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COALGEBRAS OVER THE HIGHER RANK SYMPLECTIC GROUPS

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ABSTRACT. In these notes we generalize the construction obtained for the deformation of the symplectic group $\mathrm{Sp}(2)$ to the case of any N even: $N = 2k$. We characterize the bialgebras $A_q(\mathrm{Sp}(k))$ by generators and relations. We consider the deformation of the algebra of polynomials on the group $\mathrm{Sp}(k)$: $\mathrm{Pol}(\mathrm{Sp}_q(k))$ is a Hopf *-algebra and we build *-representations of it by means of a Verma module construction.

1. INTRODUCTION

Quantum groups from the algebraic point of view as the spectrum of a Hopf algebra were first introduced by Drinfeld in 1986 [1]. Another different approach is by Jimbo [2] where the quantum groups are considered as the deformations of the universal enveloping algebra of a semisimple Lie algebra. In [3, 12] the term quantum group was introduced by Woronowicz from a different point of view: a quantum group is to be considered as a deformation of the C^*-algebra of the continuous functions on a compact group. In these notes we will use Woronowicz's approach to quantum groups. Theorem 3.1 gives an explicit characterization of the quantum group $\mathrm{Sp}_q(k)$ and it provides the justification of why it is a twisted symplectic group. We consider $\mathrm{Pol}(\mathrm{Sp}_q(k))$ as a deformation of the algebra of polynomials on $\mathrm{Sp}(k)$. It has a structure of a Hopf*-algebra.

We prove a Poincaré-Birkoff-Witt theorem for the bialgebra $A_q(\mathrm{Sp}(k))$ and we construct a basis of $\mathrm{Pol}(\mathrm{Sp}_q(k))$. In §3 we construct irreducible *-representations by means of a Verma module construction.

2. NOTATION AND DEFINITIONS

Let us state the definitions of bialgebra, Hopf algebra, and Hopf *-algebra. Let A be an algebra over $\mathbb{C}$ with unit I; we denote the multiplication by

$$m: A \otimes A \to A, \qquad a \otimes b \to ab,$$

and the unit by $\varepsilon: \mathbb{C} \to A$, $z \to zI$. A is said to be a bialgebra if there exists unital algebra homomorphisms

$$\varphi: A \to A \otimes A, \qquad e: A \to \mathbb{C},$$

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so that

$$(2.1) \qquad \begin{aligned}(\varphi \otimes \mathrm{id})\varphi &= (\mathrm{id} \otimes \varphi)\varphi\,,\\ (e \otimes \mathrm{id})\varphi &= \mathrm{id} = (\mathrm{id} \otimes e)\varphi.\end{aligned}$$

φ is called the comultiplication and e the counit. Property (2.1) is called the coassociativity. The bialgebra A is a Hopf algebra if there exists a linear map $K\colon A \to A$ called the antipode such that

$$(m \circ (K \otimes \mathrm{id}) \circ \varphi)(a) = (\varepsilon \circ e)(a)I = (m \circ (\mathrm{id} \otimes K) \circ \varphi)(a)$$

for every $a \in A$. We say that A is a Hopf*-algebra if there exists an involutive antilinear mapping $*\colon A \to A$, $a \to a^*$, such that the comultiplication and the counit are *-homomorphisms and $K \circ *$ is involutive, i.e., $K \circ * \circ K \circ * = \mathrm{id}$.

3. The bialgebra $A_q(\mathrm{Sp}(k))$ and the Hopf*-algebra $\mathrm{Pol}(\mathrm{Sp}_q(k))$

In this section we consider the bialgebra $A_q(\mathrm{Sp}(k))$ and the Hopf*-algebra $\mathrm{Pol}(\mathrm{Sp}_q(k))$. We prove a theorem that gives a characterization of $A_q(\mathrm{Sp}(k))$ by generators and relations. The bialgebra $A_q(\mathrm{Sp}(k))$ may also be viewed as a specialization of the algebra of functions on the q-deformations of the symplectic group as defined in [11].

Fix $0 < q < 1$ and let $A_q(\mathrm{Sp}(k))$ be the unital algebra generated by the matrix elements of $u = (u_{ij})_{i,j=1,\dots,N}$, with $N = 2k$. There exist unique unital algebra homomorphisms

$$\varphi\colon A_q(\mathrm{Sp}(k)) \to A_q(\mathrm{Sp}(k)) \otimes A_q(\mathrm{Sp}(k))$$

and

$$e\colon A_q(\mathrm{Sp}(k)) \to \mathbb{C}$$

so that $e(u_{ij}) = \delta_{ij}$ and $\varphi(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$. Then $A_q(\mathrm{Sp}(k))$ is a bialgebra with comultiplication φ and counit e. We define the quantum determinant by

