CESÀRO MEANS OF FOURIER TRANSFORMS
AND MULTIPLIERS ON $L^1(\mathbb{R})$

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ABSTRACT. We prove that the Cesàro mean $\sigma$ of a multiplier $\lambda$ on $L^1(\mathbb{R})$ is
also a multiplier on $L^1(\mathbb{R})$. In the particular cases when (i) $\lambda$ is odd, we prove
that $\sigma$ is the Fourier transform of an odd function in the Hardy space $H^1(\mathbb{R})$,
and (ii) $\lambda$ is even, we give a necessary and sufficient condition in order that $\sigma$
be a Fourier transform of an even function in $L^1(\mathbb{R})$. As a corollary, we obtain
a nontrivial condition for $\lambda$ in order to be a multiplier on $L^1(\mathbb{R})$; namely,

$$\int_0^\infty \frac{1}{t} \left| \int_0^t (\lambda(\xi) - \lambda(-\xi)) \, d\xi \right| \frac{dt}{t} < \infty.$$ 

We also prove Hardy type inequalities for multipliers and Hilbert transforms.

1. Preliminary results

Let $f$ be a function defined on the real line $\mathbb{R} := (-\infty, \infty)$. We recall that
if $f$ is Lebesgue integrable over $\mathbb{R}$, in sign : $f \in L^1(\mathbb{R})$, then the Fourier
transform of $f$ is defined by

$$\hat{f}(t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-itx} \, dx, \quad t \in \mathbb{R},$$

while the Hilbert transform of $f$ is defined as an improper integral by

$$\tilde{f}(x) := -\pi \lim_{\delta \downarrow 0} \int_{\delta}^{\infty} \frac{f(x + t) - f(x - t)}{t} \, dt.$$ 

Hille and Tamarkin [5] proved that $\tilde{f}(x)$ exists for almost all $x$ in $\mathbb{R}$. They
also proved that if $\tilde{f} \in L^1(\mathbb{R})$, then we necessarily have

$$(1.1) \quad \int_0^\infty \frac{t}{t} < \infty.$$ 

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search under grant #234.
We note that (1.1) is the extension of Hardy's inequality (see, e.g., [7, Volume 1, p. 286]) from Fourier series to Fourier transforms. To honor Hardy, by \( H^1(\mathbb{R}) \) we denote the space of functions \( f \) in \( L^1(\mathbb{R}) \) whose Hilbert transform \( \hat{f} \) is also in \( L^1(\mathbb{R}) \).

We say that a function \( \lambda \), measurable and bounded on \( \mathbb{R} \), is a multiplier on \( L^1(\mathbb{R}) \) if for every \( f \in L^1(\mathbb{R}) \) there exists a function \( f_1 \in L^1(\mathbb{R}) \) such that

\[
\lambda(t)\hat{f}(t) = \hat{f}_1(t), \quad t \in \mathbb{R}.
\]

As is known (see, e.g., [1, p. 269]), a necessary and sufficient condition for a function \( \lambda \) to be a multiplier on \( L^1(\mathbb{R}) \) is that \( \lambda \) is the Fourier-Stieltjes transform of a function \( \mu \) of bounded variation on \( \mathbb{R} \); i.e.,

\[
(1.2) \quad \lambda(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-itx} d\mu(x), \quad t \in \mathbb{R}.
\]

We remind the reader that the integral on the right-hand side of (1.2) is a Lebesgue-Stieltjes integral (see, e.g., [6, Chapter 1]), while \( \mu(x) \) is a complex-valued function of bounded variation, i.e., whose total variation over \( \mathbb{R} \) is finite (see, e.g., [6, Chapter 6]). Then the one-sided limits \( \mu(x - 0) \) and \( \mu(x + 0) \) exist at each \( x \in \mathbb{R} \), and the limits

\[
\mu(\infty) := \lim_{x \to \infty} \mu(x) \quad \text{and} \quad \mu(-\infty) := \lim_{x \to -\infty} \mu(x)
\]

also exist and are finite. For the sake of definiteness, we always assume that \( \mu \) is continuous on the left. (Without loss of generality, we may also assume that \( \mu(-\infty) = 0 \).)

