ON SMOOTH AND ANALYTIC DISKS IN $C^2$
WITH COMMON BOUNDARY

KANG-TAE KIM

Abstract. We construct explicitly a real analytic embedded real two-dimensional disk in $C^2$ totally real except at exactly one elliptic complex tangent point, which shares the common boundary with an analytic disk in the same $C^2$, but does not contain this analytic disk in its envelope of holomorphy. The same proof further yields an explicit example of a holomorphic re-embedding of the standard two-sphere into $C^2$ in such a way that the new embedding shows some exceptional properties: It bounds a real three-dimensional Levi flat cell in $C^2$ foliated by analytic disks, which is not polynomially convex. In particular, this new embedding of the standard two-sphere cannot be a subset of any compact strongly pseudoconvex surface in $C^2$ or a subset of any strongly pseudoconvex graph in $C^2$ in the sense of Bedford and Gaveau.

The main results of this article are the following two theorems.

Theorem 1. There exists a $C^\infty$ smooth, real two-dimensional smooth disk $M$ in $C^2$ which is totally real except at exactly one elliptic complex tangent point such that:

(a) $\partial M = \{(e^{i\theta}, 0) \in C^2 | \theta \in R\}$ and

(b) there exists a contractible bounded strongly pseudoconvex domain $\Omega$ with a $C^\infty$ smooth boundary that contains $M$ but does not contain the complex analytic disk $\{(z, 0) \in C^2 | |z| < 1\}$. In fact, $\Omega$ can be chosen such that it is biholomorphic to a strongly convex domain.

Theorem 2. There exists an embedded real analytic sphere $\Sigma \subset C^2$ of real dimension two which bounds a real three-dimensional Levi flat surface $\Gamma \subset C^2$ and admits a complex analytic disk $A$ such that $\partial A \subset \Sigma = \partial \Gamma$ but $A \not\subset \Gamma$. Furthermore, $\Sigma$ admits exactly two complex tangent and, hence, elliptic, points.

In the above and throughout this paper, by an analytic disk we mean (the image of) an injective holomorphic embedding of the open unit disk in $C$ into $C^2$ which extends continuously across the boundary. The smooth real disk is...
(the image of) a smooth embedding of the open unit disk in \( \mathbb{C} \) into \( \mathbb{C}^2 \) which admits a continuous extension across the boundary.

The definition of the term elliptic (complex tangent) point is due to Bishop [4]. The elliptic points on a real surface are special because they initiate a family of analytic disks with boundaries in the real surface forming a Levi flat surface in its neighborhood ([1, 2, 4, 6, 7, 10, 11], et al.).

**Remarks.** (1) Notice that our results are in contrast to the theorems by Bedford and Gaveau [2], Bedford and Klingenberg [3], Gromov [7], and others concerning global envelopes of holomorphy.

(2) Our results, however, do not answer the question raised by Forstnerič in [5] since the smooth disk \( \mathcal{M} \) and the analytic disk \( \mathcal{A} \) in Theorem 1 meet at a common interior point. After the first version of this article was written, Forstnerič informed us that the methods presented in [5] can also prove our main result in a somewhat implicit way. However, our construction in this article is still of separate interest because it is more elementary and explicit and does not depend on the methods of [5].

**Proof of Theorems 1 and 2.** Denote \( E^3 := \{(z_1, z_2) \in \mathbb{C}^2 \mid \text{Im} z_2 = 0\} \) and write the coordinate system of \( \mathbb{C}^2 = \mathbb{R}^4 \) as

\[
z_1 = x_1 + \sqrt{-1} y_1 := (x_1, y_1), \quad z_2 = x_2 + \sqrt{-1} y_2 := (x_2, y_2).
\]

Consequently, \( (x_1, y_1; x_2) \) denotes a real coordinate system for \( E^3 \). Consider the plane \( P \) generated by \( x_2 \) and \( y_1 \) axes and the ellipse

\[
\left(\frac{x_2}{1 + \eta}\right)^2 + \left(\frac{y_1}{\eta}\right)^2 = \frac{1}{\lambda} \quad (\lambda \geq 1)
\]

rotated by a small angle \( \alpha > 0 \). Call this rotated ellipse \( Q \). Then \( Q \) is represented by

\[
\left(\frac{cx_2 - sy_1}{1 + \eta}\right)^2 + \left(\frac{sx_2 + cy_1}{\eta}\right)^2 = \frac{1}{\lambda}
\]

where

(a) \( c = \cos \alpha, \ s = \sin \alpha \) and

(b) \( \lambda > 1 \) is chosen so that the \( x_2 \)-intercepts of \( Q \) are \( \pm 1 \).