$$D = \sum_{\sigma \in S_N} (-q)^{l(\sigma)} u_{1,\sigma(1)} \cdots u_{N,\sigma(N)}\,,$$

where S_N is the permutation group of $\{1, \dots, N\}$, and $l(\sigma)$ is the length of the permutation. The quantum minor D^{ij} is defined to be the $(N-1)$ by $(N-1)$ quantum determinant of the matrix obtained from u by deleting the ith row and the jth column. Thus,

$$D^{ij} = \sum_{\sigma \in S_{N-1}} (-q)^{l(\sigma)} u_{1,\sigma(1)} \cdots u_{i-1,\sigma(i-1)} u_{i+1,\sigma(i+1)} \cdots u_{N,\sigma(N)}\,,$$

where, indicating by a hat the index to be omitted,

$$\sigma\colon \{1, \dots, \hat{\imath}, \dots, N\} \to \{1, \dots, \hat{\jmath}, \dots N\}.$$

By developing the quantum determinant along a row or a column we have the following equations:

$$(3.1) \qquad \delta_{ij} D = \sum_{k=1}^{N} (-q)^{k-j} u_{i,k} D^{jk}\,, \qquad \delta_{ij} D = \sum_{k=1}^{N} (-q)^{i-k} D^{ki} u_{k,j}\,,$$

$$(3.2) \qquad \delta_{ij} D = \sum_{k=1}^{N} (-q)^{j-k} D^{jk} u_{i,k}\,, \qquad \delta_{ij} D = \sum_{k=1}^{N} (-q)^{k-i} u_{k,j} D^{ki}.$$

It is known [11], that in the case of $A_q(\mathrm{GL}(N))$ the quantum determinant generates the center of the algebra. We see that $D \in \mathrm{centre}(A_q(\mathrm{Sp}(k)))$. The comultiplication and the counit act on D as follows:

$$\varphi(D) = D \otimes D, \qquad e(D) = 1.$$

Define the algebra $\mathrm{Pol}(\mathrm{Sp}_q(k))$ as the extension of $A_q(\mathrm{Sp}(k))$, with the element D^{-1} which satisfies

$$DD^{-1} = I = D^{-1}D.$$

The comultiplication φ and the counit e can be extended to $\mathrm{Pol}(\mathrm{Sp}_q(k))$ by putting

$$\varphi(D^{-1}) = D^{-1} \otimes D^{-1}, \qquad e(D^{-1}) = 1.$$

We define

$$K(u_{ij}) = (-q)^{i-j} D^{ji} D^{-1} \quad \text{and} \quad K(D^{-1}) = D,$$

K extends to $\mathrm{Pol}(\mathrm{Sp}_q(k))$ as a unital linear antimultiplicative mapping. Thus $\mathrm{Pol}(\mathrm{Sp}_q(k))$ is a Hopf-algebra with comultiplication φ, counit e, and antipode K.

We introduce a *-operation on $\mathrm{Pol}(\mathrm{Sp}_q(k))$ by letting

$$u_{ij}^* = K(u_{ji}) = (-q)^{j-i} D^{ij} D^{-1} \quad \text{and} \quad (D^{-1})^* = D.$$

Thus D is unitary,

$$D^*D = I = DD^*.$$

So, $\mathrm{Pol}(\mathrm{Sp}_q(k))$ is a Hopf *-algebra. By (3.1) and (3.2) we have

$$\sum_{k=1}^{N} u_{ik} u_{jk}^* = \delta_{ij} I = \sum_{k=1}^{N} u_{ki}^* u_{kj},$$

i.e., u is unitary.

We now characterize the Hopf *-algebra $\mathrm{Pol}(\mathrm{Sp}_q(k))$ following Woronowicz's definition of quantum group. Let $\{e_1, \dots, e_N\}$ be the standard basis in $\mathbb{C}^N$ over $\mathbb{C}$, where $N = 2k$. Let q be a real parameter $0 < q < 1$; we define the vectors

$$\xi_{(i)} = e_i \otimes e_{k+i} - q e_{k+i} \otimes e_i, \qquad i = 1, \dots, k.$$

Thus the ξ_i belong to the $\mathbb{C}$-vector space of q-alternating 2-tensors $\Lambda^2(\mathbb{C}^N)$, for $i = 1, \dots, k$.