2. MAIN RESULTS: CESÁRO MEANS OF MULTIPLIERS ON \( L^1(\mathbb{R}) \)

We distinguish between the cases of odd and even multipliers.

Case (i). Let \( \lambda \) be an odd multiplier on \( L^1(\mathbb{R}) \). Then there exists a function \( \mu \) of bounded variation on \( \mathbb{R} \) such that (1.2) is satisfied. An integration by substitution gives

\[
(2.1) \quad \lambda(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx} d\mu(-x).
\]

The oddness of \( \lambda \) implies that \( \mu \) is also odd: \( \mu(-x) = -\mu(x), \ x \in \mathbb{R} \). In particular, \( \mu(0) = 0 \).

We define the Cesáro mean of \( \lambda \) by

\[
(2.2) \quad \sigma(t) := \frac{1}{t} \int_{0}^{t} \lambda(\xi) d\xi, \quad t \in \mathbb{R}, \ t \neq 0.
\]

Clearly, \( \sigma(t) \) is also an odd function. We will prove the following

Theorem 1. The Cesáro mean of an odd multiplier \( \lambda \) on \( L^1(\mathbb{R}) \) is the Fourier transform of an odd function in \( H^1(\mathbb{R}) \). In particular, it is also a multiplier on \( L^1(\mathbb{R}) \).

Combining Theorem 1 with (1.1) yields the following
Corollary 1. If $\lambda$ is an odd multiplier on $L^1(\mathbb{R})$, then
\[ \int_0^\infty \frac{1}{t} \int_t^\infty \lambda(\xi) \, d\xi \, \frac{dt}{t} < \infty. \]

If, in addition, $\lambda(t) \geq 0$ for $t \geq 0$, then $\int_0^\infty \frac{\lambda(\xi)}{\xi} \, d\xi < \infty$.

The next corollary is due to the fact that if $\lambda$ is a multiplier on $L^1(\mathbb{R})$, then $\lambda(t) - \lambda(-t)$ is an odd multiplier on $L^1(\mathbb{R})$.

Corollary 2. If $\lambda$ is a multiplier on $L^1(\mathbb{R})$, then
\[ \int_0^\infty \frac{1}{t} \int_t^\infty \{\lambda(\xi) - \lambda(-\xi)\} \, d\xi \, \frac{dt}{t} < \infty. \]

Case (ii). Now let $\lambda$ be an even multiplier on $L^1(\mathbb{R})$. Then there exists a function $\mu$ of bounded variation on $\mathbb{R}$ such that (1.2) is satisfied. An integration by substitution gives (2.1). The evenness of $\lambda$ implies that $\mu$ is also even: $\mu(-x) = \mu(x)$, $x \in \mathbb{R}$.

We consider the Cesàro mean $\sigma$ of $\lambda$ defined by (2.2). This time $\sigma$ is an even function. We will prove the following

Theorem 2. The Cesàro mean $\sigma$ of an even multiplier $\lambda$ on $L^1(\mathbb{R})$ is also a multiplier on $L^1(\mathbb{R})$. Furthermore, $\sigma$ is a Fourier transform if and only if $\mu(x)$ is continuous at $x = 0$, where $\mu$ is the function of bounded variation associated with $\lambda$ according to (1.2).

Since any multiplier can be written as the sum of an even and an odd multiplier, from Theorems 1 and 2 we conclude the following

Corollary 3. The Cesàro mean of a multiplier on $L^1(\mathbb{R})$ is also a multiplier on $L^1(\mathbb{R})$.

3. Proofs of Theorems 1 and 2

We begin with an auxiliary result on Lebesgue-Stieltjes integrals.

