A COMPARISON BETWEEN EULER AND CESÀRO METHODS OF SUMMABILITY

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ABSTRACT. It is well known that there are sequences that are summable by every Cesàro method $C_r$ ($r > 0$) but are not summable by any Euler method $E_p$ ($0 < p < 1$). It is proved here that on the other hand there are sequences that are summable by every Euler method $E_p$ ($0 < p < 1$) but are not summable by any Cesàro method.

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We consider the regular Cesàro methods $C_r$ ($r > 0$) and the regular Euler methods $E_p$ ($0 < p < 1$), where the latter is defined as in [5]; that is, the $E_p$-transform of any sequence $\{x_k\}$ is given by $\{t_n(p)\}$ where

\[ t_n(p) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} x_k \quad (n = 0, 1, \ldots). \]

Let $(C_r)$ and $(E_p)$ denote the summability fields of $C_r$ and $E_p$ respectively. It is well known that

\[ (E_p) \subseteq (C_r) \quad \text{and} \quad (C_r) \not\subseteq (E_p), \quad 0 < p < 1, \ r > 0 \]

(see, for instance, [5, Satz 64.V]). It is also well known that if $0 < p < 1$ and $r > 0$,

\[ m \cap \bigcup_{0 < q < 1} (E_q) = m \cap (E_p) \subseteq m \cap (C_r) = m \cap \bigcup_{s > 0} (C_s), \]

where $m$ is the set of all bounded sequences. Further, it is known that

\[ \bigcap_{r > 0} (C_r) \not\subseteq \bigcup_{0 < p < 1} (E_p); \]

that is, there exist sequences $x$ that are summable by every Cesàro method $C_r$ ($r > 0$) but not by any Euler method $E_p$ [2, p. 251; 1, p. 213]. Such a sequence $x$ can be made to satisfy certain additional conditions too, like being bounded,
or unbounded, or that the series $\sum a_n = \sum (x_n - x_{n-1})$ have gaps of a certain type (see [3]). But the question whether

$$\bigcap_{0 < p < 1} (E_p) \subset (C_r) \quad \text{for some } r > 0$$

seems to have remained open. The object of the present paper is to prove that the answer to the question is in the negative. In fact, we prove rather more.

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**Theorem.** (a) There exist sequences that are $E_p$-summable for every $0 < p < 1$ but are not summable by any Cesàro method; that is,

$$\bigcap_{0 < p < 1} (E_p) \not\subset \bigcup_{r > 0} (C_r).$$

(b) There exists a sequence $\{V^{(m)}\}$ of row-finite regular matrix methods such that

$$V^{(1)} \not\subset \bigcup_{r > 0} (C_r) \quad \text{and} \quad (V^{(1)}) \subset (V^{(2)}) \subset \cdots \subset \bigcap_{0 < p < 1} (E_p).$$

**Proof.** (a) Let the sequence $\{x_n\}$ be defined by

$$(2) \quad x_k = (-1)^k \sum_{r=0}^{k} c_r [k], \quad (k = 0, 1, \ldots)$$

where $[k]_r := k(k-1)\cdots(k-r+1)$ ($r$ factors) (and is interpreted as 0 when $r = 0$ or $k < r$) and where $\{c_r\}$ is some fixed sequence of positive numbers which satisfy a certain condition to be imposed later. For $0 < p < 1$, we have for each $n = 0, 1, \ldots$,

$$t_n(p) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} (-1)^k \sum_{r=0}^{k} c_r [k],$$

$$= \sum_{k=0}^{n} c_r \sum_{k=r}^{n} \binom{n}{r} p^k (1-p)^{n-k} (-1)^k [k],$$

$$= \sum_{r=0}^{n} c_r \sum_{k=0}^{r} \binom{n}{k} p^k (1-p)^{n-k} (-1)^k [k],$$

since $[k]_r = 0$ for $k < r$. Now

$$(4) \quad \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} x^k = [px + (1-p)]^n.$$

Differentiating each side of this expression $r$ times (with respect to $x$) and then putting $x = -1$, we see that the inner sum in (3) is equal to

$$(-1)^r [n]_r p^r (1-2p)^{n-r}.$$  

Hence

$$(5) \quad t_n(p) = \sum_{r=0}^{n} c_r (-1)^r [n]_r p^r (1-2p)^{n-r} = \sum_{r=0}^{n} c_r a_{nr}(p), \quad \text{say.}$$
Now let \( \{a_r\}, \{b_r\} \) be (fixed) sequences of positive numbers decreasing to 0, with \( a_0 < 1/2 \), and let \( \sum b_r < \infty \). We note that for a given \( r \), \( a_{nr}(p) \to 0 \) as \( n \to \infty \), uniformly for \( p \) in any closed interval contained in \((0, 1)\). It is bounded for any fixed \( n \); hence it is bounded for all \( n \), uniformly in \( p \) for \( p \) in any such interval. Hence we can choose \( \{c_r\} \) so that for each \( r \),

\[
c_r |a_{nr}(p)| < b_r \quad ( a_r \leq p \leq 1 - a_r, \text{ for all } n ).
\]

