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REDUCIBLE HILBERT SCHEME OF SMOOTH CURVES WITH POSITIVE BRILL-NOETHER NUMBER

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ABSTRACT. In this paper we demonstrate various reducible examples of the scheme $\mathcal{I}'_{d,g,r}$ of smooth curves of degee d and genus g in $\mathbb{P}^r$ with positive Brill-Noether number. An example of a reducible $\mathcal{I}'_{d,g,r}$ with positive $\rho(d, g, r)$, namely, the example $\mathcal{I}'_{2g-8,g,g-8}$, has been known to some people and seems to have first appeared in the literature in Eisenbud and Harris, *Irreducibility of some families of linear series with Brill-Noether number* -1, Ann. Sci. École Norm. Sup. (4) **22** (1989), 33–53. The purpose of this paper is to add a wider class of examples to the list of such reducible examples by using general k-gonal curves. We also show that $\mathcal{I}'_{d,g,r}$ is irreducible for the range of $d \geq 2g-7$ and $g-d+r \leq 0$.

INTRODUCTION

In [S] Severi has asserted with an incomplete proof that the subscheme $\mathcal{I}'_{d,g,r}$ which is the union of the irreducible components of the Hilbert scheme $\mathcal{H}_{d,g,r}$ whose general points correspond to smooth, irreducible, and nondegenerate curves of degree d and genus g in $\mathbb{P}^r$ is irreducible if $d \geq g+r$. Also in [H] it has been conjectured that $\mathcal{I}'_{d,g,r}$ is irreducible if the Brill-Noether number $\rho(d, g, r) := g-(r+1)(g-d+r)$ is positive.

In this paper we demonstrate various reducible examples of the subscheme $\mathcal{I}'_{d,g,r}$ with positive Brill-Noether number. Indeed an example of a reducible $\mathcal{I}'_{d,g,r}$ with positive $\rho(d, g, r)$, namely the example $\mathcal{I}'_{2g-8,g,g-8}$ (or other variations of it), has been known to some people (including the author), but it seems to have first appeared in the literature in [EH]. The purpose of this paper is to add a wider class of examples to the list of such reducible examples by using general k-gonal curves. We also show that $\mathcal{I}'_{d,g,r}$ is irreducible for the range of $d \geq 2g-7$ and $g-d+r \leq 0$. Throughout we will be working over the field of complex numbers.

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1. Terminology, notation, and some preliminary results

We first recall that, given nonnegative integers r, d, for every point p of the moduli space $\mathscr{M}_g$ of smooth curves of genus g and any sufficiently small connected neighborhood U of p, there are a smooth connected variety $\mathscr{M}$, a finite ramified covering

$$h : \mathscr{M} \to U,$$

and two varieties, proper over $\mathscr{M}$,

$$\xi : \mathscr{C} \to \mathscr{M}, \qquad \pi : \mathscr{G}^r_d \to \mathscr{M}$$

with the following properties:

(1) $\mathscr{C}$ is a universal curve over $\mathscr{M}$, i.e., for every $p \in \mathscr{M}$, $\xi^{-1}(p)$ is a smooth curve of genus g whose isomorphism class is $h(p)$.

(2) $\mathscr{G}^r_d$ parametrizes pairs $(p, \mathscr{D})$, where $p \in \mathscr{M}$ and $\mathscr{D}$ is a linear system (possibly incomplete) of degree d and of dimension r, which is denoted by g^r_d, on $C = \xi^{-1}(p)$.

Let $\mathscr{G}$ be the union of irreducible components of $\mathscr{G}^r_d$ whose general element corresponds to pairs $(p, \mathscr{D})$ such that $\mathscr{D}$ is a very ample linear system on $\xi^{-1}(p) = C$, i.e., $\mathscr{D}$ induces an embedding of C into $\mathbb{P}^r$.

In order to show the irreducibility of $\mathscr{I}'_{d,g,r}$, it is sufficient to demonstrate the irreducibility of $\mathscr{G}$ since the open subset of $\mathscr{I}'_{d,g,r}$ consisting of points corresponding to smooth curves is a $\mathbb{P}GL_{r+1}$ bundle over an open subset of $\mathscr{G}$. Also we will utilize the following fact which is basic in the theory for our purposes; see [AC1] or [H] for detailed discussion and proof.