Theorem 3.1. *Let $y' = (y_q \otimes I)$ be the vector belonging to the $\mathbb{C}$-vector space of q-alternating N-tensors $\Lambda^N(\mathbb{C}^N) \otimes \mathrm{Pol}(\mathrm{Sp}_q(k))$. Then for any N by N matrix,*

$$u^*u = I = uu^*,$$

with entries u_{ij} belonging to the algebra $\mathrm{Pol}(\mathrm{Sp}_q(k))$, the following two systems of conditions are equivalent:

(I) *The matrix $u = (u_{ij})_{i,j=1,\dots,N}$ satisfies the following conditions:*

(a) *quantum symplectic invariance condition*

$$(u \otimes I)(I \otimes u)x = x$$

where $x = \sum_{i=1}^{k} \xi_i \otimes \mathrm{id}_{\mathrm{Pol}(\mathrm{Sp}_q(k))}$;

(b) *quantum determinant condition*

$$((u\otimes I)(I\otimes u)\otimes\cdots\otimes(u\otimes I)(I\otimes u))y'=y'$$

where the tensor product is repeated N times.

(II) *u is of the form*

$$u=\begin{pmatrix} A & -qB^* \\ B & A^* \end{pmatrix}$$

where A, B are k by k matrices with entries in $\mathrm{Pol}(\mathrm{Sp}_q(k))$, *and the u_{ij} satisfy the following relations:*

(A) *Lie brackets*

$$[u_{ij}, u_{lm}]=(q-1/q)u_{lj}u_{im}, \quad \text{for } 1\le i, j\le k, k+1\le l, m\le N.$$

(B) *q-relations*

(a) $u_{ij}u_{il}=qu_{il}u_{ij}$, *for* $1\le i\le N$, $1\le j\le k$, $k+1\le l\le N$.

(b) *Set* $(z_{ij})=(u_{ij})^T$, *so that* $z_{ij}=u_{ji}$; *then*

$$z_{ij}z_{il}=qz_{il}z_{ij}, \quad \text{for } 1\le i\le N, 1\le j\le k, k+1\le l\le N.$$

(C) *Commutation relations*

$$\begin{aligned} u_{ij}u_{lm}&=u_{lm}u_{ij}, \quad \text{for } j\ge 1, l\le k, k+1\le i, m\le N,\\ u_{ii}u_{jj}&=u_{jj}u_{ii}, \quad \text{for } 1\le i, j\le k. \end{aligned}$$

(D) *Zero divisors*

(a)

$$u_{ij}u_{i+1,l}=0=u_{i+1,l}u_{ij}, \quad \text{for } i=1,\dots,N-1, i \text{ odd}, \\ 1\le j\le k, k+1\le l\le N,$$

$$u_{ij}u_{i-1,l}=0=u_{i-1,l}u_{ij}, \quad \text{for } i=1,\dots,N, i \text{ even}, \\ 1\le j\le k, k+1\le l\le N.$$

(b) *Set* $(z_{ij})=(u_{ji})^T$, *so that* $z_{ij}=u_{ji}$; *then*

$$\begin{aligned} z_{ij}z_{i+1,l}=0=z_{i+1,l}z_{ij}, &\quad \text{for } i=1,\dots,N-1, i \text{ odd},\\ &\quad 1\le j\le k, k+1\le l\le N,\\ z_{ij}z_{i-1,l}=0=z_{i-1,l}z_{ij}, &\quad \text{for } i=1,\dots,N, i \text{ even},\\ &\quad 1\le j\le k, k+1\le l\le N. \end{aligned}$$

(c)

$$\begin{aligned} u_{ij}u_{lm}&=0=u_{lm}u_{ij}, \quad \text{for } j\ge 1, l\ge k, 1\le i, m\le k,\\ u_{ij}u_{lm}&=0=u_{lm}u_{ij}, \quad \text{for } j\ge 1, l\ge k, k+1\le i, m\le N. \end{aligned}$$

Proof. (I) $\Rightarrow$ (II). We use the unitarity of the matrix u to simplify the equation (a). Hence we get

$$(I\otimes u)x=(u^*\otimes I)x.$$

By rewriting this equation in the matrix form using the basis of $\mathbb{C}^N\otimes\mathbb{C}^N$ composed of $\{e_i\otimes e_j\}$, where $\{e_i\}$ represents the standard basis of $\mathbb{C}^N$, we get the desired form of the matrix.

To prove that (b) gives all the relations between the u_{ij} in (II) we use induction on the dimension k. For $N=4$, i.e., $k=2$, it is true in view of Theorem 1 of [6]. Assume then that it is true for some $k \in \mathbb{Z}$. We prove it for $k+1$.

