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# CESÀRO MEANS OF FOURIER TRANSFORMS AND MULTIPLIERS ON $L^1(\mathbf{R})$

DANG VU GIANG AND FERENC MÓRICZ

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ABSTRACT. We prove that the Cesàro mean $\sigma$ of a multiplier $\lambda$ on $L^1(\mathbf{R})$ is also a multiplier on $L^1(\mathbf{R})$. In the particular cases when (i) $\lambda$ is odd, we prove that $\sigma$ is the Fourier transform of an odd function in the Hardy space $H^1(\mathbf{R})$, and (ii) $\lambda$ is even, we give a necessary and sufficient condition in order that $\sigma$ be a Fourier transform of an even function in $L^1(\mathbf{R})$. As a corollary, we obtain a nontrivial condition for $\lambda$ in order to be a multiplier on $L^1(\mathbf{R})$; namely,

$$\int_0^\infty \left| \frac{1}{t} \int_0^t \{\lambda(\xi) - \lambda(-\xi)\}\, d\xi \right| \frac{dt}{t} < \infty.$$

We also prove Hardy type inequalities for multipliers and Hilbert transforms.

## 1. PRELIMINARY RESULTS

Let $f$ be a function defined on the real line $\mathbf{R} := (-\infty, \infty)$. We recall that if $f$ is Lebesgue integrable over $\mathbf{R}$, in sign : $f \in L^1(\mathbf{R})$, then the Fourier transform of $f$ is defined by

$$\hat{f}(t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-itx}\, dx, \qquad t \in \mathbf{R},$$

while the Hilbert transform of $f$ is defined as an improper integral by

$$\tilde{f}(x) := -\frac{1}{\pi} \lim_{\delta \downarrow 0} \int_\delta^\infty \frac{f(x+t) - f(x-t)}{t}\, dt.$$

Hille and Tamarkin [5] proved that $\tilde{f}(x)$ exists for almost all $x$ in $\mathbf{R}$. They also proved that if $\tilde{f} \in L^1(\mathbf{R})$, then we necessarily have

$$\int_0^\infty \frac{|\hat{f}(t)|}{t} < \infty. \tag{1.1}$$

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We note that (1.1) is the extension of Hardy's inequality (see, e.g., [7, Volume 1, p. 286]) from Fourier series to Fourier transforms. To honor Hardy, by $H^1(\mathbf{R})$ we denote the space of functions $f$ in $L^1(\mathbf{R})$ whose Hilbert transform $\tilde{f}$ is also in $L^1(\mathbf{R})$.

We say that a function $\lambda$, measurable and bounded on $\mathbf{R}$, is a multiplier on $L^1(\mathbf{R})$ if for every $f \in L^1(\mathbf{R})$ there exists a function $f_1 \in L^1(\mathbf{R})$ such that

$$\lambda(t)\hat{f}(t) = \hat{f}_1(t), \qquad t \in \mathbf{R}.$$

As is known (see, e.g., [1, p. 269]), a necessary and sufficient condition for a function $\lambda$ to be a multiplier on $L^1(\mathbf{R})$ is that $\lambda$ is the Fourier-Stieltjes transform of a function $\mu$ of bounded variation on $\mathbf{R}$; i.e.,

$$\lambda(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-itx}\, d\mu(x), \qquad t \in \mathbf{R}. \tag{1.2}$$

We remind the reader that the integral on the right-hand side of (1.2) is a Lebesgue-Stieltjes integral (see, e.g., [6, Chapter 1]), while $\mu(x)$ is a complex-valued function of bounded variation, i.e., whose total variation over $\mathbf{R}$ is finite (see, e.g., [6, Chapter 6]). Then the one-sided limits $\mu(x-0)$ and $\mu(x+0)$ exist at each $x \in \mathbf{R}$, and the limits

$$\mu(\infty) := \lim_{x\to\infty} \mu(x) \quad \text{and} \quad \mu(-\infty) := \lim_{x\to-\infty} \mu(x)$$

also exist and are finite. For the sake of definiteness, we always assume that $\mu$ is continuous on the left. (Without loss of generality, we may also assume that $\mu(-\infty) = 0$.)