Lemma. If a function $\nu$ is continuous and $\mu$ is nondecreasing on the interval $[0, \infty)$ with $\mu(\infty) < \infty$, then
\[ \lim_{\delta \to 0} \int_0^\delta \nu(t) \, d\mu(t) = \int_0^\infty \nu(t) \, d\mu(t) - \nu(0)\{\mu(+\delta) - \mu(0)\}. \]

Proof. Consider an approximating sum for the integral on the left-hand side of (3.1):
\[ \nu(\delta)\{\mu(\delta_1) - \mu(\delta)\} + \nu(\delta_1)\{\mu(\delta_2) - \mu(\delta_1)\} + \cdots \]
where $0 < \delta < \delta_1 < \delta_2 < \cdots$ is a partition of the interval $[\delta, \infty)$. If we add the extra term $\nu(\delta)\{\mu(\delta) - \mu(0)\}$ to (3.2), then we obtain an approximating sum for the integral on the right-hand side of (3.1). Now letting $\delta$ tend to 0 and at the same time making the partition finer and finer, we conclude that
\[ \lim_{\delta \to 0} \left\{ \nu(0)\{\mu(\delta) - \mu(0)\} + \int_\delta^\infty \nu(t) \, d\mu(t) \right\} = \int_0^\infty \nu(t) \, d\mu(t). \]
This is equivalent to (3.1) to be proved.
Proof of Theorem 1. First we prove that \( \sigma \) is the Fourier transform of an odd function in \( L^1(\mathbb{R}) \). Indeed, the function in question is defined by

\[
h(x) := \begin{cases} 
\int_x^\infty \frac{d\mu(\xi)}{\xi} & \text{if } x > 0, \\
-h(-x) & \text{if } x < 0,
\end{cases}
\]

and let \( h(0) = 0 \). Clearly, \( h(x) \) is odd.

We show that \( h \in L^1(\mathbb{R}) \). By Fubini's theorem,

\[
\int_0^\infty |h(x)| \, dx \leq \int_0^\infty dx \int_x^\infty \frac{d|\mu|(\xi)}{\xi} = \int_0^\infty \frac{d|\mu|(\xi)}{\xi} \int_0^x dx = \int_0^\infty d|\mu|(\xi) < \infty.
\]

Here and in the sequel, by \( |\mu|(x) \) we denote the total variation of \( \mu \) over \((-\infty, x]\).

Second we prove that \( \sigma \) is the Fourier transform of \( h \). By Fubini's theorem and the lemma,

\[
\hat{\sigma}(t) = -i \sqrt{\frac{2}{\pi}} \int_0^\infty h(x) \sin tx \, dx = -i \sqrt{\frac{2}{\pi}} \int_0^\infty \left\{ \int_x^\infty \frac{d\mu(\xi)}{\xi} \right\} \sin tx \, dx
\]

\[
= -i \sqrt{\frac{2}{\pi}} \int_0^\infty \left\{ \int_0^x \frac{\sin tx}{x} \, dx \right\} d\mu(\xi) = -i \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1 - \cos tx}{tx} d\mu(\xi).
\]

On the other hand,

\[
\hat{\sigma}(t) = -i \sqrt{\frac{2}{\pi}} \frac{1}{t} \int_0^t \left\{ \int_0^\infty \sin \xi x d\mu(x) \right\} d\xi
\]

\[
= -i \sqrt{\frac{2}{\pi}} \frac{1}{t} \int_0^\infty \left\{ \int_0^t \sin x\xi d\xi \right\} d\mu(x)
\]

\[
= -i \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1 - \cos tx}{tx} d\mu(x).
\]

Comparing (3.5) and (3.6) shows that \( \sigma(t) = \hat{h}(t) \), \( t \in \mathbb{R} \).

Third we prove that \( \hat{h} \), the Hilbert transform of \( h \), also belongs to \( L^1(\mathbb{R}) \). Without loss of generality, we may assume that \( \mu \) is nondecreasing on \( \mathbb{R} \). By definition,

\[
\pi \hat{h}(x) = \lim_{\delta \downarrow 0} \int_0^\infty \frac{h(x - t) - h(x + t)}{t} \, dt
\]

\[
= \lim_{\delta \downarrow 0} \int_0^{x/2} + \int_{x/2}^\infty =: H_1(x) + H_2(x), \text{ say}.
\]

Clearly, \( H_1(x) \geq 0 \) and \( H_2(x) \geq 0 \) for \( x > 0 \).