Proposition 1.1. *There exists a unique component $\mathscr{G}_0$ of $\mathscr{G}$ which dominates $\mathscr{M}$ (or $\mathscr{M}_g$) if the Brill-Noether number $\rho(d, g, r)$ is positive. Furthermore in this case, for any possible component $\mathscr{G}'$ of $\mathscr{G}$ other than $\mathscr{G}_0$, a general element $(p, \mathscr{D})$ of $\mathscr{G}'$ is such that $\mathscr{D}$ is a special linear system on $C = \xi^{-1}(p)$.*

Remark 1.2. In the Brill-Noether range, i.e., in the range $\rho(d, g, r) > 0$, we call the unique component $\mathscr{G}_0$ of $\mathscr{G}$ which dominates $\mathscr{M}$ the principal component. We call other possible components exceptional components.

The following facts will also turn out to be useful for our purposes; see [AC2] for the proof.

Proposition 1.3. (i) *Any component of $\mathscr{G}^r_d$ has dimension at least* $3g - 3 + \rho(d, g, r)$.

(ii) *Suppose $g > 0$, and let X be a component of $\mathscr{G}^2_d$ whose general element $(p, \mathscr{D})$ is such that $\mathscr{D}$ is a linear system on $C = \xi^{-1}(p)$ which is not composed with an involution. Then*

$$\dim X = 3g - 3 + \rho(d, g, 2) = 3d + g - 9.$$

(iii) *The variety $\mathscr{G}^1_d$ is smooth of dimension*

$$\rho(d, g, 1) + \dim \mathscr{M}_g.$$

By using Proposition 1.3(ii), one can prove the following fact regarding a subvariety of $\mathscr{G}^r_d$ consisting of birationally very ample linear series; see [KK].

Proposition 1.4. *Let $\mathscr{W}$ be an irreducible closed subvariety of $\mathscr{G}^r_d$, $r \geq 2$, whose general element $(p, \mathscr{D})$ is such that $\mathscr{D}$ is complete, special, and birationally very ample on $C = \xi^{-1}(p)$. Then*

$$\dim \mathscr{W} \leq 3d + g - 4r - 1.$$

Corollary 1.5. *Whenever*

$$2g + 1 + \frac{3-3g}{r} < d \leq 2g - 2 \qquad (r \geq 3),$$

$\mathscr{G}$ (and hence $\mathscr{I}'_{d,g,r}$) is irreducible with the expected dimension $3g - 3 + \rho(d, g, r)$.

Proof. Suppose there exists an exceptional component $\mathscr{G}'$ of $\mathscr{G}$. Since we are in the Brill-Noether range, by Proposition 1.1 there is an open set $\mathscr{V}$ of $\mathscr{G}'$ whose elements consist of pairs $(p, \mathscr{D})$ such that $\mathscr{D}$ is a special very ample linear system on $C = \xi^{-1}(p)$. Consider the map

$$\psi : \mathscr{V} \to \mathscr{G}^\alpha_d$$

defined by $\psi(p, \mathscr{D}) = (p, |D|)$, where $D \in \mathscr{D}$, $\alpha = \dim |D|$. Then by Proposition 1.4 and by noting the fact that the dimension of a fiber of ψ over a point in $\psi(\mathscr{V})$ is $\dim \mathbb{G}(r, \alpha)$, we have $\dim \mathscr{G}' = \dim \mathscr{V} \leq 3d + g - 4\alpha - 1 + (r+1)(\alpha - r) = 3d + g - 1 - r^2 - r + (r-3)\alpha$.

On the other hand, by Castelnuovo theory the largest possible α in case $d \geq g$ is $\frac{2d-g+1}{3}$. Thus the above inequality implies

$$\dim \mathscr{G}' \leq 3d + g - 1 - r^2 - r + (r-3)\frac{2d-g+1}{3} < 3g - 3 + \rho(d, g, r),$$

which is contradictory to Proposition 1.3(i). □

Remark 1.6. (i) Corollary 1.5 was also known to Ein; see [E1]. He later gave a wider range of d, g, r for which $\mathscr{I}'_{d,g,r}$ is irreducible when $r \geq 5$; see [E2].

