EVERY 3-MANIFOLD WITH BOUNDARY EMBEDS IN Triod x Triod x I

LI ZHONGMOU

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Abstract. Let M be a compact, connected 3-manifold with nonempty boundary. Then M embeds in $T \times T \times I$, where $T$ is a triod and $I = [0, 1]$.

1. Introduction

Gillman and Rolfsen [2] proved that every compact, orientable 3-manifold with boundary embeds in $c(p) \times c(q) \times c(r)$, where $c(n)$ denotes the cone on $n$ points. In this paper, we improve their argument to the following:

Theorem. Let M be a compact, connected 3-manifold with nonempty boundary. Then M embeds in Triod x Triod x I.

We use the term fake 3-ball for a compact, contractible 3-manifold which is not homeomorphic to the 3-ball $I^3$; and let $\cong$ and $\leftrightarrow$ denote homeomorphism and embedding, respectively. According to the theorem, the 3-dimensional Poincaré conjecture is equivalent to (also see [2]):

Conjecture. There is no fake 3-ball in $T \times T \times I$.

Gillman [1] proved that if a compact 3-dimensional $M$ embeds in $c(p) \times I \times I$ and has trivial rational homology, then $M \cong I^3$. This fact, indicates that the above conjecture is true for 3-manifolds in $T \times I \times I$. On the other hand, our theorem answers Gillman's question in [1] in the negative.

2. A Lemma

Let $K = T \times I$, $PQ$ be the binding $v \times I$ for the vertex $v$ of order 3 of $T$ and $\alpha, \beta, \gamma$ be the three pages (i.e., rectangles) of $T \times I$. We will use $A_i A_j$ to denote the line segment whose endpoints are $A_i, A_j$.

Definition. Suppose disjoint annuli $G_1, G_2, \ldots, G_n$, lie in $K$. If $(\bigcup_{i=1}^n G_i) \cap PQ$ is some line segment and each of the line segments has a regular neighborhood which consists of two discs such that each of the discs exactly lie in a page, then we say that $G_1, G_2, \ldots, G_n$ are transversal in $K$ and call the discs of $(\bigcup_{i=1}^n G_i) \cap \alpha$, $(\bigcup_{i=1}^n G_i) \cap \beta$, $(\bigcup_{i=1}^n G_i) \cap \gamma$ band discs.

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Definition. Let annuli $G_1, G_2, \ldots, G_n$ be transversal in $K$. Suppose that $K$ lies in $R^3$ and $\alpha$ is above $\beta \cup \gamma$. Then we say that $K$ lies in $R^3$ standardly and call $\bigcup_{i=1}^{n} \text{Bd} G_i$ a band link in $K \subset R^3$.

Definition. Let annuli $G_1, G_2, \ldots, G_n$ be transversal in $K$ and band disc $X \subset \alpha$. Suppose $\bigcup_{i=1}^{n} G_i$ does not intersect the shade region of Figure 1. Then we move $X$ to $X'$ as in Figure 1 to get new annuli. We call such a move an elementary move. Compositions of elementary moves are called $I$-moves.

Clearly, suppose that annuli $\bigcup_{i=1}^{n} G_i \subset K$ $I$-moves to $\bigcup_{i=1}^{n} G_i'$; then there is an embedding $H: (\bigcup_{i=1}^{n} G_i) \times I \rightarrow K \times I$ such that $H((\bigcup_{i=1}^{n} G_i) \times 0) = \bigcup_{i=1}^{n} G_i \subset K \times 0$ and $H((\bigcup_{i=1}^{n} G_i) \times 1) = \bigcup_{i=1}^{n} G_i' \subset K \times 1$.

Lemma. Let annuli $G_1, G_2, \ldots, G_n$ be transversal in $K \times A_1$, where $A_1$ is a point. Then there are $n$ disjoint 3-balls $B_1, B_2, \ldots, B_n$ in $K \times (A_1A_2 \cup A_2A_3 \cup A_3A_4)$ such that $G_i \subset \text{Bd} B_i$ ($i = 1, 2, \ldots, n$).

Proof. Suppose that if we let $K \times A_1$ lie in $R^3$ standardly, then we only need change the crossing relations of $m$ pairs of band discs $(X_1, X_1'), \ldots, (X_m, X_m')$ to get new annuli whose central curve is a trivial link in $R^3$.