## 2. Main results: Cesàro means of multipliers on $L^1(\mathbf{R})$

We distinguish between the cases of odd and even multipliers.

*Case* (i). Let $\lambda$ be an odd multiplier on $L^1(\mathbf{R})$. Then there exists a function $\mu$ of bounded variation on $\mathbf{R}$ such that (1.2) is satisfied. An integration by substitution gives

$$\lambda(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx}\, d\mu(-x). \tag{2.1}$$

The oddness of $\lambda$ implies that $\mu$ is also odd: $\mu(-x) = -\mu(x)$, $x \in \mathbf{R}$. In particular, $\mu(0) = 0$.

We define the Cesàro mean of $\lambda$ by

$$\sigma(t) := \frac{1}{t} \int_0^t \lambda(\xi)\, d\xi, \qquad t \in \mathbf{R},\ t \neq 0. \tag{2.2}$$

Clearly, $\sigma(t)$ is also an odd function. We will prove the following

**Theorem 1.** *The Cesàro mean of an odd multiplier $\lambda$ on $L^1(\mathbf{R})$ is the Fourier transform of an odd function in $H^1(\mathbf{R})$. In particular, it is also a multiplier on $L^1(\mathbf{R})$.*

Combining Theorem 1 with (1.1) yields the following

**Corollary 1.** *If $\lambda$ is an odd multiplier on $L^1(\mathbf{R})$, then*

$$\int_0^\infty \left| \frac{1}{t} \int_0^t \lambda(\xi)\, d\xi \right| \frac{dt}{t} < \infty .$$

*If, in addition, $\lambda(t) \geq 0$ for $t \geq 0$, then $\int_0^\infty \frac{\lambda(\xi)}{\xi}\, d\xi < \infty$.*

The next corollary is due to the fact that if $\lambda$ is a multiplier on $L^1(\mathbf{R})$, then $\lambda(t) - \lambda(-t)$ is an odd multiplier on $L^1(\mathbf{R})$.

**Corollary 2.** *If $\lambda$ is a multiplier on $L^1(\mathbf{R})$, then*

$$\int_0^\infty \left| \frac{1}{t} \int_0^t \{\lambda(\xi) - \lambda(-\xi)\}\, d\xi \right| \frac{dt}{t} < \infty .$$

*Case* (ii). Now let $\lambda$ be an even multiplier on $L^1(\mathbf{R})$. Then there exists a function $\mu$ of bounded variation on $\mathbf{R}$ such that (1.2) is satisfied. An integration by substitution gives (2.1). The evenness of $\lambda$ implies that $\mu$ is also even: $\mu(-x) = \mu(x)$, $x \in \mathbf{R}$.

We consider the Cesàro mean $\sigma$ of $\lambda$ defined by (2.2). This time $\sigma$ is an even function. We will prove the following

**Theorem 2.** *The Cesàro mean $\sigma$ of an even multiplier $\lambda$ on $L^1(\mathbf{R})$ is also a multiplier on $L^1(\mathbf{R})$. Furthermore, $\sigma$ is a Fourier transform if and only if $\mu(x)$ is continuous at $x = 0$, where $\mu$ is the function of bounded variation associated with $\lambda$ according to* (1.2).

Since any multiplier can be written as the sum of an even and an odd multiplier, from Theorems 1 and 2 we conclude the following

**Corollary 3.** *The Cesàro mean of a multiplier on $L^1(\mathbf{R})$ is also a multiplier on $L^1(\mathbf{R})$.*

## 3. Proofs of Theorems 1 and 2

We begin with an auxiliary result on Lebesgue-Stieltjes integrals.

