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ON THE THEORY OF THE KONTOROVICH-LEBEDEV TRANSFORMATION ON DISTRIBUTIONS

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ABSTRACT. A new proof of the representation theorem for the Kontorovich-Lebedev transformation on distributions is given.

As is known from Bremermann [1] and Brychkov and Prudnikov [2], a variety of integral transformations have been extended to various classes of distributions. These problems were considered by Buggle [3], Glaeske and Heß [6], and Zemanian [10] for the Kontorovich-Lebedev transformation which was introduced by Lebedev [7] and generalized, for example, by Yakubovich [9]. Some applications of the Kontorovich-Lebedev transformation on distributions were given by Forristall and Ingram [5].

In this paper we give a simpler proof of the representation theorem for the Kontorovich-Lebedev transformation on distributions than that which was given by Zemanian [10], by defining the double integral for this transformation under suitable continuity and integrability conditions on the ordinary function f as

$$f(x) = \frac{1}{\pi^2} \lim_{\varepsilon \to 0+} \int_0^\infty \int_{-\infty}^\infty \tau \sinh(\pi\tau) K_{i\tau}(y) f(\tau)\, d\tau K_{ix}(y) y^{\varepsilon - 1}\, dy, \tag{1}$$

where $K_{ix}(y)$ is the Macdonald function; see Erdelyi, Magnus, Oberhettinger, and Tricomi [4].

Let $\mathscr{E}(R)$ and $\mathscr{E}'(R)$ as usual denote the customary spaces of test functions and distributions encountered in distribution theory; see Bremermann [1]. We will show that equation (1) continues to hold when f is in $\mathscr{E}'(R)$ with the inner integral suitably interpreted, the outer limit being in the weak topology of $\mathscr{E}'(R)$.

We define the generalized Kontorovich-Lebedev transform $F(y)$ of any f in $\mathscr{E}'(R)$ by

$$F(y) = \langle f(\tau), \tau \sinh(\pi\tau) K_{i\tau}(y) \rangle. \tag{2}$$

From the representation

$$K_{i\tau}(y) = \int_0^\infty e^{-y\cosh u} \cos(\tau u)\, du, \tag{3}$$

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it follows that $K_{i\tau}(y)$ is an entire function of τ in R for each fixed $y > 0$ and an analytic function of $y > 0$ for every fixed τ. The Kontorovich-Lebedev transformation $F(y)$ in (2) is also an analytic function. Indeed, $\mathscr{E}(R)$ is a metrizable locally convex space with the topology generated by the following collection of seminorms

$$\gamma_{p,K}(\phi) \equiv \sup_{x\in K}\{|D_x^p\phi(x)|\} < \infty, \tag{4}$$

where p is a nonnegative integer, K is a compact set in R, and $D_x = \partial/\partial x$. Thus with $\Delta y \neq 0$ we have

$$\frac{F(y+\Delta y)-F(y)}{\Delta y} - \langle f(\tau), \tau\sinh(\pi\tau)D_yK_{i\tau}(y)\rangle = \langle f(\tau), \psi_{\Delta y}(\tau)\rangle, \tag{5}$$

where

$$\psi_{\Delta y}(\tau) = \tau\sinh(\pi\tau)\int_0^\infty \cos(\tau u)\left[\frac{e^{-y\cosh u}(e^{-\Delta y\cosh u}-1)}{\Delta y} - D_y e^{-y\cosh u}\right]du.$$

It is straightforward to show that, as $|\Delta y| \to 0$, $\psi_{\Delta y}(\tau) \to 0$ in $\mathscr{E}(R)$ and (5) tends to zero.

Theorem. *Let f be in $\mathscr{E}'(R)$, and put*

$$F(y) = \langle f(\tau), \tau\sinh(\pi\tau)K_{i\tau}(y)\rangle.$$

Then for every $\phi(x)$ in $\mathscr{E}(R)$ such that $\phi(x)/x$ together with its derivatives are bounded functions we have the inversion formula

$$\frac{\langle f_\tau, \phi(\tau)\rangle + \langle f_{-\tau}, \phi(-\tau)\rangle}{2} = \lim_{\varepsilon\to 0+}\left\langle \pi^{-2}\int_0^\infty F(y)K_{ix}(y)y^{\varepsilon-1}\,dy, \phi(x)\right\rangle, \tag{6}$$

where the inner integral is absolutely convergent for any $\varepsilon > 0$.

