AUSLANDER'S $\delta$-INVARIANTS OF GORENSTEIN LOCAL RINGS

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Abstract. Let $(R, m, k)$ be a Gorenstein local ring with associated graded ring $G(R)$. It is conjectured that for any integer $n > 0$, Auslander's $\delta$-invariant $\delta(R/m^n)$ of $R/m^n$ equals 1 if and only if $m^n$ is contained in a parameter ideal of $R$. In an earlier paper we showed that the conjecture holds if $G(R)$ is Cohen-Macaulay. In this paper we prove that the conjecture has an affirmative answer if depth $G(R) = \text{dim } R - 1$ and $R$ is gradable. We also prove that if $R$ is not regular and depth $G(R) > \text{dim } R - 1$, then $\delta(R/m^2) = 1$ if and only if $R$ has minimal multiplicity.

Introduction

Throughout this paper we assume that $(R, m, k)$ is a commutative Noetherian Gorenstein local ring (or a graded Gorenstein $k$-algebra with unique maximal graded ideal $m$), and all modules are finitely generated. For an $R$-module $M$, there is an exact sequence of $R$-modules

$$0 \rightarrow Y_M \rightarrow X_M \overset{\varphi}{\rightarrow} M \rightarrow 0$$

where $X_M$ is a maximal Cohen-Macaulay module, $Y_M$ is a module of finite projective dimension, and $\varphi$ is right minimal, i.e., all endomorphisms $\alpha : X_M \rightarrow X_M$ with $\varphi \circ \alpha = \varphi$ are isomorphisms (see [1, 2]). This sequence is called the minimal Cohen-Macaulay approximation of $M$. It is uniquely determined (up to isomorphism) by $M$. The theory of Cohen-Macaulay approximations was initiated by Auslander and Buchweitz. The rank of a maximal free direct summand of $X_M$ is denoted by $\delta_R(M)$ (or simply $\delta(M)$) and is called the $\delta$-invariant of $M$ (over $R$) by Auslander.

One of the main questions is how the $\delta$-invariant of $M$ reflects the structures of $M$ and of $R$. If $M$ is a module of finite projective dimension, the minimal Cohen-Macaulay approximation of $M$ is just the minimal free resolution of $M$. If an $R$-module $M$ has a factor module which is of finite projective dimension, then $\delta(M) > 0$ [1, 3]. Thus the $\delta$-invariant of $M$ can be regarded as a measure of how far the module $M$ is from being of finite projective dimension. It was first conjectured that for an $R$-module $M$, $\delta(M) > 0$ if and only if $M$ has a factor module of finite projective dimension. Unfortunately this is not the
case in general (see the example in §2). However, the modules of the form $M = R/\mathfrak{m}^n$, $n \geq 1$, seem special. Following [3] and [5] we showed in [4] that if the associated graded ring $G(R)$ of $R$ is Cohen-Macaulay then $\delta(R/\mathfrak{m}^n) = 1$ if and only if $\mathfrak{m}^n$ is contained in a parameter ideal of $R$. Here an ideal $I$ is called a parameter ideal if it is generated by a system of parameters of $R$. We then conjectured that this is true in general, i.e., without the assumption on $G(R)$. In this paper we give further evidence for the conjecture. We show that if $R$ is a graded Gorenstein $k$-algebra such that depth $G(R) \geq \dim R - 1$, then the conjecture holds for $R$. We call a local ring $(R, \mathfrak{m}, k)$ gradable if the completion $\hat{R}$ of $R$ at its maximal ideal $\mathfrak{m}$ is isomorphic to the completion of a graded $k$-algebra at the graded maximal ideal. Since the $\delta$-invariants behave well under faithfully flat ring extensions [1], we have that the conjecture holds for a Gorenstein local ring $R$ if $R$ is gradable and $\dim G(R) \geq \dim R - 1$.

It is known that $\delta(R/\mathfrak{m}) = 1$ if and only if $R$ is a regular local ring [3]. We show that for a Gorenstein local ring $R$ with depth $G(R) \geq \dim R - 1$, $\delta(R/\mathfrak{m}^2) = 1$ if and only if $\mathfrak{m}^2$ is contained in a parameter ideal of $R$. If $R$ is not regular, then by a result of Sally [6], $R$ has minimal multiplicity.

The results in this paper are based on a particular description of the minimal Cohen-Macaulay approximation of a module $M$ with depth $M = \dim R - 1$.

