PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 122, Number 2, October 1994

ON THE DISTANCE OF THE RIEMANN-LIOUVILLE OPERATOR FROM COMPACT OPERATORS

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. We consider generalized Hardy operators

$$Tf(x) = \int_a^x \varphi(x, y) f(y)\, dy, \qquad x \in (a, b) \subset \mathbb{R},$$

acting between two weighted Lebesgue spaces $X = L^p(a, b; v)$ and $Y = L^q(a, b; w)$, $1 < p \le q < \infty$, and present lower and upper bounds on the distance of T from the space of all compact linear operators P, $P: X \to Y$. The conditions on the kernel $\varphi(x, y)$ are patterned in such a way that the above mentioned class of operators T contains the Riemann-Liouville fractional operators of orders equal to or greater than one.

1. INTRODUCTION

Let $-\infty \le a < b \le \infty$. By $\mathscr{W}(a, b)$ we denote the set of all weights, i.e., the set of all functions measurable, positive, and finite almost everywhere on (a, b). For $p \in (1, \infty)$ and $v \in \mathscr{W}(a, b)$ the weighted Lebesgue space $L^p(a, b; v)$ is defined as the set of all measurable functions on (a, b) with a finite norm

$$\|u\|_{p,(a,b),v} = \left(\int_a^b |u(x)|^p v(x)\, dx\right)^{1/p}.$$

(If $v \equiv 1$, we write simply $L^p(a, b)$ and $\|\cdot\|_{p,(a,b)}$ instead of $L^p(a, b; v)$ and $\|\cdot\|_{p,(a,b),v}$, respectively.)

Let $1 < p \le q < \infty$, v, $w \in \mathscr{W}(a, b)$. Several authors (cf. [6, 8, 2]) have established necessary and sufficient conditions for the inequality

$$\|Tf\|_{q,(a,b),w} \le C\|f\|_{p,(a,b),v} \tag{1.1}$$

to hold on $L^p(a, b; v)$, where the operator T is given by

$$Tf(x) = \int_a^x \varphi(x, y) f(y)\, dy \tag{1.2}$$

with a kernel φ satisfying various assumptions. Here we consider the kernel φ

Received by the editors January 12, 1993.
1991 *Mathematics Subject Classification*. Primary 47B38, 26A33; Secondary 26D10.

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0002-9939/94 \$1.00 + \$.25 per page

with the following properties (cf. [2]):

$$
(1.3)\qquad
\begin{aligned}
&\text{(i)} && \varphi\colon \{(x, y) \in \mathbb{R}^2 \colon x, y \in (a, b),\ x > y\} \to (0, \infty);\\
&\text{(ii)} && \varphi(x, y) \text{ is nondecreasing in } x \text{ and nonincreasing in } y;\\
&\text{(iii)} && \text{there exists } D > 0 \text{ such that}\\
& && \varphi(x, y) \le D[\varphi(x, z) + \varphi(z, y)] \text{ if } y < z < x.
\end{aligned}
$$

The Riemann-Liouville operator T_σ, given by

$$
(1.4)\qquad T_\sigma f(x) = \int_0^x (x-y)^{\sigma-1} f(y)\, dy, \qquad x > 0, \sigma \ge 1,
$$

is an example of an operator T from (1.2) whose kernel satisfies (1.3)(i)–(1.3)(iii). Note that in all papers mentioned above it is assumed that $a = 0$, $b = \infty$, and the kernel φ is such that conditions (1.3)(i)–(1.3)(iii) are fulfilled.

Let $p' = p/(p-1)$, $q' = q/(q-1)$, $a \le \alpha < \beta \le b$, $t \in (\alpha, \beta)$, $i \in \{0, 1\}$. Set

(1.5)

$$
B^{(i)}_{(\alpha,\beta)}(t) = \left(\int_t^\beta \varphi(x, t)^{qi} w(x)\, dx\right)^{1/q} \left(\int_\alpha^t \varphi(t, x)^{p'(1-i)} v(x)^{1-p'}\, dx\right)^{1/p'},
$$

$$
B^{(i)}_{(\alpha,\beta)} = \sup_{\alpha<t<\beta} B^{(i)}_{(\alpha,\beta)}(t), \qquad B_{(\alpha,\beta)} = \max_{i=0,1} B^{(i)}_{(\alpha,\beta)}.
$$