For band discs $X_1, X_1'$, we have an $I$-move, $I_1$, a pair of new band discs $(Y_1, Y_1')$, and annuli $G_1' = I_1(G_1)$ ($i = 1, 2, \ldots, n$) as in Figure 2. Note that we can replace changing the crossing relation between $X_1$ and $X_1'$ with changing the crossing relation between $Y_1$ and $Y_1'$. For $I_1(X_2), I_1(X_2')$, using a similar method, we get $I_2, (Y_2, Y_2'),$ and $G_2', \ldots, G_n', \ldots$. For $I^{m-1}I^{m-2}\ldots I_1(X_m)$ and $I^{m-1}I^{m-2}\ldots I_1(X_m')$, we get $I_m, (Y_m, Y_m'),$ and $G_1^{(m)}, \ldots, G_n^{(m)}$.

Let $I' = I^{m-1}I^{m-2}\ldots I_1$. Then $G_i^{(m)} = I'(G_i)$ ($i = 1, 2, \ldots, n$). We still use $Y_i, X_i'$ to denote $I_1I_1\ldots I_i(Y_i), I_1I_1\ldots I_i(X_i')$, respectively ($i = 1, 2, \ldots, m$).

Suppose that $K \times A_1$ lies in $R^3$ standardly. If we change the crossing relations of the pairs $(Y_1, Y_1'), \ldots, (Y_m, Y_m')$ in $R^3$ to get new annuli $H_1, H_2, \ldots, H_n$, then the central curve of new annuli is a trivial link in $R^3$. Suppose that $H_i$ has $k_i$ twists in $R^3$ ($i = 1, 2, \ldots, n$). Then we $I$-move $G_i^{(m)}, \ldots, G_n^{(m)}$ to get annuli $L_1, L_2, \ldots, L_n$ as Figure 3. We denote this $I$-move as $I'$ and denote the shade band discs of Figure 3 as $Y_{m+1}, Y_{m+2}, \ldots, Y_{m'}$ ($m' = m+k_1+k_2+\ldots+k_n$). Note that in Figure 3, if we change the band discs $Y_1, Y_2, \ldots, Y_{m'}$ of $L_1, L_2, \ldots, L_n$ under $\beta \cup \gamma$ in $R^3$, then we obtain annuli whose boundary curve is a trivial link in $R^3$. 
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Figure 2

Let $I$-move $I = I''I'$. Then there is an embedding $H: (\bigcup_{i=1}^{n} G_i) \times A_1 A_2 \hookrightarrow K \times A_1 A_2$ such that

$$H\left(\left(\bigcup_{i=1}^{n} G_i\right) \times A_1\right) = \left(\bigcup_{i=1}^{n} G_i\right) \subset K \times A_1$$

and

$$H\left(\left(\bigcup_{i=1}^{n} G_i\right) \times A_2\right) = \left(\bigcup_{i=1}^{n} L_i\right) \subset K \times A_2$$
Consider $\alpha \times (A_2A_3 \cup A_4A_2) \cong I^3$. For $Y_1, Y_2, \ldots, Y_{m'}$, we give band discs $Z_1, Z_2, \ldots, Z_{m'}$ in $PQ \times A_4A_2$; for the other band discs of $(\bigcup_{i=1}^{n} L_i) \cap (\alpha \times A_2)$, we give band discs $Z_{m'+1}, Z_{m'+2}, \ldots, Z_{m''} \subset PQ \times A_3$. Then we get $m''$ small annuli in $\alpha \times (A_2A_3 \cup A_4A_2)$. Note that there are $m''$ disjoint 3-balls $B_1, B_2, \ldots, B_{m''}$ in $\alpha \times (A_2A_3 \cup A_4A_2)$ such that the small annuli lie in the boundary surfaces of the 3-balls, respectively. (See Figure 4.)

Let $L' = (\bigcup_{i=1}^{m''} Z_i) \cup (\bigcup_{i=1}^{n} L_i) \cap (\beta \cup \gamma)$. Then $L'$ is $n$ annuli $L'_1, L'_2, \ldots, L'_n$ and whose boundary curve is a trivial link in $(\beta \cup \gamma) \times (A_2A_3 \cup A_4A_2) \cong I^3$. Therefore, there are $n$ disjoint 3-balls $B'_1, B'_2, \ldots, B'_n$ in $(\beta \cup \gamma) \times (A_2A_3 \cup A_4A_2)$ such that these annuli lie in the boundary surfaces of the 3-balls, respectively.