By using the form of the matrix u, we can write the following block decomposition obtained from blocks of lower dimensions for which the induction hypothesis applies:

$$\begin{pmatrix} A_0 & a_1 & B_0^* & b_1^* \\ a_2 & a_3 & b_2^* & b_3^* \\ B_0 & b_1 & A_0^* & a_1^* \\ b_2 & b_3 & a_2^* & a_3^* \end{pmatrix}.$$

Let us see how the induction hypothesis works to give all the relations. Observe that

$$a_i = \begin{pmatrix} a_i^1 \\ \vdots \\ a_k^i \end{pmatrix}, \qquad i = 1, 2,$$

which is a $k \times 1$ matrix, and similarly

$$b_i = \begin{pmatrix} b_i^1 \\ \vdots \\ b_k^i \end{pmatrix}, \qquad i = 1, 2.$$

The zero divisors are obtained by using the induction hypothesis on the 2×2 matrix whose blocks are

$$\begin{pmatrix} A_0 & a_1 \\ a_2 & a_3 \end{pmatrix}, \quad \begin{pmatrix} B_0 & b_1 \\ b_2 & b_3 \end{pmatrix}, \quad \begin{pmatrix} A_0^* & a_1^* \\ a_2^* & a_3^* \end{pmatrix}, \quad \begin{pmatrix} B_0^* & b_1^* \\ b_2^* & b_3^* \end{pmatrix}.$$

This implies $A_0 a_i = 0$, $i = 1, 2, 3$, where in this proof this stands for a product of $a_{rs} a_j^i$, for every j, r, s, $r \neq s$.

We define $a_i A_0 \equiv (A_0^* a_i)^*$, so that we also get $a_i A_0 = 0$. In a similar way we proceed to obtain $B_0 b_i = 0$, $i = 1, 2, 3$. For the relations $A_0 b_i^* = 0$, $i = 2, 3$, we proceed as above and obtain the relations by the induction hypothesis; similarly for $b_i^* A_0 = (A_0^* b_i)^* = 0$. Also,

$$A_0 b_i = 0 = b_i A_0, \qquad i = 1, 3.$$

By the induction hypothesis on the matrix of order $2k$ obtained by suitably deleting rows and columns from the original matrix of order $2k+2$, the q-relations are

$$A_0 B_0^* = q B_0^* A_0.$$

Similarly by suitably choosing $k \times k$ blocks in the original matrix we have

$$a_i b_i^* = q b_i^* a_i, \qquad i = 2, 3.$$
$$a_1 B_0^* = q B_0^* a_1.$$

We proceed in a similar way to verify $A_0 B_0 = q B_0 A_0$.

The commutation relations follow by the induction hypothesis in the case $B_0 B_0^* = B_0^* B_0$, and by the induction hypothesis and by suitably choosing subblocks for the relations $B_0 b_i^* = b_i^* B_0$.

For the Lie bracket relations, to verify that

$$[A_0, A_0^*] = (q - q^{-1}) B_0 B_0^*,$$

use the q-relations

$$A_0A_0^* = qB_0B_0^*, \qquad A_0^*A_0 = qB_0^*B_0,$$

which imply the above Lie brackets relations. On summing the latter formulae, we obtain

$$A_0a_1^* = qa_1^*A_0, \qquad i = 1, 2, 3.$$

Hence the result holds.

(II) $\Rightarrow$ (I). This converse implication works in the same way as for the case $n = 4$ (see Theorem 1 of [6]). □

By considering the form of the matrix u, the relations can be equivalently rewritten as

(A) $\quad [a_{ij}, a_{kl}^*] = (q - 1/q)b_{kj}b_{il}^*,\ 1 \le i, j \le k,\ k+1 \le l, m \le N.$

(B1) $\quad a_{ij}b_{il}^* = qb_{il}^*a_{ij},\ 1 \le i \le N,\ 1 \le j \le k,\ k+1 \le l \le N.$

(B2) $\quad a_{ji}b_{li} = qb_{li}a_{ji},\ 1 \le i \le N,\ 1 \le j \le k,\ k+1 \le l \le N.$

(C) $\quad b_{ij}b_{lm}^* = b_{lm}^*b_{ij},\ j \ge 1,\ l \le k,\ k+1 \le i, m \le N,$
$\quad a_{ii}a_{jj} = a_{jj}a_{ii}$ (i.e., the diagonal entries of A).

(D1)
$a_{ij}b_{i+1,l}^* = 0 = b_{i+1,l}^*a_{ij},\ 1 \le i \le N-1,\ i$ odd, $1 \le j \le k,\ k+1 \le l \le N,$ or
$b_{ij}a_{i+1,l}^* = 0 = a_{i+1,l}^*b_{ij},\ 1 \le i \le N-1,\ i$ odd, $1 \le j \le k,\ k+1 \le l \le N.$

(D2)
$a_{ij}b_{i-1,l}^* = 0 = b_{i-1,l}^*a_{ij},\ 2 \le i \le N,\ i$ even, $1 \le j \le k,\ k+1 \le l \le N,$ or
$b_{ij}a_{i-1,l}^* = 0 = a_{i-1,l}^*b_{ij},\ 2 \le i \le N,\ i$ even, $1 \le j \le k,\ k+1 \le l \le N.$