Applying the method of [6] one can prove

Theorem 1.1. *Let* $1 < p \le q < \infty$, *and let the function* φ *satisfy* (1.3)(i)–(1.3)(iii). *Then there exists a positive constant* C *such that inequality* (1.1) *holds for all* $f \in L^p(a, b; v)$ *if, and only if,*

$$
(1.6)\qquad B_{(a,b)} < \infty.
$$

Moreover, if C *is the least constant for which* (1.1) *holds on* $L^p(a, b; v)$, *then*

$$
(1.7)\qquad B_{(a,b)} \le C \le k(q, p, D) B_{(a,b)},
$$

where

$$
(1.8)\qquad k(q, p, D) = 2^{1/q'}(D+1)^2\left[D^q\left(1 + \frac{q}{p'}\right)\left(1 + \frac{p'}{q}\right)^{q/p'} + 1\right]^{1/q}
$$

with D *from* (1.3)(iii).

In other words, Theorem 1.1 provides a necessary and sufficient condition for the boundedness of T from $X := L^p(a, b; v)$ into $Y := L^q(a, b; w)$, $1 < p \le q < \infty$. The aim of the paper is to establish lower and upper bounds on the distance of T from the space of all compact linear operators from X into Y provided $T\colon X \to Y$ is bounded and the function φ, in addition to conditions (1.3)(i)–(1.3)(iii), satisfies

(1.3)(iv) For any $\gamma \in (a, b)$ the function φ is uniformly continuous on the set $\{(x, y) \in \mathbb{R}^2 \colon x, y \in (a, b),\ y < x < \gamma\}$.

(Note that for the Riemann-Liouville operator T_σ from (1.4) condition (1.3)(iv) is again fulfilled.) In particular, from such bounds one easily obtains a characterization of those operators T that are compact. The main result of the paper is Theorem 3.1. Our work is patterned on [10, 5, 4], where operators $P: L^p(a, b) \to L^q(a, b)$, given by

$$Pf(x) = v(x) \int_a^x v(y) f(y)\, dy, \qquad x \in (a, b),$$

with v, $w \in \mathscr{W}(a, b)$, are considered. We also use some techniques drawn from [9, 3].

2. Preliminaries

Let X, Y be Banach spaces. The symbol $\mathscr{B}(X, Y)$ is used to denote the space of all bounded linear maps from X into Y. The subset of $\mathscr{B}(X, Y)$ consisting of all compact maps or maps having a finite-dimensional range is denoted by $\mathscr{K}(X, Y)$ or $\mathscr{F}_r(X, Y)$, respectively. Evidently, for any $P \in \mathscr{B}(X, Y)$,

$$\operatorname{dist}(P, \mathscr{K}(X, Y)) \le \operatorname{dist}(P, \mathscr{F}_r(X, Y)).$$

It is easy to see that the map

$$\Phi: L^q(a, b) \to L^q(a, b; w) \qquad (1 \le q < \infty,\ w \in \mathscr{W}(a, b)) \tag{2.1}$$

given by $\Phi(u) = uw^{-1/q}$ is an isometric isomorphism. Using this map and Corollaries V.5.2, V.5.3 from [3] we see that the following analogue of Corollary V.5.4 of [3] is true.

Lemma 2.1. *Let $P \in \mathscr{B}(X, Y)$, where $Y = L^q(a, b; w)$, $1 \le q < \infty$, $w \in \mathscr{W}(a, b)$. Then $\operatorname{dist}(P, \mathscr{K}(X, Y)) = \operatorname{dist}(P, \mathscr{F}_r(X, Y))$.*

Similarly, one can verify that the following analogue of Lemma V.5.6 from [3] holds.