**Lemma.** *If a function $\nu$ is continuous and $\mu$ is nondecreasing on the interval $[0, \infty)$ with $\mu(\infty) < \infty$, then*

$$\lim_{\delta \downarrow 0} \int_\delta^\infty \nu(t)\, d\mu(t) = \int_0^\infty \nu(t)\, d\mu(t) - \nu(0)\{\mu(+0) - \mu(0)\}. \tag{3.1}$$

*Proof.* Consider an approximating sum for the integral on the left-hand side of (3.1):

$$\nu(\delta)\{\mu(\delta_1) - \mu(\delta)\} + \nu(\delta_1)\{\mu(\delta_2) - \mu(\delta_1)\} + \cdots \tag{3.2}$$

where $0 < \delta < \delta_1 < \delta_2 < \cdots$ is a partition of the interval $[\delta, \infty)$. If we add the extra term $\nu(0)\{\mu(\delta) - \mu(0)\}$ to (3.2), then we obtain an approximating sum for the integral on the right-hand side of (3.1). Now letting $\delta$ tend to 0 and at the same time making the partition finer and finer, we conclude that

$$\lim_{\delta \downarrow 0} \left\{ \nu(0)\{\mu(\delta) - \mu(0)\} + \int_\delta^\infty \nu(t)\, d\mu(t) \right\} = \int_0^\infty \nu(t)\, d\mu(t).$$

This is equivalent to (3.1) to be proved.

*Proof of Theorem* 1. First we prove that $\sigma$ is the Fourier transform of an odd function in $L^1(\mathbf{R})$. Indeed, the function in question is defined by

$$h(x) := \begin{cases} \int_x^\infty \frac{d\mu(\xi)}{\xi} & \text{if } x > 0, \\ -h(-x) & \text{if } x < 0, \end{cases} \tag{3.3}$$

and let $h(0) = 0$. Clearly, $h(x)$ is odd.

We show that $h \in L^1(\mathbf{R})$. By Fubini's theorem,

$$\begin{aligned} \int_0^\infty |h(x)|\, dx &\le \int_0^\infty dx \int_x^\infty \frac{d|\mu|(\xi)|}{\xi} \\ &= \int_0^\infty \frac{d|\mu|(\xi)}{\xi} \int_0^\xi dx = \int_0^\infty d|\mu|(\xi) < \infty. \end{aligned} \tag{3.4}$$

Here and in the sequel, by $|\mu|(x)$ we denote the total variation of $\mu$ over $(-\infty, x]$.

Second we prove that $\sigma$ is the Fourier transform of $h$. By Fubini's theorem and the lemma,

(3.5)

$$\begin{aligned} \hat{h}(t) &= -i\sqrt{\frac{2}{\pi}} \int_0^\infty h(x) \sin tx\, dx = -i\sqrt{\frac{2}{\pi}} \int_0^\infty \left\{ \int_x^\infty \frac{d\mu(\xi)}{\xi} \right\} \sin tx\, dx \\ &= -i\sqrt{\frac{2}{\pi}} \int_0^\infty \left\{ \int_0^\xi \frac{\sin tx}{\xi}\, dx \right\} d\mu(\xi) = -i\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1-\cos t\xi}{t\xi}\, d\mu(\xi). \end{aligned}$$

On the other hand,

$$\begin{aligned} \sigma(t) &= -i\sqrt{\frac{2}{\pi}} \frac{1}{t} \int_0^t \left\{ \int_0^\infty \sin \xi x\, d\mu(x) \right\} d\xi \\ &= -i\sqrt{\frac{2}{\pi}} \frac{1}{t} \int_0^\infty \left\{ \int_0^t \sin x\xi\, d\xi \right\} d\mu(x) \\ &= -i\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1-\cos tx}{tx}\, d\mu(x). \end{aligned} \tag{3.6}$$

Comparing (3.5) and (3.6) shows that $\sigma(t) = \hat{h}(t)$, $t \in \mathbf{R}$.

Third we prove that $\tilde{h}$, the Hilbert transform of $h$, also belongs to $L^1(\mathbf{R})$. Without loss of generality, we may assume that $\mu$ is nondecreasing on $\mathbf{R}$. By definition,

$$\begin{aligned} \pi\tilde{h}(x) &= \lim_{\delta \downarrow 0} \int_\delta^\infty \frac{h(x-t) - h(x+t)}{t}\, dt \\ &= \lim_{\delta \downarrow 0} \int_\delta^{x/2} + \int_{x/2}^\infty =: H_1(x) + H_2(x), \qquad \text{say}. \end{aligned} \tag{3.7}$$

Clearly, $H_1(x) \ge 0$ and $H_2(x) \ge 0$ for $x > 0$.