(D3)
$a_{ji}b_{l,i+1}^* = 0 = b_{l,i+1}^*a_{ji},\ 1 \le i \le N-1,\ i$ odd, $1 \le j \le k,\ k+1 \le l \le N,$ or
$b_{ji}a_{l,i+1}^* = 0 = a_{l,i+1}^*b_{ji},\ 1 \le i \le N-1,\ i$ odd, $1 \le j \le k,\ k+1 \le l \le N.$

(D4)
$a_{ji}b_{l,i-1}^* = 0 = b_{i-1,l}^*a_{ji},\ 2 \le i \le N,\ i$ even, $1 \le j \le k,\ k+1 \le l \le N,$ or
$b_{ji}a_{l,i-1}^* = 0 = a_{l,i-1}^*b_{ji},\ 2 \le i \le N,\ i$ even, $1 \le j \le k,\ k+1 \le l \le N.$

(D5)
$a_{ij}b_{lm} = 0 = b_{lm}a_{ij}$ for $j \ge 1,\ l \ge k,\ 1 \le i, m \le k,$ or
$a_{ij}a_{lm} = 0 = a_{lm}a_{ij}$ for $j \ge 1,\ l \ge k,\ 1 \le i, m \le k,$ or
$b_{ij}b_{lm} = 0 = b_{lm}b_{ij}$ for $j \ge 1,\ l \ge k,\ 1 \le i, m \le k.$

(D6)
$a_{ij}b_{lm} = 0 = b_{lm}a_{ij}$ for $j \ge 1,\ l \ge k,\ k+1 \le i, m \le N,$ or
$a_{ij}a_{lm} = 0 = a_{lm}a_{ij}$ for $j \ge 1,\ l \ge k,\ k+1 \le i, m \le N,$ or
$b_{ij}b_{lm} = 0 = b_{lm}b_{ij}$ for $j \ge 1,\ l \ge k,\ k+1 \le i, m \le N.$

4. The Poincaré-Birkhoff-Witt theorem and Verma module construction

We want to construct a basis for the bialgebra $A_q(\mathrm{Sp}(k))$ and for the Hopf *-algebra $\mathrm{Pol}(\mathrm{Sp}_q(k))$ by proving a Poincaré-Birkhoff-Witt theorem for

$A_q(\mathrm{Sp}(k))$. By following [14] we say that if a pair (u_{ab}, u_{cd}) of matrix elements satisfies relations (A) it is "bad". For a product of matrix elements

$$x = u_{i_1 j_1} \cdots u_{i_p j_p},$$

we define the badness $b(x)$ of x by

$$b(x) = \#\{1 \le r, s \le p : (u_{i_r j_r}, u_{i_s j_s}) \text{ is bad}\}.$$

We introduce a special ordering on the matrix elements by $u_{ij} <_0 u_{kl}$ if $i < k$ or if $i = k$ and $j > l$.

Lemma 4.1. *For the ordering $<_0$ on the elements u_{ij} there is a basis of $A_q(\mathrm{Sp}(k))$ consisting of*

$$\{u_{i_1 j_1}^{r_1} \cdots u_{i_m j_m}^{r_m} : m = N^2, r_1 \in \mathbb{Z}_+, u_{i_1 j_1} <_0 u_{i_2 j_2} <_0 \cdots <_0 u_{i_m j_m}\}.$$

Proof. Set $X = \{u_{ij} : 1 \le i, j \le N\}$. Let $\mathbb{C}\langle X\rangle$ be the free associative algebra generated by X. We define a special ordering on the matrix elements by $u_{ij} <_0 u_{kl}$ if $i < k$ or if $i = k$ and $j > l$. We extend $<_0$ to monomials of $\mathbb{C}\langle X\rangle$ by first ordering the degree and for monomials with the same degree by the lexicographical ordering. We introduce a reduction system S for the free associative algebra $\mathbb{C}\langle X\rangle$ as follows. Consider

$$u = (u_{ij}) = \begin{pmatrix} A & -qB^* \\ B & A^* \end{pmatrix}$$

as in Theorem 3.1, where A and B are k by k matrices. We now define the operator L by

$$\begin{aligned}
&L(AB^*) = qB^*A, \text{ where for } AB^* \text{ we mean a product of } a_{ij}b^*_{kl} \text{ if } i = k,\\
&L(AA^*) = A^*A + (q - q^{-1})BB^*,\\
&L(AB) = qBA, \text{ for } a_{ij}b_{kl} \text{ if } j = l,\\
&L(AA\backslash\text{diagonal entries}) = 0,\\
&L(BB) = 0,\\
&L(AB^*) = 0, \text{ for } a_{ij}b^*_{kl}, \text{ if } i \ne k,\\
&L(BB^*) = B^*B,\\
&L(AA) = AA, \text{ for diagonal entries},\\
&L(AB) = 0, \text{ for } a_{ij}b_{kl}, \text{ if } j \ne l.
\end{aligned}$$