Lemma 2.2. *Set $Y = L^q(a, b; w)$, where $1 \le q < \infty$, $w \in \mathscr{W}(a, b)$. Let $P \in \mathscr{F}_r(X, Y)$ and $\epsilon > 0$. Then there exist $R \in \mathscr{F}_r(X, Y)$ and $[\bar{a}, \bar{b}] \subset (a, b)$ such that $\|P - R\| < \epsilon$ and $\operatorname{supp} Rf \subset [\bar{a}, \bar{b}]$ for all $f \in X$.*

Further, we shall need

Proposition 2.3 (see [1, Theorem 2.21]). *Let $1 \le q < \infty$. A bounded set $S \subset L^q(a, b)$ is precompact in $L^q(a, b)$ if, and only if, for every $\epsilon > 0$ there exist a number $\delta > 0$ and an interval $G = [\bar{a}, \bar{b}] \subset (a, b)$ such that for every $u \in S$ and every $h \in \mathbb{R}$ with $|h| < \delta$ we have*

$$\int_a^b |u(x + h) - u(x)|^q\, dx \le \epsilon^q \tag{2.2}$$

and

$$\int_{(a, b)\setminus G} |u(x)|^q\, dx \le \epsilon^q. \tag{2.3}$$

(We define $u(x) = 0$ for $x \notin (a, b)$).

3. Results

First, let us formulate the main result of the paper.

Theorem 3.1. *Suppose that $1 < p \le q < \infty$, $-\infty \le a < b \le \infty$, v, $w \in \mathscr{W}(a, b)$, $X = L^p(a, b; v)$, $Y = L^q(a, b; w)$. Let T be the operator* (1.2) *with the kernel* φ *satisfying* (1.3)(i)–(1.3)(iv). *Moreover, let* $B_{(a,b)} < \infty$. *Then*

$$\tfrac{1}{2} J \le \operatorname{dist}(T, \mathscr{K}(X, Y)) \le k(q, p, D) J, \tag{3.1}$$

where

$$J = \lim_{\alpha \to a_+} B_{(a,\alpha)} + \lim_{\beta \to b_-} B_{(\beta, b)} \tag{3.2}$$

and $k(q, p, D)$ *is given by* (1.8).

Corollary 3.2. *Let all assumptions of Theorem* 3.1 *be fulfilled. Then*

$$T \in \mathscr{K}(X, Y) \iff J = 0.$$

Theorem 3.1 is a consequence of Lemmas 3.4 and 3.5.

From this point on we shall assume that symbols a, b, p, q, T, X, and Y have the same meaning as in Theorem 3.1. Further, we shall need a certain decomposition of the operator T.

For any $\alpha, \beta \in (a, b)$, $\alpha < \beta$, the operator T can be written as

$$T = T_1 + T_2 + T_3, \tag{3.3}$$

where

$$T_1 f(x) = \chi_{(a,\alpha]}(x) \int_a^x \varphi(x, y) f(y) \chi_{(a,\alpha]}(y)\, dy, \tag{3.4}$$

$$T_2 f(x) = \chi_{(\alpha,b)}(x) \int_a^x \varphi(x, y) f(y) \chi_{(a,\beta]}(y)\, dy, \tag{3.5}$$

$$T_3 f(x) = \chi_{[\beta,b)}(x) \int_a^x \varphi(x, y) f(y) \chi_{[\beta,b)}(y)\, dy \tag{3.6}$$

for $x \in (a, b)$; here χ_M is the characteristic function of the set M.

We have

Lemma 3.3. *Suppose that the function* φ *satisfies* (1.3)(i)–(1.3)(iv). *Moreover, let* $B_{(a,b)} < \infty$. *Then* $T_2 \in \mathscr{K}(X, Y)$.

Proof. The proof can be done by means of Proposition 2.3 (cf. the proof of Theorem 7.3 in [7] or the proof of Theorem 3 of [9]). Let us point out that assumption (1.3)(iv) is essential for this proof.

Now it is easy to establish an upper bound on $\operatorname{dist}(T, \mathscr{K}(X, Y))$.

Lemma 3.4. *Suppose that the function* φ *satisfies* (1.3)(i)–(1.3)(iv). *Let* $B_{(a,b)} < \infty$. *Then*

$$\operatorname{dist}(T, \mathscr{K}(X, Y)) \le k(q, p, D) J. \tag{3.7}$$

Proof. Let T_2 be the operator from (3.5). According to Lemma 3.3, $T_2 \in \mathscr{K}(X, Y)$. Thus, using (3.3), we have

$$\begin{aligned} \operatorname{dist}(T, \mathscr{K}(X, Y)) &= \inf\{\|T - P\| : P \in \mathscr{K}(X, Y)\} \\ &\le \|T_1 + T_3\| \le \|T_1\| + \|T_3\| \end{aligned}$$

with T_1, T_3 from (3.4), (3.6). However, each of the operators T_1, T_3 defines a bounded operator of the type given by (1.2). Therefore, by Theorem 1.1,

$$\|T_1\| \le k(q, p, D) B_{(a,\alpha)}, \qquad \|T_3\| \le k(q, p, D) B_{(\beta,b)}.$$