By (3.3)

$$H_1(x) = \lim_{\delta \downarrow 0} \int_0^{x/2} \frac{dt}{t} \int_{x-t}^{x+t} \frac{d\mu(\xi)}{\xi}. \tag{3.8}$$

Let $\delta > 0$ be fixed. By Fubini's theorem,

$$\begin{aligned}\int_{2\delta}^{\infty} dx \int_{\delta}^{x/2} \frac{dt}{t} \int_{x-t}^{x+t} \frac{d\mu(\xi)}{\xi} &= \int_{\delta}^{\infty} \frac{dt}{t} \int_{2t}^{\infty} dx \int_{x-t}^{x+t} \frac{d\mu(\xi)}{\xi} \\ &= \int_{\delta}^{\infty} \frac{dt}{t} \int_{t}^{\infty} \frac{d\mu(\xi)}{\xi} \int_{\max(\xi-t,\,2t)}^{\xi+t} dx \\ &\le 2\int_{\delta}^{\infty} dt \int_{t}^{\infty} \frac{d\mu(\xi)}{\xi} = 2\int_{\delta}^{\infty} \frac{d\mu(\xi)}{\xi} \int_{\delta}^{\xi} dt.\end{aligned}$$

By Fatou's lemma, $H_1 \in L^1(\mathbf{R})$ and

$$\int_0^{\infty} H_1(x)\, dx \le 2\{\mu(\infty) - \mu(+0)\}. \tag{3.9}$$

Now we consider $H_2$, which we decompose as

$$H_2(x) = -\int_{x/2}^{\infty} \frac{h(x+t)}{t}\, dt + \int_{x/2}^{\infty} \frac{h(x-t)}{t}\, dt =: \beta_1(x) + \beta_2(x), \quad \text{say}. \tag{3.10}$$

By (3.4) it is not hard to check that

(3.11)

$$\begin{aligned}\int_0^{\infty} |\beta_1(x)|\, dx &= \int_0^{\infty} dx \int_{3x/2}^{\infty} \frac{h(u)}{u-x}\, du \\ &= \int_0^{\infty} h(u)\, du \int_0^{2u/3} \frac{dx}{u-x} = \int_0^{\infty} h(u)\, du \int_{u/3}^{u} \frac{dt}{t} < \infty.\end{aligned}$$

By simple substitutions and manipulations,

(3.12)

$$\begin{aligned}\beta_2(x) &= \left\{\int_{x/2}^{x} + \int_{x}^{\infty}\right\} \frac{h(x-t)}{t}\, dt \\ &= \int_{x/2}^{x} \frac{dt}{t} \int_{x-t}^{\infty} \frac{d\mu(\xi)}{\xi} - \int_{x}^{\infty} \frac{dt}{t} \int_{t-x}^{\infty} \frac{d\mu(\xi)}{\xi} \\ &= \int_0^{\infty} \frac{d\mu(\xi)}{\xi} \int_{\max(x-\xi,\,x/2)}^{x} \frac{dt}{t} - \int_0^{\infty} \frac{d\mu(\xi)}{\xi} \int_{x}^{x+\xi} \frac{dt}{t} \\ &= \int_0^{x/2} \ln\left(\frac{x}{x-\xi}\right) \frac{d\mu(\xi)}{\xi} + (\ln 2) \int_{x/2}^{\infty} \frac{d\mu(\xi)}{\xi} - \int_0^{\infty} \ln\left(\frac{x+\xi}{x}\right) \frac{d\mu(\xi)}{\xi} \\ &= \int_0^{x/2} \ln\left(\frac{x^2}{x^2-\xi^2}\right) \frac{d\mu(\xi)}{\xi} + (\ln 2) \int_{x/2}^{\infty} \frac{d\mu(\xi)}{\xi} - \int_{x/2}^{\infty} \ln\left(\frac{x+\xi}{x}\right) \frac{d\mu(\xi)}{\xi} \\ &=: \beta_3(x) + \beta_4(x) + \beta_5(x), \quad \text{say}.\end{aligned}$$