By (3.3)

\[
H_1(x) = \lim_{\delta \downarrow 0} \int_0^{x/2} \frac{dt}{t} \int_{x-t}^{x+t} \frac{d\mu(\xi)}{\xi}.
\]
Let $\delta > 0$ be fixed. By Fubini's theorem,
\[
\int_{2\delta}^{\infty} dx \int_{x-t}^{x/2} \frac{dt}{t} \int_{x-t}^{x+t} \frac{d\mu(\xi)}{\xi} = \int_{\delta}^{\infty} dt \int_{2t}^{\infty} dx \int_{x-t}^{x+t} \frac{d\mu(\xi)}{\xi} = \int_{\delta}^{\infty} dt \int_{2t}^{\infty} \frac{d\mu(\xi)}{\xi} \int_{\max(x-t, 2t)}^{\xi+t} dx \leq 2 \int_{\delta}^{\infty} dt \int_{2t}^{\infty} \frac{d\mu(\xi)}{\xi} \int_{\delta}^{\xi} dt.
\]

By Fatou's lemma, $H_1 \in L^1(\mathbb{R})$ and
\[(3.9) \quad \int_{0}^{\infty} H_1(x) \, dx \leq 2\{\mu(\infty) - \mu(0)\}.
\]

Now we consider $H_2$, which we decompose as
\[(3.10) \quad H_2(x) = - \int_{x/2}^{\infty} \frac{h(x + t)}{t} \, dt + \int_{x/2}^{\infty} \frac{h(x - t)}{t} \, dt =: \beta_1(x) + \beta_2(x), \text{ say.}
\]

By (3.4) it is not hard to check that
\[(3.11) \quad \int_{0}^{\infty} |\beta_1(x)| \, dx = \int_{0}^{\infty} dx \int_{3x/2}^{\infty} \frac{h(u)}{u - x} \, du = \int_{0}^{\infty} h(u) \, du \int_{0}^{u} \frac{dt}{u} < \infty.
\]

By simple substitutions and manipulations,
\[(3.12) \quad \beta_2(x) = \left\{ \int_{x/2}^{x} + \int_{x}^{\infty} \right\} \frac{h(x - t)}{t} \, dt = \int_{x/2}^{x} \frac{dt}{t} \int_{x-t}^{x} \frac{d\mu(\xi)}{\xi} - \int_{x}^{\infty} \frac{dt}{t} \int_{x-x}^{\infty} \frac{d\mu(\xi)}{\xi} = \int_{0}^{\infty} \frac{d\mu(\xi)}{\xi} \int_{\max(x-x, x/2)}^{x} \frac{dt}{t} - \int_{0}^{\infty} \frac{d\mu(\xi)}{\xi} \int_{x}^{x+\xi} \frac{dt}{t} \leq \int_{0}^{x/2} \ln \left( \frac{x}{x - \xi} \right) \frac{d\mu(\xi)}{\xi} + (\ln 2) \int_{x/2}^{\infty} \frac{d\mu(\xi)}{\xi} - \int_{0}^{\infty} \ln \left( \frac{x + \xi}{x} \right) \frac{d\mu(\xi)}{\xi} \]
\[= \int_{0}^{x/2} \ln \left( \frac{x^2}{x^2 - \xi^2} \right) \frac{d\mu(\xi)}{\xi} + (\ln 2) \int_{x/2}^{\infty} \frac{d\mu(\xi)}{\xi} - \int_{x/2}^{\infty} \ln \left( \frac{x + \xi}{x} \right) \frac{d\mu(\xi)}{\xi} =: \beta_3(x) + \beta_4(x) + \beta_5(x), \text{ say.}
\]