(ii) It is quite easy to show that, in case $d \geq 2g - 1$, $\mathscr{I}'_{d,g,r}$ is empty if $r > d - g$ and is irreducible if $r \leq d - g$; see [H, p. 61].

2. Irreducibility of $\mathscr{I}'_{d,g,r}$ with large d

Theorem 2.1. *$\mathscr{I}'_{d,g,r}$ is irreducible for $d \geq 2g - 7$ and $g + r \leq d$, $r \geq 3$.*

Proof. For the case $d \geq 2g-2$, it is a consequence of Corollary 1.5 and Remark 1.6(ii). For the case $2g - 7 \leq d \leq 2g - 3$, we proceed as follows. Let $d = 2g - 2 - k$, where $1 \leq k \leq 5$. Suppose there exists an exceptional component $\mathscr{G}'$ of $\mathscr{G}$. Then by Proposition 1.1, a general element $(p, \mathscr{D}) \in \mathscr{G}'$ is such that $\mathscr{D}$ is a special linear system on $C = \xi^{-1}(p)$, i.e., $\dim |\mathscr{D}| > d - g$. Let $\mathscr{V}$ be an open subset of $\mathscr{G}'$ consisting of elements $(p, \mathscr{D})$ with $\dim |\mathscr{D}| = \alpha > d - g$. Consider the map

$$\Psi : \mathscr{V} \to \mathscr{G}^{k+\alpha+1-g}_k$$

defined by $\Psi(p, \mathscr{D}) = (p, |K - D|)$, where $D \in \mathscr{D}$ and K is a canonical divisor on $C = \xi^{-1}(p)$. Then by noting the fact that the dimension of a fiber of Ψ over a point in $\mathscr{G}^{k+\alpha+1-g}_k$ is at most $\dim \mathbb{G}(r, \alpha)$, we have

$$\dim \mathscr{G}' = \dim \mathscr{V} \leq \dim \mathscr{G}^{k+\alpha+1-g}_k + (r+1)(\alpha - r). \tag{2.1.1}$$

By Clifford's theorem and the inequality $\alpha > d - g = g - 2 - k$, we have $g - 1 - k \leq \alpha \leq g - 1 - \frac{k}{2}$. Thus only the following pairs for (k, α)'s are possible:

(i) $\{(k, g-k-1); 1 \leq k \leq 5\}$;
(ii) $\{(k, g-k); 2 \leq k \leq 5\}$;
(iii) $\{(k, g-k+1); k = 4 \text{ or } 5\}$.

For case (i), by inequality (2.1.1) and the hypothesis $g + r \leq d$,

$$\dim \mathcal{G}' \leq \dim \mathcal{G}_k^0 + (r+1)(g-k-1-r) < 3g - 3 + \rho(2g-2-k, g, r),$$

which is a contradiction.

For case (ii), again by (2.1.1) and the hypothesis $g + r \leq d$, we have

$$\begin{aligned}\dim \mathcal{G}' &\leq \dim \mathcal{G}_k^1 + (r+1)(g-k-r)\\ &= 2g - 5 + 2k + (r+1)(g-k-r) < 3g - 3 + \rho(d, g, r),\end{aligned}$$

which is a contradiction.

For case (iii), suppose $(k, \alpha) = (4, g-3)$. Because $\alpha = \frac{d}{2}$, any $(p, \mathcal{D}) \in \mathcal{V}$ is such that $C = \xi^{-1}(p)$ is a hyperelliptic curve by Clifford's theorem. But this is a contradiction since a hyperelliptic curve cannot have a birationally very ample special linear system.

Suppose $(k, \alpha) = (5, g-4)$. Consider $\Psi(p, \mathcal{D}) = (p, |K - D|) \in \mathcal{G}_5^2$. If the complete $|K - D|$ has no base point, $|K - D|$ induces a birational map on $C = \xi^{-1}(p)$ and $g(C) \leq 6$, contrary to the hypothesis $g + r \leq d$ and $r \geq 3$. Thus $|K - D|$ has a base point, and there exists a g_4^2 on C whence C is a hyperelliptic curve by Clifford's theorem. Again this is a contradiction because there cannot exist a birationally very ample special linear system on a hyperelliptic curve. □

To demonstrate the reducibility of $\mathcal{I}'_{2g-8, g, g-8}$, we do need the following lemma whose elementary proof we omit here.