We begin with $\beta_3$. Making use of Fubini's theorem, the substitution $x = \xi t$, and the inequality

$$\ln(1+u) \le u, \qquad u \ge 0, \tag{3.13}$$

we find

$$\begin{aligned}
\int_0^\infty |\beta_3(x)|\,dx &= \int_0^\infty \frac{d\mu(\xi)}{\xi}\int_{2\xi}^\infty \ln\left(\frac{x^2}{x^2-\xi^2}\right)dx \\
&= \int_0^\infty d\mu(\xi)\int_2^\infty \ln\left(\frac{t^2}{t^2-1}\right)dt \qquad (3.14)\\
&\le \int_0^\infty d\mu(\xi)\int_2^\infty \frac{1}{t^2-1}\,dt < \infty .
\end{aligned}$$

Since

$$\beta_4(x) = (\ln 2)h(x/2), \qquad x>0, \tag{3.15}$$

and $h$ is an odd function, by (3.4) we see that $\beta_4 \in L^1(\mathbf{R})$. Finally, applying Fubini's theorem and the substitutions $x=\xi t$ and $u=\frac{1}{t}$, in turn we get

$$\begin{aligned}
\int_0^\infty |\beta_5(x)|\,dx &= \int_0^\infty \frac{d\mu(\xi)}{\xi}\int_0^{2\xi} \ln\left(\frac{x+\xi}{x}\right)dx \\
&= \int_0^\infty d\mu(\xi)\int_0^2 \ln\left(\frac{t+1}{t}\right)dt \qquad (3.16)\\
&= \int_0^\infty d\mu(\xi)\int_{1/2}^\infty \frac{\ln(1+u)}{u^2}\,du < \infty .
\end{aligned}$$

Combining (3.7), (3.9)–(3.12), (3.14)–(3.16), and the fact that $\tilde{h}$ is an even function (since $h$ is odd), we conclude that $\tilde{h} \in L^1(\mathbf{R})$. This completes the proof of Theorem 1.

*Proof of Theorem* 2. First we prove that $\sigma$ is the Fourier transform of an even function, up to an additive constant. In particular, it follows that $\sigma$ is a multiplier on $L^1(\mathbf{R})$. In fact, the function in question is defined by

$$g(x) := \begin{cases} \displaystyle\int_x^\infty \frac{d\mu(\xi)}{\xi} & \text{if } x>0, \\ g(-x) & \text{if } x<0 \end{cases} \tag{3.17}$$

(cf. (3.3)). Clearly, $g$ is even. Similarly to (3.4), we deduce that $g$ is in $L^1(\mathbf{R})$.

Second we consider $\hat{g}$, the Fourier transform of $g$,

$$\hat{g}(t) = \sqrt{\frac{2}{\pi}}\int_0^\infty g(x)\cos tx\,dx. \tag{3.18}$$

Again fix $\delta>0$. By (3.17) and Fubini's theorem,

(3.19)

$$\begin{aligned}
\int_\delta^\infty g(x)\cos tx\,dx &= \int_\delta^\infty \cos tx\,dx\int_x^\infty \frac{d\mu(\xi)}{\xi} = \int_\delta^\infty \frac{d\mu(\xi)}{\xi}\int_\delta^\xi \cos tx\,dx \\
&= \int_\delta^\infty \frac{\sin t\xi}{t\xi}\,d\mu(\xi) - \frac{\sin t\delta}{t}\int_\delta^\infty \frac{d\mu(\xi)}{\xi}.
\end{aligned}$$

Without loss of generality, we may assume that $\mu$ is nondecreasing on $(0,\infty)$. Then for $0<\delta<\delta_1$ we may write that

$$\begin{aligned}
0 \le \frac{\sin t\delta}{t}\int_\delta^\infty \frac{d\mu(\xi)}{\xi} &\le \delta\left\{\int_\delta^{\delta_1} + \int_{\delta_1}^\infty\right\}\frac{d\mu(\xi)}{\xi} \\
&\le \{\mu(\delta_1)-\mu(\delta)\} + o(1) \to 0 \quad \text{as } \delta\to 0.
\end{aligned}$$