Consequently, for any α, $\beta \in (a, b)$ with $\alpha < \beta$,

$$\operatorname{dist}(T, \mathscr{K}(X, Y)) \le k(q, p, D)\,[B_{(a,\alpha)} + B_{(\beta,b)}],$$

and (3.7) follows.

The following lemma provides a lower bound on $\operatorname{dist}(T, \mathscr{K}(X, Y))$.

Lemma 3.5. *Assume that the function* φ *satisfies* (1.3)(i)–(1.3)(iii). *Moreover, let* $B_{(a,b)} < \infty$. *Then*

$$\tfrac{1}{2} J \le \operatorname{dist}(T, \mathscr{K}(X, Y)). \tag{3.8}$$

Proof. Let $\lambda > \operatorname{dist}(T, \mathscr{K}(X, Y))$. By Lemma 2.1 there exist $P \in \mathscr{F}_r(X, Y)$ such that

$$\|T - P\| < \lambda. \tag{3.9}$$

Set $\epsilon = (\lambda - \|T - P\|)/2$. Then, according to Lemma 2.2, there exist $R \in \mathscr{F}_r(X, Y)$ and $[\bar{a}, \bar{b}] \subset (a, b)$ such that

$$\|P - R\| < \epsilon \tag{3.10}$$

and

$$\operatorname{supp} Rf \subset [\bar{a}, \bar{b}] \quad \text{for every } f \in X. \tag{3.11}$$

Inequalities (3.9) and (3.10) yield $\|T - R\| < \lambda$, and consequently,

$$\|Tf - Rf\|_Y \le \lambda \|f\|_X \quad \text{for all } f \in X.$$

This together with (3.11) implies that for all $f \in X$

$$\int_a^{\bar{a}} |Tf(x)|^q w(x)\, dx + \int_{\bar{b}}^b |Tf(x)|^q w(x)\, dx \le \lambda^q \|f\|_X^q. \tag{3.12}$$

I. Take $\beta \in [\bar{b}, b)$, $r \in (\beta, b)$, $f \in X$, $f \ge 0$, and $i \in \{0, 1\}$. Then

$$\begin{aligned}\int_r^b |Tf(x)|^q w(x)\, dx &= \int_r^b \left(\int_a^x \varphi(x, y) f(y)\, dy \right)^q w(x)\, dx \\ &\ge \int_r^b \left(\int_\beta^r \varphi(x, y) f(y)\, dy \right)^q w(x)\, dx.\end{aligned}$$

By (1.3)(ii) we have

$$\varphi(x, y) \ge \varphi(r, y)^{1-i} \varphi(x, r)^i$$

for $x > r > y$ and hence

$$\int_r^b |Tf(x)|^q w(x)\, dx \ge \left(\int_r^b \varphi(x, r)^{qi} w(x)\, dx \right) \left(\int_\beta^r \varphi(r, y)^{1-i} f(y)\, dy \right)^q. \tag{3.13}$$

Now set

$$f(y) = \varphi(r, y)^\mu v(y)^{1-p'} \chi_{(\beta, r)}(y), \qquad y \in (a, b),$$

where

$$\mu = (1 - i)/(p - 1). \tag{3.14}$$

Then

$$\|f\|_X^q = \left(\int_\beta^r \varphi(r, y)^{p'(1-i)} v(y)^{1-p'}\, dy \right)^{q/p} \tag{3.15}$$

and, by (3.13),

$$\tag{3.16} \begin{aligned} &\int_r^b |Tf(x)|^q w(x)\,dx \\ &\quad \geq \left(\int_r^b \varphi(x\,,\,r)^{qi} w(x)\,dx\right)\left(\int_\beta^r \varphi(r\,,\,y)^{p'(1-i)} v(y)^{1-p'}\,dy\right)^q. \end{aligned}$$