We begin with $\beta_3$. Making use of Fubini's theorem, the substitution $x = \xi t$, and the inequality
\[(3.13) \quad \ln(1 + u) \leq u, \quad u \geq 0,
\]
we find
\[ \int_0^\infty |\beta_3(x)| \, dx = \int_0^\infty \frac{d\mu(\xi)}{\xi} \int_{2\xi}^{\infty} \ln \left( \frac{x^2}{x^2 - \xi^2} \right) \, dx \]
(3.14)
\[ = \int_0^\infty d\mu(\xi) \int_2^{\infty} \ln \left( \frac{t^2}{t^2 - 1} \right) \, dt \]
\[ \leq \int_0^\infty d\mu(\xi) \int_2^{\infty} \frac{1}{t^2 - 1} \, dt < \infty. \]

Since
\[ (3.15) \quad \beta_4(x) = (\ln 2)h(x/2), \quad x > 0, \]
and \( h \) is an odd function, by (3.4) we see that \( \beta_4 \in L^1(\mathbb{R}) \). Finally, applying Fubini’s theorem and the substitutions \( x = \xi t \) and \( u = \frac{1}{t} \), in turn we get
\[ \int_0^\infty |\beta_4(x)| \, dx = \int_0^\infty d\mu(\xi) \int_0^{2\xi} \ln \left( \frac{x + \xi}{x} \right) \, dx \]
(3.16)
\[ = \int_0^\infty d\mu(\xi) \int_0^{2} \ln \left( \frac{t+1}{t} \right) \, dt \]
\[ = \int_0^\infty d\mu(\xi) \int_{1/2}^{\infty} \frac{\ln(1+u)}{u^2} \, du < \infty. \]

Combining (3.7), (3.9)-(3.12), (3.14)-(3.16), and the fact that \( \hat{h} \) is an even function (since \( h \) is odd), we conclude that \( \hat{h} \in L^1(\mathbb{R}) \). This completes the proof of Theorem 1.

**Proof of Theorem 2.** First we prove that \( \sigma \) is the Fourier transform of an even function, up to an additive constant. In particular, it follows that \( \sigma \) is a multiplier on \( L^1(\mathbb{R}) \). In fact, the function in question is defined by
\[ (3.17) \quad g(x) := \begin{cases} \int_x^\infty \frac{d\mu(\xi)}{\xi} & \text{if } x > 0, \\ g(-x) & \text{if } x < 0. \end{cases} \]
(cf. (3.3)). Clearly, \( g \) is even. Similarly to (3.4), we deduce that \( g \) is in \( L^1(\mathbb{R}) \).

Second we consider \( \hat{g} \), the Fourier transform of \( g \),
\[ (3.18) \quad \hat{g}(t) = \frac{\sqrt{2}}{\pi} \int_0^\infty g(x) \cos tx \, dx. \]
Again fix \( \delta > 0 \). By (3.17) and Fubini’s theorem,
\[ (3.19) \quad \int_\delta^\infty g(x) \cos tx \, dx = \int_\delta^\infty \cos tx \, dx \int_x^\infty \frac{d\mu(\xi)}{\xi} = \int_\delta^\infty \frac{d\mu(\xi)}{\xi} \int_\delta^\xi \cos tx \, dx \]
\[ = \int_\delta^\infty \frac{\sin t\xi}{t\xi} \, d\mu(\xi) - \frac{\sin t\delta}{t} \int_\delta^\infty \frac{d\mu(\xi)}{\xi}. \]