Thus, from (3.18), (3.19), and the lemma it follows that

$$\begin{aligned}\hat{g}(t) &= \sqrt{\frac{2}{\pi}} \lim_{\delta \downarrow 0} \int_\delta^\infty \frac{\sin t\xi}{t\xi}\, d\mu(\xi) \\ &= \sqrt{\frac{2}{\pi}} \left\{ \int_0^\infty \frac{\sin t\xi}{t\xi}\, d\mu(\xi) - \{\mu(+0) - \mu(0)\} \right\}.\end{aligned} \tag{3.20}$$

On the other hand, the Cesàro mean $\sigma$ of $g$ defined in (2.2) can be rewritten as

$$\begin{aligned}\sigma(t) &= \sqrt{\frac{2}{\pi}} \frac{1}{t} \int_0^t d\xi \int_0^\infty \cos \xi x \, d\mu(x) \\ &= \sqrt{\frac{2}{\pi}} \frac{1}{t} \int_0^\infty d\mu(x) \int_0^t \cos x\xi \, d\xi = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin tx}{tx}\, d\mu(x).\end{aligned} \tag{3.21}$$

Combining (3.20) and (3.21) gives

$$\sigma(t) = \hat{g}(t) + \sqrt{\frac{2}{\pi}} \{\mu(+0) - \mu(0)\}, \tag{3.22}$$

whence the necessity and sufficiency part in Theorem 2 is obvious. This completes the proof of Theorem 2.

## 4. Related results

(i) We point out that in the case of an even multiplier $\lambda$ on $L^1(\mathbf{R})$ we cannot state that the function $g$ defined in (3.17) belongs to $H^1(\mathbf{R})$. Nevertheless, we are able to deduce a Hardy type inequality for the Hilbert transform $\tilde{g}$. More exactly, the following is true.

**Theorem 3.** *If $\lambda$ is an even multiplier on $L^1(\mathbf{R})$, $\mu$ is the function of bounded variation associated with $\lambda$ according to* (1.2), *$\sigma$ is the Cesàro mean of $\lambda$ defined by* (2.2), *and $g$ is defined by* (3.17), *then*

$$\int_0^\infty \left| \pi \tilde{g}(x) - \frac{2}{x} \int_0^{x/2} d\mu(\xi) \right| dx \le C \int_0^\infty d|\mu|(\xi), \tag{4.1}$$

*where $C$ is an absolute constant.*

The next corollary is an immediate consequence of Theorem 3.

**Corollary 4.** *If $\lambda$ is an even multiplier on $L^1(\mathbf{R})$ and $\mu$ is the function of bounded variation associated with $\lambda$ according to* (1.2), *then the function $g$ defined by* (3.17) *belongs to $H^1(\mathbf{R})$ if and only if*

$$\int_0^\infty \left| \frac{1}{x} \int_0^x d\mu(\xi) \right| dx < \infty. \tag{4.2}$$

*Proof of Theorem* 3. Similarly to the case of an odd multiplier on $L^1(\mathbf{R})$ (cf. (3.7)–(3.12)), we can represent the Hilbert transform $\tilde{g}$ by

$$\pi \tilde{g}(x) = G_1(x) + \beta_1(x) + \gamma_3(x) + \beta_4(x) + \beta_5(x), \tag{4.3}$$

where

$$G_1(x) := \lim_{\delta\downarrow 0}\int_\delta^{x/2}\frac{dt}{t}\int_{x-t}^{x+t}\frac{d\mu(\xi)}{\xi},$$

$$\beta_1 := -\int_{x/2}^{\infty}\frac{g(x+t)}{t}\,dt, \qquad \gamma_3(x) := \int_0^{x/2}\ln\left(\frac{x+\xi}{x-\xi}\right)\frac{d\mu(\xi)}{\xi}$$