1. A CONSTRUCTION OF MINIMAL COHEN-MACALAY APPROXIMATION

In this section we assume that $(R, \mathfrak{m}, k)$ is a Gorenstein local ring. We denote by $\mu(M)$ the minimal number of generators of an $R$-module $M$. Let $M$ be an $R$-module with depth $M = \dim R - 1$. The following description of the minimal Cohen-Macaulay approximation of $M$ seems well known.

Proposition 1.1 [1]. Let $M$ be an $R$-module with depth $M = \dim R - 1$, and let $0 \to Y_M \to X_M \to M \to 0$ be the minimal Cohen-Macaulay approximation of $M$. Then $Y_M \cong R^n$, where $n = \mu(\text{Ext}^1(M, R))$.

Proof. Since $M$ is not a maximal Cohen-Macaulay module, we have

$$\text{Ext}^1(M, R) \neq (0).$$

Let $\xi_1, \ldots, \xi_n$ be a minimal set of generators of $\text{Ext}^1(M, R)$, and let $E$ be the extension of $M$ by $R^n$ corresponding to the element $(\xi_1, \ldots, \xi_n)$ of $\text{Ext}^1(M, R^n) \cong \bigoplus_n \text{Ext}^1(M, R)$. Applying the functor $\text{Hom}(\ , R)$ to the exact sequence

$$0 \to R^n \to E \to M \to 0$$

yields a long exact sequence

$$0 \to \text{Hom}(M, R) \to \text{Hom}(E, R) \to \text{Hom}(R^n, R) \to \text{Ext}^1(M, R) \to \text{Ext}^1(E, R) \to 0,$$
and $\text{Ext}^i(E, R) = (0)$ for $i \geq 2$. By the choice of $E$, the map $\psi$ is surjective. Therefore, $\text{Ext}^1(E, R) = (0)$. Hence $E$ is a maximal Cohen-Macaulay $R$-module. Since $n = \mu(\text{Ext}^1(M, R))$, (1) in fact is the minimal Cohen-Macaulay approximation of $M$. \(\square\)

As an application of Proposition 1.1, we have the following result on the $\delta$-invariant of $M$.

**Proposition 1.2.** Let $M$ be an $R$-module with depth $M = \dim R - 1$. Then

$$\delta(M) = \mu(M) + \mu(\text{Ext}^1(M, R)) - \mu(\text{Hom}(\Omega(M), R))$$

where $\Omega(M)$ is the first syzygy of $M$.

**Proof.** We put $s = \mu(M)$, $n = \mu(\text{Ext}^1(M, R))$, and $t = \mu(\text{Hom}(\Omega(M), R))$. We have the short exact sequence $0 \to \Omega(M) \to R^s \to M \to 0$. Let $0 \to R^n \to X_M \to M \to 0$ be the minimal Cohen-Macaulay approximation of $M$. We can construct the following pullback diagram:

$$
\begin{array}{c}
0 & \to & 0 \\
\downarrow & & \downarrow \\
R^n & = & R^n \\
\downarrow & & \downarrow \\
0 & \to & \Omega(M) \\
\downarrow & & \downarrow \\
0 & \to & F \\
\downarrow & & \downarrow \\
0 & \to & X_M \\
\end{array}
$$

Thus we have $F \cong R^n \oplus R^s$. On the other hand, $\Omega(M)$ is a maximal Cohen-Macaulay $R$-module. Applying the functor $\text{Hom}(\ , R)$ to the middle row, we obtain an exact sequence

$$0 \to \text{Hom}(X_M, R) \to \text{Hom}(F, R) \to \text{Hom}(\Omega(M), R) \to 0.$$ 

Therefore, $F \cong R^t \oplus R^{\delta(M)}$. Hence, we get $\delta(M) + t = s + n$. \(\square\)

2. **One-dimensional rings**

In this section $(R, m, k)$ will always be a 1-dimensional Gorenstein local ring. For any $R$-module $M$, we denote by $\text{soc}(M)$ the socle of $M$, i.e., $\text{soc}(M) = \{x \in M \mid mx = 0\}$. If $M$ is of finite length, we denote by $l(M)$ the length of $M$. We put $M^* = \text{Hom}(M, R)$. In this section we always assume that $I$ is a regular ideal of $R$, i.e., the ideal $I$ contains a regular element of $R$. In this section we illustrate and apply the results in §1 to the cyclic modules over $R$.