Lemma 2.2. *Let C be a trigonal curve of genus $g \geq 8$ with the trigonal pencil g_3^1. Then $|K - 2g_3^1|$ is very ample, and any g_6^2 is equal to $2g_3^1$.*

Theorem 2.3. (i) *For $r < \frac{2g-7}{3}$, $r \leq g-8$, and $r \geq 3$, $\mathcal{I}'_{2g-8, g, r}$ is irreducible.*

(ii) *For $\frac{2g-7}{3} \leq r \leq g - 8$ and $r \geq 3$, $\mathcal{I}'_{2g-8, g, r}$ is reducible with two components. Furthermore, a general element of the exceptional component is trigonal.*

Proof. We use all the notation used in the proof of Theorem 2.1. Let $\mathcal{G}'$ be an exceptional component of $\mathcal{G}$ and $\alpha = \dim |\mathcal{D}|$ for general $(p, \mathcal{D}) \in \mathcal{G}'$. By Clifford's theorem, we have $\alpha = g-7, g-6$, or $g-5$.

(i) If $\alpha = g-7$ or $g-6$, one can use inequality (2.1.1) and proceed exactly as in the previous theorem to show that these cases do not occur.

(ii) If $\alpha = g-5$, $|K - D| = g_6^2$, where $D \in \mathcal{D}$ for a general $(p, \mathcal{D}) \in \mathcal{G}'$. By the hypothesis $3 \leq r \leq g - 8$, the map induced by $|K - D|$ on $C = \xi^{-1}(p)$ is not birational. Instead, C may be either hyperelliptic, trigonal, or elliptic-hyperelliptic, but C cannot be hyperelliptic because a hyperelliptic curve cannot have a very ample special linear system. If C is elliptic-hyperelliptic, $|K - D| = g_6^2 = \phi^*(g_3^2)$, where ϕ is the map of degree 2 onto an elliptic curve E. Then $|\mathcal{D}| = |K - g_6^2| = g_{2g-8}^{g-5}$ is not even birationally very ample because

$|K - g_6^2 - P - Q| = g_{2g-10}^{g-6}$, where $P + Q = \phi^*(R)$, $R \in E$. Thus C cannot be elliptic-hyperelliptic.

If C is a trigonal curve, $|K - D| = g_6^2 = 2g_3^1$ and $|\mathscr{D}| = |D|$ is very ample by Lemma 2.2. Thus the only possible exceptional component of $\mathscr{G}$ may arise in this way; in other words, $\mathscr{V}$ surjects onto an open set of $\mathscr{M}_{g,3}^1$ if such $\mathscr{G}'$ exists. Hence

$$\dim \mathscr{G}' = \dim \mathscr{V} = \dim \mathscr{G}_3^1 + (r+1)(g-5-r) \geq 3g - 3 + \rho(d, g, r),$$

which proves the first half of the theorem.

On the other hand, suppose the above inequality holds, and let $\mathscr{W}$ be the closed subvariety of $\mathscr{G}_d^r$ whose general element $(p, \mathscr{D})$ is such that p corresponds to a trigonal curve and $\mathscr{D}$ is a general r-dimensional subspace of $|K - 2g_3^1|$ on $C = \xi^{-1}(p)$; i.e., $\mathscr{W}$ is just the locus in $\mathscr{G}_{2g-8}^r$ over trigonal curves. By the preceding discussion, $\mathscr{W}$ is indeed a component of $\mathscr{G}$ other than $\mathscr{G}_0$ because $\mathscr{D}$ is very ample. Furthermore, the uniqueness of such an exceptional component $\mathscr{G}' = \mathscr{W}$ is also obvious from the preceding discussion. $\square$

3. Exceptional components over general k-gonal curves

We now construct more examples of reducible $\mathscr{I}'_{d,g,r}$ with positive Brill-Noether number by using general k-gonal curves. We need the following lemma due to Ballico [B, Proposition 1].