Without loss of generality, we may assume that \( \mu \) is nondecreasing on \((0, \infty)\). Then for \( 0 < \delta < \delta_1 \) we may write that
\[ 0 \leq \frac{\sin t\delta}{t} \int_\delta^\infty \frac{d\mu(\xi)}{\xi} \leq \delta \left( \int_\delta^{\delta_1} + \int_{\delta_1}^\infty \right) \frac{d\mu(\xi)}{\xi} \]
\[ \leq \{ \mu(\delta_1) - \mu(\delta) \} + o(1) \to 0 \quad \text{as } \delta \to 0. \]
Thus, from (3.18), (3.19), and the lemma it follows that

\begin{equation}
\hat{g}(t) = \sqrt{\frac{2}{\pi}} \lim_{\delta \to 0} \int_{-\infty}^{\infty} \frac{\sin t \xi}{t \xi} d\mu(\xi)
\end{equation}

\begin{equation}
= \sqrt{\frac{2}{\pi}} \left\{ \int_{0}^{\infty} \frac{\sin t \xi}{t \xi} d\mu(\xi) - \{\mu(+0) - \mu(0)\} \right\}.
\end{equation}

On the other hand, the Cesàro mean \( \sigma \) of \( g \) defined in (2.2) can be rewritten as

\begin{equation}
\sigma(t) = \sqrt{\frac{2}{\pi}} \int_{t}^{0} d\xi \int_{0}^{\infty} \cos \xi x d\mu(x)
\end{equation}

\begin{equation}
= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} d\mu(x) \int_{0}^{t} \cos \xi x d\xi = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \frac{\sin tx}{tx} d\mu(x).
\end{equation}

Combining (3.20) and (3.21) gives

\begin{equation}
\sigma(t) = \hat{g}(t) + \sqrt{\frac{2}{\pi}} \{\mu(+0) - \mu(0)\},
\end{equation}

whence the necessity and sufficiency part in Theorem 2 is obvious. This completes the proof of Theorem 2.

4. Related results

(i) We point out that in the case of an even multiplier \( \lambda \) on \( L^1(\mathbb{R}) \) we cannot state that the function \( g \) defined in (3.17) belongs to \( H^1(\mathbb{R}) \). Nevertheless, we are able to deduce a Hardy type inequality for the Hilbert transform \( \hat{g} \). More exactly, the following is true.

**Theorem 3.** If \( \lambda \) is an even multiplier on \( L^1(\mathbb{R}) \), \( \mu \) is the function of bounded variation associated with \( \lambda \) according to (1.2), \( \sigma \) is the Cesàro mean of \( \lambda \) defined by (2.2), and \( g \) is defined by (3.17), then

\begin{equation}
\int_{0}^{\infty} \left| \pi \hat{g}(x) - \frac{2}{x} \int_{0}^{x/2} d\mu(\xi) \right| dx \leq C \int_{0}^{\infty} d|\mu|(\xi),
\end{equation}

where \( C \) is an absolute constant.

The next corollary is an immediate consequence of Theorem 3.

**Corollary 4.** If \( \lambda \) is an even multiplier on \( L^1(\mathbb{R}) \) and \( \mu \) is the function of bounded variation associated with \( \lambda \) according to (1.2), then the function \( g \) defined by (3.17) belongs to \( H^1(\mathbb{R}) \) if and only if

\begin{equation}
\int_{0}^{\infty} \left| \frac{1}{x} \int_{0}^{x} d\mu(\xi) \right| dx < \infty.
\end{equation}

**Proof of Theorem 3.** Similarly to the case of an odd multiplier on \( L^1(\mathbb{R}) \) (cf. (3.7)–(3.12)), we can represent the Hilbert transform \( \hat{g} \) by

\begin{equation}
\pi \hat{g}(x) = G_1(x) + \beta_1(x) + \gamma_3(x) + \beta_4(x) + \beta_5(x),
\end{equation}
where

\[ G_1(x) := \lim_{\delta \to 0} \int_{x-i\delta}^{x+i\delta} \frac{d\mu(\xi)}{\xi}, \]

\[ \beta_1 := -\int_{x/2}^{\infty} \frac{g(x+t)}{t} dt, \quad \gamma_3(x) := \int_{0}^{x/2} \ln \left( \frac{x+\xi}{x-\xi} \right) \frac{d\mu(\xi)}{\xi} . \]