First we have

**Proposition 2.1.** Let $I$ be an ideal of $R$. Then

$$\delta(R/I) = 1 + l(\text{soc}(R/I)) - \mu(I^*).$$

In particular, we have $\delta(R/I) > 0$ if and only if $l(\text{soc}(R/I)) = \mu(I^*)$. 

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Proof. By Proposition 1.2 it suffices to show that $l(soc(M)) = \mu(Ext^1(M, R))$ for an $R$-module $M$. Let $N$ be the maximal finite length submodule of $M$. Applying the functor $\text{Hom}(\_ , R)$ to the exact sequence $0 \to N \to M \to M/N \to 0$, we obtain an isomorphism $Ext^1(N, R) \cong Ext^1(M, R)$ since $M/N$ is a maximal Cohen-Macaulay module. Therefore, $l(soc(M)) = \mu(Ext^1(N, R)) = \mu(Ext^1(M, R))$. Since $\delta(R/I) \leq 1$ for any ideal $I$ of $R$ [1], we get the second statement. \[\square\]

Let $x \in m$ be an arbitrary $R$-regular element. The formula in Proposition 2.1 can be written in the following form.

**Corollary 2.2.** Let $x \in m$ be $R$-regular. Then

$$\delta(R/I) = 1 + l(soc(R/I)) - l(soc(I/\bar{x}I)),$$

and $\delta(R/I) = 1$ if and only if $l(soc(R/I)) = l(soc(I/\bar{x}I))$.

**Proof.** We need only show that $\mu(I^*) = l(soc(I/\bar{x}I))$ for any regular element $x$ of $R$. The exact sequence $0 \to R \xrightarrow{x} R \to R/\bar{x}R \to 0$ gives rise to an exact sequence $0 \to I^* \xrightarrow{\delta} I^* \to \text{Hom}_R(I/\bar{x}I, R/\bar{x}R) \to 0$. Since $x \in m$, we have $\mu(I^*) = l(\text{Hom}(I/\bar{x}I, R/\bar{x}R)) = l(soc(I/\bar{x}I))$. \[\square\]

As a consequence of the Proposition 2.1 we have

**Corollary 2.3.** Let $I$ be a Gorenstein ideal of $R$. Then $\delta(R/I) = 1$ if and only if $I$ is a principal ideal.

**Proof.** Since $R/I$ has a simple socle, we have $\delta(R/I) = 1$ if and only if $I^*$ is generated by one element. Hence $I^*$ is a free $R$-module of rank one. Applying the functor $\text{Hom}(\_ , R)$ to the exact sequence $0 \to I \to R \to R/I \to 0$ gives an exact sequence $0 \to R \xrightarrow{x} R \to Ext^1(R/I, R) \to 0$. Therefore, $Ext^1(R/I, R) \cong R/(x)$ for some regular element $x$ of $R$. Since $I$ is a Gorenstein ideal, we get $R/I \cong Ext^1(R/I, R)$, and this implies that $I = (x)$. \[\square\]

Now we consider another case. Let $Q(R)$ be the total ring of quotients of $R$ with respect to the $R$-regular elements, and let $I^{-1} = \{z \in Q(R) \mid zI \subseteq R\}$. Then we know that there exists a natural $R$-isomorphism $\phi_I: I^{-1} \to I^*$ given by sending $z \in I^{-1}$ to the morphism in $I^*$ defined as multiplication by $z$. Also we have $I \subseteq II^{-1} \subseteq R$. If $II^{-1} = R$, then $I$ is a principal ideal and so $\delta(R/I) = 1$. We now consider the case where $II^{-1} = I$. We first give the following criterion.

**Proposition 2.4.** $\delta(R/I) = 1$ if and only if $R \subset mI^{-1}$.

**Proof.** We have the following commutative exact diagram

$$\begin{array}{ccccccccc}
0 & \to & R & \xrightarrow{\alpha} & I^{-1} & \to & \text{coker} \alpha & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & R^* & \to & I^* & \to & Ext^1(R/I, R) & \to & 0 \\
\end{array}$$

where $\alpha$ is the natural inclusion and all the vertical arrows are isomorphisms. By Proposition 2.1, $\delta(R/I) = 1$ if and only if $\mu(I^*) = \mu(Ext^1(R/I, R))$. Hence we have $\delta(R/I) = 1$ if and only if $\text{Im} \alpha \subset mI^{-1}$. \[\square\]

As a consequence we have
Corollary 2.5. If $II^{-1} = I$, then $\delta(R/I) = 0$.