Let $U = \{A\}$, $V = \{B^*\}$. Define

$$\begin{aligned}
f_1 &: U \to V \otimes U, & x &\to 1 \otimes x,\\
f_2 &: V \to V \otimes U, & y &\to y \otimes 1,
\end{aligned}$$

which are well-defined maps. Then there exists a mapping

$$F : U \times V \to V \otimes U, \qquad F(x, y) = f_1(x) f_2(y)$$

by the universality of the tensor product $\widetilde{F} : U \otimes V \to V \otimes U$; set $L = \widetilde{F}$. Hence L is well defined.

The reduction system S is given by $\{u_{ij}u_{kl}, L(u_{ij}u_{kl})\}$, for $u_{1l} <_0 u_{ij}$. The ordering is compatible with the reduction system, i.e., for every

$$x = u_{ij}u_{kl}, \qquad L(u_{ij}u_{kl}) <_0 u_{ij}u_{kl}$$

because of the lexicographical ordering for monomials. Every element of $\mathbb{C}\langle X\rangle$ is reduction finite since the number of monomials smaller than a given monomial is finite.

If we prove that all ambiguities are resolvable then one of the equivalent conditions of the diamond lemma [16, Theorem 1.2B] holds. This proves Lemma 4.1. We observe that there are only overlapping ambiguities. Consider

$$x = u_{rs}u_{kl}u_{ij}$$

with $u_{rs} <_0 u_{kl} <_0 u_{ij}$. To prove that they are resolvable we need to prove that there exists composition of reductions r and r' such that

$$rL_{(12)}(x) = r'L_{(23)}(x)$$

where

$$L_{(12)}(x) = L(u_{rs}u_{kl})u_{ij}, \qquad L_{(23)}(x) = u_{rs}L(u_{kl}u_{ij}).$$

We have to consider different cases. If $b(x) = 0$,

$$L_{(12)}L_{(23)}L_{(12)}(x) = L_{(23)}L_{(12)}L_{(23)}(x),$$

i.e., everything moves at the price of a constant. We proceed similarly if $b(x) = 1$. The case $b(x) = 3$ does not occur, since only pairs of the form (a_{ij}, a^*_{kl}) satisfy the (A) relations, and pairs of the form (a_{ij}, a_{kl}) are zero divisors or, if they are on the diagonal of A, they commute. Then either the relation holds trivially or it gives the case $b(x) = 2$. We are thus left to discuss this case. Here

$$x = \begin{cases} a^*_{ij}a^*_{kl}a_{rs}, \\ a_{ij}a^*_{kl}a^*_{rs}, \\ a_{ij}a^*_{kl}a_{rs}, \\ a^*_{ij}a_{kl}a_{rs}, \\ a_{ij}a_{kl}a^*_{rs}. \end{cases}$$

If any pair of the form (a^*_{ij}, a^*_{kl}) or (a_{ij}, a_{kl}) is a zero divisor then it holds trivially. If it is not a zero divisor, i.e., we consider the diagonal entries of A, then we move one a^*_{kl} or a_{kl} and we are left with something that commutes and we are back to $b(x) = 1$. Thus the result holds. □

The following result then follows from this and [14].

Theorem 4.2. *For any total ordering $<$ on the matrix elements u_{ij} there exists a basis for $A_q(\mathrm{Sp}(k))$ consisting of*

$$\{u^{r_1}_{i_1j_1}\cdots u^{r_m}_{i_mj_m} : m = n^2,\ r_i \in \mathbb{Z}_+,\ u_{i_1j_1} < u_{i_2j_2} < \cdots < u_{i_mj_m}\}.$$

Remark. The theorem yields a basis for every ordering.