(there is an essential difference between \( \beta_3 \) and \( \gamma_3 \), due to the evenness of \( g \), in contrast to the oddness of \( h \)),

\[ \beta_4(x) := (\ln 2) \int_{x/2}^{\infty} \frac{d\mu(\xi)}{\xi}, \quad \beta_5(x) := \int_{x/2}^{\infty} \ln \left( \frac{x+\xi}{x} \right) \frac{d\mu(\xi)}{\xi} . \]

Repeating analogous reasonings as in the proof of Theorem 1, we conclude that the functions \( G_1, \beta_1, \beta_4, \) and \( \beta_5 \) are in \( L^1(\mathbb{R}) \). Thus, from (4.3) it follows that

\[ \int_{0}^{\infty} |\pi g(x) - \gamma_3(x)| dx \leq C \int_{0}^{\infty} d|\mu|(\xi). \]  

On the basis of (3.13) and the reversed inequality \( \ln(1 + u) \geq u - u^2/2, \) \( u \geq 0 \), we see that the inequality

\[ 0 \leq \frac{2}{x-\xi} - \frac{1}{\xi} \ln \left( \frac{x+\xi}{x-\xi} \right) \leq \frac{4\xi}{(x-\xi)^2} \]

holds true for \( 0 \leq \xi \leq \frac{x}{2} \). Consequently, by setting

\[ \gamma_4(x) := \int_{0}^{x/2} \frac{d\mu(\xi)}{x-\xi}, \]

from (4.4) it follows that

\[ \int_{0}^{\infty} |\pi g(x) - 2\gamma_4(x)| dx \leq \int_{0}^{\infty} d|\mu|(\xi) \int_{2\xi}^{\infty} \left| \frac{1}{\xi} \ln \left( \frac{x+\xi}{x-\xi} \right) - \frac{2}{x-\xi} \right| dx \]

\[ \leq \int_{0}^{\infty} d|\mu|(\xi) \int_{2\xi}^{\infty} \frac{4\xi}{(x-\xi)^2} dx = 4 \int_{0}^{\infty} d|\mu|(\xi). \]

Analogously, by Fubini's theorem,

\[ \int_{0}^{\infty} \left| \gamma_4(x) - \frac{1}{x} \int_{0}^{x/2} d\mu(\xi) \right| dx = \int_{0}^{\infty} \left| \int_{0}^{x/2} \frac{\xi}{x(x-\xi)} d\mu(\xi) \right| dx \]

\[ \leq \int_{0}^{\infty} d|\mu|(\xi) \int_{2\xi}^{\infty} \frac{\xi}{x(x-\xi)} dx = \int_{0}^{\infty} \frac{dt}{t(t-1)} \int_{0}^{\infty} d|\mu|(\xi). \]

Combining (4.5) and (4.6) results in (4.1). This completes the proof of Theorem 3.

(ii) Adopting the relevant parts in the proofs of Theorems 1 and 2 with appropriate modifications, we can achieve an analogous result on the Cesàro mean \( \sigma \) of the Fourier transform \( \hat{f} \) defined by

\[ \sigma(t) := \frac{1}{t} \int_{0}^{t} \hat{f}(\xi) d(\xi), \quad t \in \mathbb{R}, \ t \neq 0 \]

(cf. definition (2.2)), where \( f \in L^1(\mathbb{R}) \) is a given function.
Corollary 5. If $f \in L^1(\mathbb{R})$ and $\sigma$ is defined by (4.7), then $\sigma$ is also the Fourier transform of a function in $L^1(\mathbb{R})$.

(iii) On closing, we note that Hardy [4], Goes [3], and Georgakis [2] achieved similar results on the arithmetic means of the Fourier-Stieltjes coefficients of periodic functions $\mu$ of bounded variation on the torus $[-\pi, \pi]$.

Note added in proof

After submitting this paper, it came to our knowledge that Georgakis [8] has proved our Theorems 1–2 and Corollaries 3–4 in a more general setting. However, our estimates in the proofs are different from those in his proofs.

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