Lemma 3.1. *Fix positive integers g, k, ℓ with $k \geq 2$, $g \geq 2k - 2$, and $1 \leq \ell \leq [\frac{g}{k-1}]$. Let $|E| = g_k^1$ be the unique pencil of degree k on a general k-gonal curve of genus g. Then $\dim|\ell E| = \ell$.*

Corollary 3.2. *Fix positive integers g, k, ℓ with $k \geq 3$, $g \geq 2k - 2$, and $1 \leq \ell \leq [\frac{g}{k-1}] - 2$. Let $|E| = g_k^1$ be the unique pencil of degree k on a general k-gonal curve C of genus g. Then for any $P, Q \in C$, $\dim|\ell E + P + Q| = \ell$.*

Proof. We first claim that $\dim|\ell E + P| = \ell$ for any $P \in C$. Suppose $\dim|\ell E + P| = \ell + 1$ for some $P \in C$. By Lemma 3.1, $\dim|(\ell+1)E| = \dim|\ell E + P + E' - P| = \dim|\ell E + P| = \ell + 1$, $E' \in |E|$. Then $E' - P \succ 0$ is the base locus of $|(\ell+1)E|$, which is in fact base-point-free.

Suppose that $\dim|\ell E + P + Q| = \ell + 1$ for some $P, Q \in C$. By the first claim and Lemma 3.1, we have $\ell + 1 = \dim|\ell E + P + Q| = \dim|(\ell+1)E| = \dim|(\ell+1)E + P| = \dim|\ell E + P + Q + E'' - Q|$, where $E'' \in |E|$ and $E'' - Q \succ 0$. Then $E'' - Q$ is the base locus of the linear system $|(\ell+1)E + P|$, but this is a contradiction because the actual base locus of $|(\ell+1)E + P|$ is P. $\square$

Lemma 3.2 implies the following immediate corollary.

Corollary 3.3. *Let g, k, ℓ be positive integers such that $k \geq 3$, $g \geq 2k - 2$, and $1 \leq \ell \leq [\frac{g}{k-1}] - 2$. Let C be a general k-gonal curve with the unique pencil $|E|$ of degree k. Then $|K - \ell E|$ is very ample.*

Theorem 3.4. *Let g, k, ℓ, r be integers such that $k \geq 3$, $r \geq 3$, $2 \leq \ell \leq [\frac{g}{k-1}] - 2$, $\frac{2g+2-2k}{\ell+1} - 1 < r \leq g - 2 - \ell k$, and $d = 2g - 2 - \ell k$. Then $\mathscr{I}'_{d,g,r}$ is reducible with at least one exceptional component containing the family of general k-gonal curves.*

Proof. By Corollary 3.3, there exists a family $\mathscr{A}$ of k-gonal curves in $\mathbb{P}^r$ of degree $2g-2-\ell k$ embedded by a general r-dimensional subsystem of $|K-\ell g_k^1|$. Furthermore

$$\begin{aligned}\dim \mathscr{A} &\geq \dim \mathscr{M}_{g,k}^1 + \dim \mathbb{G}(r, g-\ell k+\ell-1) + \dim(\operatorname{Aut}\mathbb{P}^r)\\ &> 3g-3+\rho(d, g, r)+(r+1)^2-1\end{aligned}$$

in the given range of g, k, ℓ, and r. Thus there must be an exceptional component containing the family of general k-gonal curves, and hence $\mathscr{I}'_{d,g,r}$ is reducible. □

Remark 3.5. (i) In all the examples we demonstrated so far, we deliberately chose the numbers d, g, and r so that the Brill-Noether number was positive, in particular, $\rho(d, g, r) \geq g$. On the other hand, one can come up with many examples, e.g., $\mathscr{I}'_{2g-2-\ell k, g, g-\ell k+\ell-1}$ which violate the so-called Brill-Noether-Petri Principle (see [EH, §2]) for those g, k, and ℓ in the same range as in Theorem 3.4, and in these cases the Brill-Noether number becomes negative.

(ii) If $\ell = 2$ in Theorem 3.4, one can show that the family of general k-gonal curves contained in an exceptional component of $\mathscr{I}'_{2g-2-2k, g, r}$ is indeed dense in the component.

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