \in [\alpha, \beta]$. In particular, the commutator subgroup $[\Gamma_\alpha, \Gamma_\beta]$ contains a finite-index subgroup of $\Gamma_{\alpha+\beta}$. Hence, some $x \in [\Gamma_\alpha, \Gamma_\beta]$ belongs to $\Gamma_{\alpha+\beta}\setminus\{e\}$. Lifting to $\widetilde{\Gamma}$, we see that some $\tilde{x} \in [\widetilde{\Gamma}_\alpha, \widetilde{\Gamma}_\beta]$ belongs to $\widetilde{\Gamma}_{\alpha+\beta} \setminus Z$. Let $y_\alpha Z$ and $y_\beta Z$ generate the cyclic groups $\widetilde{\Gamma}_\alpha/Z$ and $\widetilde{\Gamma}_\beta/Z$, respectively. Then $\langle y_\alpha, y_\beta, Z\rangle = \langle \widetilde{\Gamma}_\alpha, \widetilde{\Gamma}_\beta\rangle$, so Corollary 4.2 implies there is some $s \in \{y_\alpha, y_\beta\}$ such that, for all nonzero $k \in \mathbb{Z}$ and all $z \in Z$, we have $|s^k z| > [\widetilde{\Gamma}_\alpha, \widetilde{\Gamma}_\beta]$. Assume without loss of generality that $s = y_\alpha$. Since $\tilde{x} \in [\widetilde{\Gamma}_\alpha, \widetilde{\Gamma}_\beta]$, we must have $y_\alpha^k z \gg \tilde{x}$ for all nonzero $k \in \mathbb{Z}$ and all $z \in Z$. Now $y_\alpha Z$ generates $\widetilde{\Gamma}_\alpha/Z$, so each $y \in \Gamma_\alpha \setminus Z$ is of the form $y = y_\alpha^k z$ for some nonzero $k \in \mathbb{Z}$ and some $z \in Z$. Since $|y| = |y_\alpha^k z| > [\widetilde{\Gamma}_\alpha, \widetilde{\Gamma}_\beta]$ and $\tilde{x} \in [\widetilde{\Gamma}_\alpha, \widetilde{\Gamma}_\beta]$, we have $y \gg \tilde{x}$, so $\alpha > \alpha + \beta$, as desired.

Case 1. *Φ is of type A_2.* By labelling the six elements of Φ as shown in Figure 1, we can write the elements of Φ in a sequence $\alpha_1, \dots, \alpha_6$ in such a way that $\alpha_{i-1} + \alpha_{i+1} = \alpha_i$ for $i = 1, 2, \dots, 6$. From Claim A, we have either $\alpha_1 < \alpha_6$ or $\alpha_1 < \alpha_2$. Assume without loss of generality that $\alpha_1 < \alpha_2$. Since

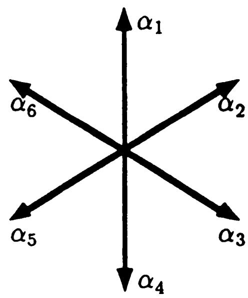

FIGURE 1. The root system A_2

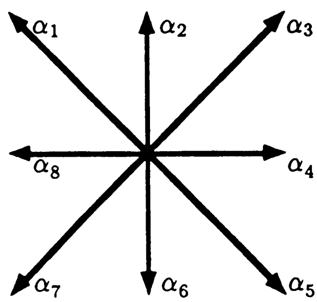

FIGURE 2. The root system B_2

$\alpha_2 \not< \alpha_1$, we conclude from Claim A that $\alpha_2 < \alpha_3$. Continuing in this fashion, we see that $\alpha_1 < \alpha_2 < \cdots < \alpha_6 < \alpha_1$. Hence $\alpha_1 < \alpha_1$, a contradiction.

Case 2. Φ *is of type* B_2. Label the eight elements of Φ as shown in Figure 2.

Claim B. *Either* $\alpha_1 < \alpha_2 < \alpha_3$ *or* $\alpha_1 < \alpha_8 < \alpha_7$. From Claim A, we know that either $\alpha_1 < \alpha_2$ or $\alpha_1 < \alpha_8$. Assume without loss of generality that $\alpha_1 < \alpha_2$. From Claim A, we see that $\alpha_2 < \alpha_8$ or $\alpha_2 < \alpha_3$. If $\alpha_2 < \alpha_3$, then we are done, so assume $\alpha_2 < \alpha_8$. Then $\alpha_8 \not< \alpha_2$, so we conclude from Claim A that $\alpha_8 < \alpha_7$. Because $\alpha_1 < \alpha_2 < \alpha_8$, this implies $\alpha_1 < \alpha_8 < \alpha_7$, as desired.

Now, from Claim B, we may assume without loss of generality that $\alpha_1 < \alpha_2 < \alpha_3$. From Claim B (and symmetry), we know that either $\alpha_3 < \alpha_2 < \alpha_1$ or $\alpha_3 < \alpha_4 < \alpha_5$. Clearly, we must have $\alpha_3 < \alpha_4 < \alpha_5$. Then, by a similar argument, it follows from Claim B (and symmetry) that $\alpha_5 < \alpha_6 < \alpha_7$. Therefore, it follows from Claim B (and symmetry) that $\alpha_7 < \alpha_8 < \alpha_1$. By transitivity, we have $\alpha_1 < \alpha_1$, a contradiction. $\square$

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