Thus from (3.12), (3.15), and (3.16) we obtain

$$\lambda \geq \left(\int_r^b \varphi(x\,,\,r)^{qi} w(x)\,dx\right)^{1/q}\left(\int_\beta^r \varphi(r\,,\,y)^{p'(1-i)} v(y)^{1-p'}\,dy\right)^{1/p'} = B^{(i)}_{(\beta\,,\,b)}(r)$$

for any $r \in (\beta\,,\,b)$, for all $\beta \in [\bar{b}\,,\,b)$, and for $i \in \{0\,,\,1\}$. This shows that $B_{(\beta\,,\,b)} \leq \lambda$ for all $\beta \in [\bar{b}\,,\,b)$. Since λ may be chosen arbitrarily close to $\operatorname{dist}(T\,,\mathscr{K}(X\,,\,Y))$, we obtain that $B_{(\beta\,,\,b)} \leq \operatorname{dist}(T\,,\mathscr{K}(X\,,\,Y))$ for every $\beta \in [\bar{b}\,,\,b)$, and consequently,

$$\tag{3.17} \lim_{\beta\to b_-} B_{(\beta\,,\,b)} \leq \operatorname{dist}\left(T\,,\mathscr{K}(X\,,\,Y)\right).$$

II. Take $\alpha \in (a\,,\,\bar{a})$, $r \in (a\,,\,\alpha)$, $f \in X$, $f \geq 0$, and $i \in \{0\,,\,1\}$. Then, using (1.3)(ii), we obtain (cf. (3.13))

$$\int_r^\alpha |Tf(x)|^q w(x)\,dx \geq \left(\int_r^\alpha \varphi(x\,,\,r)^{qi} w(x)\,dx\right)\left(\int_a^r \varphi(r\,,\,y)^{1-i} f(y)\,dy\right)^q,$$

and, taking

$$f(y) = \varphi(r\,,\,y)^\mu v(y)^{1-p'}\chi_{(a\,,\,r)}(y)\,, \qquad y \in (a\,,\,b)\,,$$

with μ from (3.14), we get

$$\tag{3.18} \begin{aligned} &\int_r^\alpha |Tf(x)|^q w(x)\,dx \\ &\quad \geq \left(\int_r^\alpha \varphi(x\,,\,r)^{qi} w(x)dx\right)\left(\int_a^r \varphi(r\,,\,y)^{p'(1-i)} v(y)^{1-p'}\,dy\right)^q; \end{aligned}$$

moreover,

$$\tag{3.19} \|f\|_X^q = \left(\int_a^r \varphi(r\,,\,y)^{p'(1-i)} v(y)^{1-p'}\,dy\right)^{q/p}.$$

Now, using (3.12), (3.18) and (3.19), we obtain (similarly as in part I)

$$\lim_{\alpha\to a_+} B_{(a\,,\,\alpha)} \leq \operatorname{dist}(T\,,\mathscr{K}(X\,,\,Y)).$$

Together with (3.17) this yields (3.8), and the proof is complete.

Remark 3.6. If we suppose that

$$\tag{3.20} \lim_{r\to a_+} B^{(i)}_{(a\,,\,b)}(r) = 0 = \lim_{r\to b_-} B^{(i)}_{(a\,,\,b)}(r)\,,$$

then it is easy to see that $J = 0$. Thus, the assumptions of Theorem 3.1 and (3.20) imply that $T \in \mathscr{K}(X\,,\,Y)$.

On the other hand, if, for example, the function φ satisfying (1.3)(i)–(1.3)(iii) is bounded, then one can prove the implication $J = 0 \implies$ (3.20) (cf. the proof of Corollary (ii) from [5]).

Moreover, let us note that, using the arguments from [9], it is possible to prove the implication $T \in \mathscr{K}(X\,,\,Y) \implies$ (3.20) provided the function φ satisfies (1.3)(i)–(1.3)(iii).

Remark 3.6 and Corollary 3.2 imply

Corollary 3.7. *Let all assumptions of Theorem* 3.1 *be fulfilled. Then* $T \in \mathscr{K}(X, Y)$ *if, and only if,* (3.20) *is true.*

The statement of Corollary 3.7 was proved by V. D. Stepanov provided $\varphi(x, y) = k(x - y)$ with a suitable function k (cf. [8]).

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