Proof. We must show that $R \notin mI^{-1}$. Suppose $R \subset mI^{-1}$; then we have $I \subset mI^{-1} \subset mI$ since $II^{-1} = I$. This implies that $I = mI$, which is impossible by Nakayama's Lemma. □

Remark. The condition $II^{-1} = I$ in fact characterizes the conductors of the overlings of $R$ in $Q(R)$. Recall that for an overring $R'$ with $R \subset R' \subset Q(R)$, the conductor $c$ of $R'$ in $R$ is defined by $c = \{ z \in Q(R) | zR' \subset R \}$. Here $c$ is an ideal of $R$. It is easy to check that if $c$ is the conductor of an overring $R'$ in $R$, then $cc^{-1} = c$. Conversely, if $II^{-1} = I$, then $I$ is the conductor of $\phi^{-1}(\text{End}(I))$ in $R$, where $\phi_I: I^{-1} \to I^*$ is the natural isomorphism and $\text{End}(I) \subset I^*$.

Now we return to the question in the introduction. If $I$ is contained in an ideal $J$ of finite projective dimension (in our situation $J$ has to be a principal ideal), then we know that $\delta(R/I) = 1$. The converse is not true in general.

Example. Let $R = k[[t^3, t^4]]$ where $k$ is a field, and let $I = (t^8 + t^9, t^{10})$. It is not hard to check by direct calculation that

(a) $I$ is not contained in any principal ideal of $R$, and
(b) $l(\text{soc}(R/I)) = l(\text{soc}(I/t^3I)) = 2$.

Therefore, $\delta(R/I) = 1 + l(\text{soc}(R/I)) - l(\text{soc}(I/t^3I)) = 1$ by Corollary 2.2, but $R/I$ has no factor module of finite projective dimension.

3. The conjecture

Let $(R, m, k)$ be a Gorenstein local ring (or a graded Gorenstein $k$-algebra with a unique graded maximal ideal $m$). We showed in [4] that if the associated graded ring $G(R)$ of $R$ is Cohen-Macaulay, then for any integer $n > 0$, $\delta(R/m^n) = 1$ if and only if there exists an $R$-sequence $x$ such that $m^n \subset (x)$. Rings with $G(R)$ Cohen-Macaulay include hypersurface rings and homogeneous Gorenstein $k$-algebras. (A graded $k$-algebra is called homogeneous if it is generated by degree 1 elements over $k$.)

Conjecture. Let $(R, m, k)$ be a Gorenstein local ring (or a graded Gorenstein $k$-algebra with a unique graded maximal ideal $m$). For any integer $n > 0$, $\delta(R/m^n) = 1$ if and only if there is an $R$-sequence $x$ such that $m^n \subset (x)$.

In this section we show two results related to the conjecture. In both cases we assume that $R$ satisfies the condition $\text{depth} G(R) \geq \dim R - 1$. We first use the criterion for $\delta(R/I) = 1$ that we developed in §2 to prove the one-dimensional case. Then we reduced the general case to the one-dimensional case by the result of [4].

Theorem 3.1. Let $R$ be a 1-dimensional graded Gorenstein $k$-algebra. For any graded ideal $I$ of $R$, $\delta(R/I) = 1$ if and only if $I \subset (x)$, where $x$ is a homogeneous regular element of $R$. In particular, the conjecture holds for $R$.

Proof. According to Proposition 2.4, $\delta(R/I) > 0$ if and only if $R \subset mI^{-1}$. In particular, $\delta(R/I) > 0$ implies that $1 \in mI^{-1}$. Suppose $1 = \sum_{i=1}^{I} z_i x_i/y_i$ where $x_i, y_i$, and $z_i$ are homogeneous elements of degree at least one and $x_i/y_i \in I^{-1}$ for all $i$. By comparing degrees we know that there is an $i$ such that
\[
\deg(z_i x_i / y_i) = \deg z_i + \deg x_i - \deg y_i = 0. \text{ Therefore, we have } 1 = u z_i x_i / y_i \text{ with } u \in k. \text{ This implies that } x_i / y_i = 1 / (uz_i). \text{ Hence we get } I \subseteq (z_i). \]

**Remark.** Let \( R \) be a 1-dimensional complete local Gorenstein domain containing an algebraically closed field \( k \). It is known that \( R \) is the completion of a graded \( k \)-algebra with respect to its irrelevant maximal ideal \([7]\). Therefore, the conjecture holds for such rings.

In \([4]\) we showed the following, which allows us to use a reduction argument.