Denote by \( L_c \) the line segment in \( P \) which is part of the line \( y_1 = c \) cut by and contained in the interior of the ellipse \( Q \). For each \( t \) choose the real numbers \( c(t) \) and \( r(t) \) such that

(c) the point \( (y_1, x_2) = (t, c(t)) \) is the midpoint of \( L_t \) and

(d) \( 2r(t) = \text{length}(L_t) \).

Then we define the ellipsoidal surface \( \mathcal{E} \) contained in \( E^3 \) defined by the equation \( x_1^2 + (x_2 - c(y_1))^2 = r(y_1)^2 \). It is easy to check that \( \mathcal{E} \) is in fact a smooth surface, and that it has exactly two points where the tangent planes are parallel to the \( x_1y_1 \)-plane. Hence, when we consider \( \mathcal{E} \) embedded in \( \mathbb{C}^2 \) by inclusion map, these two are still the only complex tangent points on \( \mathcal{E} \). It follows directly by construction that these points are in fact elliptic.

Moreover, our construction also shows that one elliptic complex tangent point belongs to the half space \( \{y_1 > 0\} \), whereas the other one is in the other open half space \( \{y_1 < 0\} \). Thus we let

\[
S := \mathcal{E} \cap \{(x_1, y_1, x_2) \in E^3 \mid y_1 > 0\}.
\]
Clearly, \( \partial S = \{(x_1, 0, x_2) \in E^3 \mid x_1^2 + x_2^2 = 1\} \).

Now consider the mapping \( \Phi: \mathbb{C}^2 \to \mathbb{C}^2 \) defined by
\[
\Phi(z, w) = (z, (1 + i)w - zw^2 - iz^2w^3)
\]
which was first introduced by Wermer in [12]. Let \( \tau(z, w) = (z + iw, z - iw) \). Notice that there exists an open subset \( U \) of \( \mathbb{C}^2 \) containing \( X := \{(x_1, 0, x_2, 0) \in \mathbb{C}^2 \mid x_1^2 + x_2^2 < 1\} \) such that \( \bar{\Phi} := \Phi \circ \tau \) restricted on \( U \) is a biholomorphism into. Without loss of generality, we may assume that \( U \) is biholomorphic to the bidisk.

Notice that by choosing a sufficiently small \( \eta > 0 \) in the above we can have \( \mathcal{E} \) contained in \( U_\delta \). Also notice that \( \bar{\Phi} \) maps \( \partial S \) onto the circle \( \Gamma = \{(z, 0) \in \mathbb{C}^2 \mid |z| = 1\} \). Thus, we let \( M = \bar{\Phi}(S) \). \( M \) is now contained in the domain \( D = \bar{\Phi}(U_\delta) \). Notice that we can also have that \( M \) is contained in the strongly pseudoconvex domain \( \Omega \) biholomorphic to a strongly convex domain by using a smooth exhaustion of the bidisk by strongly convex domains. This proves Theorem 1. Theorem 2 follows from letting \( \Sigma = \bar{\Phi}(\mathcal{E}) \) and letting \( \Gamma \) be the image under \( \bar{\Phi} \) of the obvious real three-dimensional ball bounded by \( \mathcal{E} \) in \( E^3 \subset \mathbb{C}^2 \).

**Remarks on the interior intersection of \( A \) and \( M \).** Recall that the preimage under the map \( \Phi \) of the flat analytic disk \( A \) is contained in the set represented by
\[
(1 + i)w - zw^2 - iz^2w^3 = w(1 - zw)(1 + i + izw) = 0.
\]
With a sufficiently small \( \delta \) for the domain \( D \) in the previous section, the variety defined by \( 1 + i + izw = 0 \) does not meet \( D \). Therefore, we must only look at \( w(1 - zw) = 0 \). Consequently, the preimage of \( A \) under the map \( \bar{\Phi} = \Phi \circ \tau \) lies in the algebraic variety, say \( V \), in \( \mathbb{C}^2 \) defined by \( (z_1 - iz_2)(1 - z_1^2 - z_2^2) = 0 \). We now restrict our attention to the real three-dimensional subspace \( F^3 \) given by \( \text{Im} z_2 = 0 \). It is easy to see that any surface in \( E^3 \) homeomorphic to the closed unit disk in \( \mathbb{C} \) with the boundary \( \{x_1^2 + x_2^2 = 1, \, y_1 = 0\} \) must intersect the line defined by \( x_1 = 0, \, x_2 + y_1 = 0 \). Moreover, the disks \( A \) and \( M \) we constructed in the preceding section share exactly two interior intersection points.

**Acknowledgments**

The author thanks John Anderson, Eric Bedford, and John Wermer for many inspiring discussions. He is also grateful to the referee for numerous helpful suggestions for the revision of this article, especially on the formulation of Theorem 2.

**References**


**Department of Mathematics, Brown University, Providence, Rhode Island 02912**

*E-mail address: ktkim@gauss.math.brown.edu*