Let us now specify the ordering. Let N^+, N^- be the subalgebras generated by the elements u_{ij}, $j < N+1-i$, and by u_{ij}, $j > N+1-i$, respectively. Let H be the abelian subalgebra generated by the elements $u_{i,N+1-i}$. H is generated by the elements on the antidiagonal of u, where N^+, respectively N^-, is generated by the elements above, respectively below, the antidiagonal of u. Then

$$A_q(\mathrm{Sp}(k)) = N^+ \otimes_{\mathbb{C}} H \otimes_{\mathbb{C}} N^-.$$

Now we want to construct representations of $A_q(\mathrm{Sp}(k))$ by using the Verma module construction. Let $G(N)$ be the wreath product of $\mathbb{Z}/2$ by $S(k)$, where

$S(k)$ denotes the permutation group on k letters [15]. Let $\rho \in G(N)$ be arbitrary. Here ρ represents the permutation and sign changes of $\{1, \dots, k\}$, i.e., $\rho(-k) = -\rho(k)$. Set

$$\begin{aligned} N_\rho^- &= \{u_{\rho(i),j} \colon 1 \le j < i < N\}, \\ H_\rho &= \{u_{\rho(i),i} \colon 1 \le i \le N\}, \\ N_\rho^+ &= \{u_{\rho(i),j} \colon 1 \le j,\ 1 \le i < j \le N\} \cup \{\text{zero divisors}\}. \end{aligned}$$

Choose an ordering on the matrix elements u_{ij} such that all elements of N_ρ^- are smaller than the elements of H_ρ which are smaller than the elements of N_ρ^+. By $\mathscr{N}_\rho^-$, $\mathscr{H}_\rho$, $\mathscr{N}_\rho^+$ we denote linear subspaces of $A_q(\mathrm{Sp}(k))$ generated by the basis elements, which by the theorem are products of elements of $\mathscr{N}_\rho^-$, $\mathscr{H}_\rho$, $\mathscr{N}_\rho^+$ respectively, it follows that

$$A_q(n) = \mathscr{N}_\rho^- \otimes_{\mathbf{C}} \mathscr{H}_\rho \otimes_{\mathbf{C}} \mathscr{N}_\rho^+,$$

in the sense that $n_\rho^- h_\rho n_\rho^+$ yields a basis for $A_q(\mathrm{Sp}(k))$ whenever $n_\rho^- h_\rho n_\rho^+$ runs through a basis of $\mathscr{N}_\rho^-$, $\mathscr{H}_\rho$, $\mathscr{N}_\rho^+$.

Let L_ρ be the left ideal generated by the matrix elements of N_ρ^+. Then L_ρ is the linear span of elements of the form $n_\rho^- h_\rho n_\rho^+$, with $n_\rho^+ \ne I$. Let

$$\begin{aligned} i < j \quad &\text{if } \rho(i) > \rho(j) \text{ and } u_{\rho(i),i} u_{\rho(j),j} = u_{\rho(j),j} u_{\rho(i),i} \\ &\qquad \text{or } u_{\rho(i),i} u_{\rho(j),j} = 0 = u_{\rho(j),j} u_{\rho(i),i}. \end{aligned}$$

Let

$$\begin{aligned} i < j \quad &\text{if } \rho(i) < \rho(j) \text{ and } [u_{\rho(i),i} u_{\rho(j),j}] = (q - 1/q) u_{\rho(i),j} u_{\rho(j),i} \\ &\qquad \text{or } u_{\rho(i),i} u_{\rho(j),j} = 0 = u_{\rho(j),j} u_{\rho(i),i}. \end{aligned}$$

Since

$$u_{\rho(j),i} \in N_\rho^+ \tag{4.1}$$

we have

$$u_{\rho(i),i} u_{\rho(j),j} = u_{\rho(j),j} u_{\rho(i),i} \text{ modulo } L_\rho,$$

i.e., $u_{\rho(i),i}$ commute modulo L_ρ. Let $\Gamma = (\gamma_1, \dots, \gamma_k) \in \mathbb{C}^k$ be arbitrary. Define the left ideal $F_\rho(\Gamma)$ generated by the matrix elements of N_ρ^+ and by the elements $u_{\rho(i),i} - \gamma_i I$. Then $L_\rho \subset F_\rho(\Gamma)$, $I \notin F_\rho(\Gamma)$, by using the basis.

We introduce the highest weight modules for an arbitrary element $\rho \in G(N)$ as follows. A left A_q-module M is a ρ-highest weight module if the module is generated by a ρ-highest weight vector $m \in M$. In our case a ρ-highest weight vector $m \in M$ of weight $\Gamma = (\gamma_1, \dots, \gamma_N)$ satisfies $u_{\rho(i),i} m = \gamma_i m$ and $u_{kl} m = 0$ for $u_{kl} \in N_\rho^+$. We can define

$$W^\rho(\Gamma) = A_q(\mathrm{Sp}(k))/F_\rho(\Gamma) \quad \text{and} \quad v^+ = I \ (\operatorname{mod} F_\rho(\Gamma)).$$

Thus $W^\rho(\Gamma)$ is a left A_q-module by left multiplication. Thus $W^\rho(\Gamma)$ is a ρ-highest weight module for A_q, by construction, with ρ-highest weight vector v^+ of weight Γ.