Now consider an arbitrary fixed value of \( p \), with \( 0 < p < 1 \), and let \( \varepsilon > 0 \) be arbitrarily given. Since \( a_r \to 0 \), we see that \( p \in [a_r, 1 - a_r] \) for sufficiently large \( r \). Choose \( R \) large enough for this to hold for all \( r \geq R \) such that, further, \( \sum_{r=R}^{\infty} b_r < \varepsilon \). Then for all \( n \),

\[
\sum_{r=R}^{\infty} c_r |a_{nr}(p)| < \varepsilon .
\]

But, for fixed \( r \) and \( p \), we have \( a_{nr}(p) \to 0 \) as \( n \to \infty \); so, having fixed \( R \), we have

\[
\sum_{r=0}^{R-1} c_r |a_{nr}(p)| < \varepsilon \quad \text{for all } n \text{ sufficiently large}.
\]

Then it follows from (5) and (6) that \( t_n(p) \to 0 \) as \( n \to \infty \). Thus the sequence \( x = \{x_n\} \) is summable by the method \( E_p \), for arbitrary \( p \in (0, 1) \); that is, \( x \in \bigcap_{0<p<1} (E_p) \).

On the other hand, for each fixed positive integer \( r \),

\[
|x_k| \geq c_r[k]_r = c_r k(k-1)\cdots(k-r+1) \geq (1/2)c_rk^r
\]

if \( k \) is sufficiently large. Hence \( \{x_n\} \) does not satisfy the conditions of the limitation theorem for summability by the Cesàro method \( C_r \) [1, pp. 101-102; 5, p. 104]. Since this is true for every positive integer \( r \), the sequence \( \{x_n\} \) is not summable by \( C_r \) for any real value of \( r > 0 \). This proves part (a) of the theorem.

(b) We note that all of the above remains valid if, for each \( r \geq 0 \), we replace \( c_r \) by any smaller positive number \( c'_r \). Hence, if we define the sequences \( x^{(m)} \) \((m = 1, 2, \ldots)\) by setting

\[
x_k^{(m)} = (-1)^k \sum_{r=0}^{k} c_r \left( \frac{r}{r+m} \right) [k]_r \quad (k = 0, 1, \ldots),
\]

then \( x^{(m)} \in \bigcap_{0<p<1} (E_p) \), but \( x^{(m)} \notin \bigcup_{r>0} (C_r) \) \((m = 1, 2, \ldots)\).

Now let \( y = \{y_k\} = a_1 x^{(1)} + \ldots + a_s x^{(s)} \) be an arbitrary given linear combination of the sequences \( x^{(1)}, x^{(2)}, \ldots \), where not all the \( a_j \) are zero. Then for each \( k = 0, 1, \ldots \),

\[
y_k = (-1)^k \sum_{r=0}^{k} c_r[k]_r \left( \sum_{j=1}^{s} \frac{ra_j}{r+j} \right) = (-1)^k \sum_{r=0}^{k} c_r[k]_r A(r)
\]

where \( A(x) = \sum_{j=1}^{s} x a_j / (x+j) \) \((x \geq 0)\). Now every (nonnegative) solution of the equation \( A(x) = 0 \) is a solution of the equation \( P(x) = 0 \), where \( P(x) = \sum_{j=1}^{s} x a_j \prod_{1 \leq i \leq s, \ i \neq j} (x+i) \). If the polynomial \( P(x) \) were identically zero, then
\[ P(-j) = 0 \text{ and hence } a_j = 0 \text{ for } 1 \leq j \leq s, \text{ contrary to the assumption about the numbers } a_j. \] 
Hence \( P(x) \) has only a finite number of zeros, so there exists a positive integer \( N \) such that \( P(x) \), and therefore also \( A(x) \), is nonzero and of constant sign in the interval \([N, \infty)\). From the equation \((7)\), we have for each \( k > N \),

\[
y_k = (-1)^k \sum_{r=0}^{N-1} c_r[k] A(r) + (-1)^k \sum_{r=N}^{k} c_r[k] A(r) = S_k + T_k \quad \text{(say)}.\]

Then \( S_k \) is the sum of \( N \) terms, each of which is \( O(k^{N-1}) \) as \( k \to \infty \), and hence \( S_k = O(k^{N-1}) \). On the other hand, since \( A(r) \) is nonzero and has constant sign for all \( r \geq N \) (and since \( c_r > 0 \) for all \( r \)), we see that

\[
|T_k| \geq c_N[k] |A(N)| = c_N k(k-1)(k-2) \cdots (k-N+1) |A(N)| \geq \frac{1}{2} c_N |A(N)| k^N \quad \text{for all sufficiently large } k.
\]

Hence \( T_k \neq O(k^{N-1}) \) as \( k \to \infty \). Since \( S_k = O(k^{N-1}) \), it follows that \( y_k = S_k + T_k \neq O(1) \). That is, no nontrivial linear combination of the sequences \( x^{(1)}, x^{(2)}, \ldots \) yields a bounded sequence. It follows then from Theorem 3 of [4] that for each \( m \geq 1 \) there exists a regular row-finite matrix method \( V^{(m)} \) whose summability field is the smallest sequence space containing \( c \) (the convergent sequences) and the sequences \( x^{(1)}, x^{(2)}, \ldots, x^{(m)} \). Then we have \( (V^{(1)}) \not\subseteq \bigcup_{r \geq 0} (C_r) \) and \( (V^{(1)}) \subseteq (V^{(2)}) \subseteq \cdots \subseteq \bigcap_{0 < p < 1} (E_p) \). This completes the proof of the theorem.

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**References**