**Lemma 3.2** \([4]\). Let \((R, m, k)\) be a Gorenstein local ring (or a graded Gorenstein k-algebra). Let \( x \in m \setminus m^2 \) be \( R \)-regular. Set \( \tilde{R} = R / xR \). Suppose the induced map \( \tilde{x} : m^i / m^i \rightarrow m^i / m^{i+1} \) is injective for \( i \geq 0 \). Then \( \delta_R(R/m^i) = 1 \) if and only if \( \delta_R(R/(m^i, x)) = 1 \) for \( i \geq 1 \).

Combining this lemma and Theorem 3.1 we now obtain

**Corollary 3.3.** Let \((R, m, k)\) be a graded Gorenstein k-algebra. Let \( G(R) \) be the associated graded ring of \( R \). If \( \text{depth } G(R) \geq \text{dim } R - 1 \), then the conjecture holds for \( R \).

Recall that a local ring \((R, m, k)\) is said to be gradable if the completion \( \hat{R} \) of \( R \) at its maximal ideal \( m \) is isomorphic to the completion of a graded \( k \)-algebra at its unique maximal graded ideal. Since the \( \delta \)-invariants behave well under faithfully flat ring extensions \([1]\), we have

**Corollary 3.4.** Let \((R, m, k)\) be a Gorenstein local ring. If \( R \) is gradable and \( \text{depth } G(R) \geq \text{dim } R - 1 \), then the conjecture holds for \( R \).

**Example.** Let \( R = k[[x, y, z]]/I \) where \( I = (x^3 + y^9, x^2 z^4 + y^7) \). Then \( G(R) = k[x, y, z]/J \), with \( J = (x^3, x^2 z^4, xy^7, y^{14}) \), is not Cohen-Macaulay. However, it is easily checked that \( R \) is gradable. This gives an example where \( G(R) \) is not Cohen-Macaulay, but the conjecture holds for \( R \). In this case \( \delta(R/m^i) = 0 \) for \( i < 7 \) and \( \delta(R/m^8) = 1 \).

It is known that \( \delta(R/m) = 1 \) implies that \( m \) is contained in a parameter ideal of \( R \). Therefore, \( R \) is a regular local ring. In dealing with the case where \( \delta(R/m^2) = 1 \), we have

**Theorem 3.5.** Let \((R, m, k)\) be a 1-dimensional Gorenstein local ring. Then \( \delta(R/m^2) = 1 \) if and only if \( m^2 \subseteq (x) \) for some \( x \in R \).

**Proof.** The "if" part is obvious. Now let \( \delta(R/m^2) > 0 \). Set \( I = m^2 \). By Proposition 2.4 we have that \( \delta(R/I) > 0 \) if and only if \( R \subseteq mI^{-1} \) if and only if \( 1 = \sum r_i u_i \), where \( r_i \in m \) and \( u_i \in I^{-1} \). We define

\[
l = \min \left\{ n \mid 1 = \sum_{i=1}^n r_i u_i, \quad r_i \in m, \quad u_i \in I^{-1} \right\}.
\]

To show that \( m^2 \subseteq (x) \) for some \( R \)-regular element \( x \) is equivalent to showing that \( l = 1 \). We now assume that \( l > 1 \) and \( \mu(m) \geq 3 \) and derive a contradiction. (If \( \mu(m) = 2 \), then \( R \) is a hypersurface and we know the conjecture is true.)
Let \( t \in m \setminus m^2 \) be \( \mathcal{R} \)-regular. Then \( u_i \) can be written in the form \( u_i = z_i/t^2 \) with \( z_i \in \mathcal{R} \). We assume that all \( z_i \) are in \( m \) (otherwise, we have \( l = 1 \)). Thus we have

\[
1 = \frac{\sum r_iz_i}{t^2} + \frac{\sum s_ix_i}{t}
\]

where \( z_i/t^2, x_i/t \) are in \( I^{-1} \). We assume that \( r_i, s_i, z_i, x_i \in m \) (otherwise, \( l = 1 \)) and \( z_i, s_ix_i \notin \mathcal{R} \).