$H((\bigcup_{i=1}^{n} G_i) \times A_4A_2) \cup (\bigcup_{i=1}^{m''} B_i) \cup (\bigcup_{i=1}^{n} B'_i)$ are $n$ disjoint 3-balls in $K \times (A_1A_2 \cup A_2A_3 \cup A_4A_2) \cong T \times T \times I$ and $G_1, G_2, \ldots, G_n$ lie in the boundary surfaces of these 3-balls.

### 3. A Proof of the Theorem

We only need to prove the theorem in the case that $M \cong N - \text{Int} B$ for a closed 3-manifold $N$ and a 3-ball $B \subset N$. If we let $M$ be any compact 3-manifold with boundary and $N$ be the closed 3-manifold given by attaching 3-dimensional handles to Bd $M$, then, by [3, Theorem 1.5], $M \subset N - \text{Int} C \cong N - \text{Int} B$, where $C$ is a 3-ball in $N - \text{Int} M$ and $B$ is any 3-ball in $N$.

Suppose that $(V_1, V_2)$ is a Heegaard splitting of $N$. Let $(D_1, D_2, \ldots, D_n)$ be a collection of pairwise disjoint property embedded 2-discs in $V_1$ which cut $V_1$ into a 3-ball, and let annuli $G_1, G_2, \ldots, G_n$ be disjoint regular neighborhoods of the boundary curves $J_1, J_2, \ldots, J_n$ of $D_1, D_2, \ldots, D_n$ in Bd $V_1 = \text{Bd} V_2$, respectively.

We choose a proper 2-disc $D$ in Bd $V_2$ such that $D$ does not intersect the annuli. Then we have (see Figure 5) $V_2 \cong ((\text{Bd} V_2 - \text{Int} D) \times I) \cup (\bigcup_{i=1}^{n} E_i)$, where $E_1, E_2, \ldots, E_n$ are 3-balls in $V_2$ and $(\bigcup_{i=1}^{n} \text{Bd} E_i) \cap ((\text{Bd} V_2 - \text{Int} D) \times 0)$ are annuli $F_1, F_2, \ldots, F_n$. 
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Clearly, $(\text{Bd}V_2 - \text{Int}D) \times 0$ embeds in $K \times C_1$ for a point $C_1$. Then we have $(\text{Bd}V_2 - \text{Int}D) \times I \hookrightarrow K \times C_1 C_2$ such that annuli $G_1, G_2, \ldots, G_n$ are transversal in $K \times C_2$ and annuli $F_1, F_2, \ldots, F_n$ are transversal in $K \times C_1$. Therefore, by the lemma, $\bigcup_{i=1}^n E_i \hookrightarrow (K \times C_1 A_1 \cup A_1 A_2 \cup A_3 A_1)$. Then, $M \hookrightarrow K \times C_3 C_3 \cup C_3 C_4 \cup C_2 C_3 \cup C_1 C_2 \cup C_1 A_1 \cup A_1 A_2 \cup A_3 A_1) \cong T \times H \times I$. Note that $H \times I \hookrightarrow T \times I$. (This can be seen by drawing a picture.) Therefore, $M \hookrightarrow T \times T \times I$.

**Remark.** Mr. Dale Rolfsen told me the fact that $H \times I \hookrightarrow T \times I$.

Considering the proof of the theorem, we get the following corollary easily.

**Corollary.** Suppose that $V_2$ lies in $R^3$ trivially. Then $\bigcup_{i=1}^n \text{Bd}G_i$ and $\bigcup_{i=1}^n J_i$ are links in $R^3$. If the band link $\bigcup_{i=1}^n \text{Bd}G_i$ is a trivial link, then $M \hookrightarrow T \times I \times I$; if $\bigcup_{i=1}^n J_i$ is a trivial link, then $M$ has a standard spine which embeds in $T \times I \times I$.

In particular, by [1], the band link of any Heegaard splitting of a punctured lens space is not trivial in $R^3$. On the other hand, $L(p, 1) - B$ has standard spine in $T \times I \times I$, where $B$ is a 3-ball in $L(p, 1)$.

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### References