Theorem 4.3. (i) *If* M *is any* ρ*-highest weight module with* ρ*-highest weight vector* m *of weight* Γ*, then there exists a unique surjective* A_q*-equivariant linear map* $\varphi\colon W^\rho(\Gamma)\to M$ *so that* $\varphi(v^+)=m$.

(ii) *For* $i=N,\dots,\rho(1)+1$*, we have* $u_{i1}v=0$*, for all* $v\in W^\rho(\Gamma)$.

(iii) *If* $\Gamma\in(\mathbb{C}^*)^N$*, then* $W^\rho(\Gamma)$ *has a unique proper maximal submodule.*

(iv) *For every* $\rho\in G(N)$ *and for every* $\Gamma\in(\mathbb{C}^*)^N$ *a unique irreducible* ρ*-highest weight module* $L^\rho(\Gamma)$ *exists.*

Proof. Part (i) is similar to Proposition 4.35 of [15]. Define a linear map

$$\varphi(n_\rho^- v^+)=n_\rho^- m.$$

Then φ is well defined and φ is "equivariant",

$$\varphi(u_{ij}n_\rho^- v^+)=u_{ij}\varphi(n_\rho^- v^+),$$

by using the fact that $W^\rho(\Gamma)$ is a left A_q-module. Since $M\cong\mathcal{N}_\rho^- m$ as vector spaces, the map φ is surjective. The uniqueness follows. Now $W^\rho(\Gamma)\to\mathcal{N}_\rho^- v^+$, as vector spaces. We can consider the subspace of $W^\rho(\Gamma)$ generated by the elements of the form

$$\prod_{j\le k<l\le n} u_{\rho(k),l}^{p_{kl}}v^+,\qquad p_{kl}\in\mathbb{Z}_+.$$

For $l\ge k\ge j$, $\rho(i)\ge\rho(j)$, and $i\ge j$ we have the following commutation relations:

$$(4.2)\qquad u_{\rho(i),j}u_{\rho(k),l}=\begin{cases} u_{\rho(k),l}u_{\rho(i),j} & \text{if } \rho(k)<\rho(i),\\ q u_{\rho(k),l}u_{\rho(i),j} & \text{if } \rho(k)=\rho(i),\\ u_{\rho(k),l}u_{\rho(i),j}+(q-1/q)u_{\rho(i),l}u_{\rho(k),j} & \\ & \text{if } \rho(k)>\rho(i),\\ 0 & \text{if } u_{\rho(i),j}u_{\rho(k),l} \text{ is a zero divisor.}\end{cases}$$

It follows, for $\rho(i)>\rho(j)$, $i>j$, that

$$(4.3)\qquad u_{\rho(i),j}\prod_{j\le k<l\le n} u_{\rho(k),l}^{p_{kl}}v^+=0.$$

Indeed if $\rho(i)$ is maximal with respect to $\rho(i)>\rho(j)$, $i>j$, since $k>j$, then the third commutation relation of (4.2) does not occur and we have that (4.3) follows from the fact that $u_{\rho(i),j}v^+=0$.

To prove (4.3) for general $\rho(i)$, assume that it holds for all $u_{\rho(k),j}$ with $\rho(k)>\rho(j)$, $k>j$. Then $\rho(k)>\rho(i)$. Use induction on $\sum p_{kl}=t$. Use of (4.2) gives the induction step. The third case may cause some problems but we use induction on $\rho(i)$, since $\rho(k)>\rho(i)>\rho(j)$, for $k\ge j$. Taking $j=1$, we get (ii). The zero divisors do not cause any problems since we get that (ii) holds trivially in such case. (iii) and (iv) follow from [14], [17]. By following the same argument of Proposition 3.5 of [14] we have that $L^\rho(\Gamma)$ is an irreducible highest weight module. To prove that $L^\rho(\Gamma)$ is an irreducible $\mathrm{Pol}(\mathrm{Sp}_q(\tau))$ module, we need to consider what happens to the quantum determinant D. Set

$$D^{-1}v=(\gamma_1,\dots,\gamma_k)^{-1}(-q)^{l(\rho)}v,\qquad \text{for every } v\in L^\rho(\Gamma),$$

where $l(\rho)$ represents the length of the permutation ρ. Since D is a central element it acts as scalar on $L^{\rho}(\Gamma)$, and the action on the highest weight vector v^+ is given by

$$Dv^+ = \sum_{\sigma \in S_n} (-q)^{l(\rho)} u_{\sigma(1),1} \cdots u_{\sigma(N),N} v^+ ,$$

all the terms will cancel out except for $\rho = \sigma$, since v^+ is a highest weight vector. Then the result holds and we conclude that $L^{\rho}(\Gamma)$ is irreducible. □

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