Now consider \( z_i/t^2 \in I^{-1} \) with \( z_i \in m \) and \( z_i \notin \mathcal{R} \). We have \( z_im/t = z_itm/t^2 \subset \mathcal{R} \), i.e., \( z_im \subset tR \). Therefore, under the reduction map \( \mathcal{R} \to \overline{\mathcal{R}} = \mathcal{R}/t\mathcal{R} \), we have \( \overline{z_i} \in \text{soc}(\mathcal{R}/t\mathcal{R}) \). Let \( z_j/t^2 \) be another element in \( I^{-1} \) with \( z_j \in m \) and \( z_j \notin t\mathcal{R} \). The same argument shows that \( \overline{z_j} \in \text{soc}(\mathcal{R}/t\mathcal{R}) \). Since \( \mathcal{R}/t\mathcal{R} \) has simple socle, we get \( z_j = az_j + ty_j \) where \( a \in \mathcal{R} \) and \( y_j \in \mathcal{R} \). This implies that \( y_i/t \in I^{-1} \). Hence (2) can be written in the form

\[
1 = \frac{rz}{t^2} + \frac{\sum_{i=1}^{l-1} s_ix_i}{t}
\]

where \( z/t^2, x/t \) are in \( I^{-1} \), \( r, s_i \in m \), and \( z, s_ix_i \notin t\mathcal{R} \). We assume that \( z \) and \( x_i \) are in \( m \), and both \( rz/t^2 \) and \( \sum s_ix_i/t \) are not in \( m \) (otherwise, we are done).

Since \( \sum s_ix_i \notin t\mathcal{R} \), we may assume that \( s_1x_1 \notin t\mathcal{R} \). However, we have \( s_1x_1m \subset t\mathcal{R} \), that is, under the reduction map \( \mathcal{R} \to \overline{\mathcal{R}} = \mathcal{R}/t\mathcal{R} \), \( \overline{s_1x_1} \in \text{soc}(\mathcal{R}/t\mathcal{R}) \). Since \( \mathcal{R} \) is Gorenstein, we get \( s_1x_1 = az + ty \) with \( a, y \in \mathcal{R} \) and so

\[
\frac{s_1x_1}{t} = \frac{az + ty}{t^2} + y.
\]

If \( y \in m \), we get

\[
1 - y = \frac{(r + at)z}{t^2} + \frac{\sum_{i=2}^{l-1} s_ix_i}{t},
\]

contrary to the minimality of \( l \). Therefore, \( y \) is a unit in \( \mathcal{R} \) and (3) then can be written in the form

\[
1 = \frac{rz}{t^2} + \frac{ux}{t}
\]

where \( z/t^2, x/t \) are in \( I^{-1} \), \( z, x, r, u \in m \), and \( z, x \notin t\mathcal{R} \). We now consider the two induced linear maps

\[
\overline{x} : m/m^2 \to \mathcal{R}/t\mathcal{R} \quad \text{and} \quad \overline{z} : m/m^2 \to \mathcal{R}/t^2\mathcal{R}.
\]

We have that \( \text{Im} \overline{x} \subset \text{soc}(\mathcal{R}/t\mathcal{R}) \) and \( \text{Im} \overline{z} \subset \text{soc}(\mathcal{R}/t^2\mathcal{R}) \). Therefore,

\[
\dim_k \text{Ker} \overline{x} \geq \mu(m) - 1 \quad \text{and} \quad \dim_k \text{Ker} \overline{z} \geq \mu(m) - 1.
\]

Since \( \mu(m) \geq 3 \), we have \( \text{Ker} \overline{x} \cap \text{Ker} \overline{z} \neq (0) \). Thus there exists \( s \in m \setminus m^2 \) such that \( zs \in t^2\mathcal{R} \) and \( xs \in t\mathcal{R} \). Suppose \( zs = t^2y, xs = tw \). Then \( w \in m \).

If \( y \) is a unit of \( \mathcal{R} \), then \( s \) is \( \mathcal{R} \)-regular and \( z/t^2 = y/s \). This implies that \( m^2 \subset (s) \) and \( l = 1 \). Suppose \( y \in m \); then multiplying both sides of (5) by \( s \) yields \( s = ry + uw \in m^2 \). This is a contradiction since \( s \in m \setminus m^2 \), and the proof is complete. \( \square \)
Combining this result and Lemma 3.2 we obtain the following.

**Corollary 3.6.** Let \((R, m, k)\) be a Gorenstein local ring. Suppose that
\[\text{depth } G(R) \geq \text{dim } R - 1.\]
Then \(\delta(R/m^2) = 1\) if and only if \(m^2\) is contained in a parameter ideal of \(R\). In particular, if \(R\) is not regular, then \(\delta(R/m^2) = 1\) if and only if \(R\) has minimal multiplicity.

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