(there is an essential difference between $\beta_3$ and $\gamma_3$, due to the evenness of $g$, in contrast to the oddness of $h$),

$$\beta_4(x) := (\ln 2)\int_{x/2}^{\infty}\frac{d\mu(\xi)}{\xi}, \qquad \beta_5(x) := \int_{x/2}^{\infty}\ln\left(\frac{x+\xi}{x}\right)\frac{d\mu(\xi)}{\xi}.$$

Repeating analogous reasonings as in the proof of Theorem 1, we conclude that the functions $G_1$, $\beta_1$, $\beta_4$, and $\beta_5$ are in $L^1(\mathbf{R})$. Thus, from (4.3) it follows that

$$\int_0^\infty |\pi\tilde{g}(x) - \gamma_3(x)|\,dx \le C\int_0^\infty d|\mu|(\xi). \tag{4.4}$$

On the basis of (3.13) and the reversed inequality $\ln(1+u) \ge u - u^2/2$, $u \ge 0$, we see that the inequality

$$0 \le \frac{2}{x-\xi} - \frac{1}{\xi}\ln\left(\frac{x+\xi}{x-\xi}\right) \le \frac{4\xi}{(x-\xi)^2}$$

holds true for $0 \le \xi \le \frac{x}{2}$. Consequently, by setting

$$\gamma_4(x) := \int_0^{x/2}\frac{d\mu(\xi)}{x-\xi},$$

from (4.4) it follows that

$$\begin{aligned}\int_0^\infty |\pi\tilde{g}(x) - 2\gamma_4(x)|\,dx &\le \int_0^\infty d|\mu|(\xi)\int_{2\xi}^\infty\left|\frac{1}{\xi}\ln\left(\frac{x+\xi}{x-\xi}\right) - \frac{2}{x-\xi}\right|dx \\ &\le \int_0^\infty d|\mu|(\xi)\int_{2\xi}^\infty\frac{4\xi}{(x-\xi)^2}\,dx = 4\int_0^\infty d|\mu|(\xi).\end{aligned} \tag{4.5}$$

Analogously, by Fubini's theorem,

$$\begin{aligned}\int_0^\infty\left|\gamma_4(x) - \frac{1}{x}\int_0^{x/2}d\mu(\xi)\right|dx &= \int_0^\infty\left|\int_0^{x/2}\frac{\xi}{x(x-\xi)}\,d\mu(\xi)\right|dx \\ &\le \int_0^\infty d|\mu|(\xi)\int_{2\xi}^\infty\frac{\xi}{x(x-\xi)}\,dx = \int_2^\infty\frac{dt}{t(t-1)}\int_0^\infty d|\mu|(\xi).\end{aligned} \tag{4.6}$$

Combining (4.5) and (4.6) results in (4.1). This completes the proof of Theorem 3.

(ii) Adopting the relevant parts in the proofs of Theorems 1 and 2 with appropriate modifications, we can achieve an analogous result on the Cesàro mean $\sigma$ of the Fourier transform $\hat{f}$ defined by

$$\sigma(t) := \frac{1}{t}\int_0^t \hat{f}(\xi)\,d(\xi), \qquad t \in \mathbf{R},\ t \ne 0 \tag{4.7}$$

(cf. definition (2.2)), where $f \in L^1(\mathbf{R})$ is a given function.

**Corollary 5.** *If* $f \in L^1(\mathbf{R})$ *and* $\sigma$ *is defined by* (4.7), *then* $\sigma$ *is also the Fourier transform of a function in* $L^1(\mathbf{R})$.

(iii) On closing, we note that Hardy [4], Goes [3], and Georgakis [2] achieved similar results on the arithmetic means of the Fourier-Stieltjes coefficients of periodic functions $\mu$ of bounded variation on the torus $[-\pi, \pi]$.

## Note added in proof

After submitting this paper, it came to our knowledge that Georgakis [8] has proved our Theorems 1–2 and Corollaries 3–4 in a more general setting. However, our estimates in the proofs are different from those in his proofs.

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Department of Mathematics, University of Szeged, Bolyai Institute, Aradi Vértanúk Tere 1, 6720 Szeged, Hungary

*E-mail address*, Ferenc Móricz: h6179mor@ella.hu