Proof. For each τ in R, $K_{i\tau}(y)$ is an analytic function of $y > 0$, $|K_{i\tau}(y)| \le K_0(y)$, $K_0(y) = O(\ln y)$ as $y \to 0+$, and $K_0(y) = O(y^{-1/2}e^{-y})$ as $y \to \infty$. We first establish that for each $\varepsilon > 0$, $F(y)$ is in $L(R_+; K_0(y)y^{\varepsilon-1})$; i.e., $F(y)$ belongs to the Lebesgue space of summable functions with weight $K_0(y)y^{\varepsilon-1}$. Indeed, since for each $y > 0$, $\tau\sinh(\pi\tau)K_{i\tau}(y)$ is in $\mathscr{E}(R)$, it then follows from the continuity of distributions that

$$|F(y)| < C(f)\max_{0\le p\le r}\{\gamma_{p,K}\langle\tau\sinh(\pi\tau)K_{i\tau}(y)\rangle\}, \tag{7}$$

where $C(f)$ is a constant and r is a nonnegative integer. Hence from the integral representation (3) we obtain

$$\max_{0\le p\le r}\sup_{\tau\in K}\{|D_\tau^p[\tau\sinh(\pi\tau)K_{i\tau}(y)]|\} < A_r\int_0^\infty e^{-y\cosh u}u^r\,du, \tag{8}$$

where A_r is a constant. The last integral can be investigated as follows. With $0 < y < 1$,

$$\begin{aligned}\int_0^\infty e^{-y\cosh u}u^r\,du &= \int_y^\infty e^{-x}(x^2-y^2)^{-1/2}\ln^r[x/y+(x^2/y^2-1)^{1/2}]\,dx\\ &= \left(\int_y^1+\int_1^\infty\right)e^{-x}(x^2-y^2)^{-1/2}\ln^r[x/y+(x^2/y^2-1)^{1/2}]\,dx\\ &= I_1(y)+I_2(y).\end{aligned}$$

By the second mean value theorem, it is not difficult to see that

$$I_1(y) = O(\ln^r y^{-1} \operatorname{Arch} y^{-1})$$

as $y \to 0+$ and

$$|I_2(y)| \le B \int_1^\infty e^{-x} \ln^r(x/y)\,dx = O(\ln^r y^{-1})$$

as $y \to 0+$, where B is a constant. It is straightforward to see that $F(y) = O(1)$ as $y \to \infty$. So we finally conclude that $F(y)$ is in $L(R_+; K_0(y)y^{\varepsilon-1})$, with $\varepsilon > 0$. Now according to the definition of $F(y)$ we must show that

$$\begin{aligned} I(x, \varepsilon) &= \pi^{-2} \int_0^\infty F(y) K_{ix}(y) y^{\varepsilon-1}\,dy \\ &= \left\langle f(\tau), \pi^{-2}\tau \sinh(\pi\tau) \int_0^\infty K_{ix}(y) K_{i\tau}(y) y^{\varepsilon-1}\,dy \right\rangle. \end{aligned} \tag{9}$$

In order to prove (9), we must establish that

$$\begin{aligned} I(x, \varepsilon) &= \lim_{T\to\infty} \int_{1/T}^T \Phi(x, y) \langle f(\tau), \tau \sinh(\pi\tau) K_{i\tau}(y)\rangle\,dy \\ &= \lim_{T\to\infty} \langle f(\tau), \theta_T(x, \tau)\rangle, \end{aligned} \tag{10}$$

where $\Phi(x, y) = \pi^{-2} y^{\varepsilon-1} K_{ix}(y)$ and

$$\theta_T(x, \tau) = \tau \sinh(\pi\tau) \int_{1/T}^T \Phi(x, y) K_{i\tau}(y)\,dy. \tag{11}$$

We now represent the integral (11) as the limit of the Riemann sums

$$\begin{aligned} Q(x, \tau, n) = \tau \sinh(\pi\tau) \frac{T - 1/T}{n} \sum_{m=1}^n \Phi\left(x, \frac{1}{T} + m\frac{T-1/T}{n}\right) \\ \times K_{i\tau}\left(\frac{1}{T} + m\frac{T-1/T}{n}\right). \end{aligned}$$

Then by the properties of distributions we have

$$\langle f(\tau), \theta_T(x, \tau)\rangle = \langle f(\tau), \lim_{n\to\infty} Q(x, \tau, n)\rangle, \tag{12}$$

where we note that both $\theta_T(x, \tau)$ and $Q(x, \tau, n)$ are members of $\mathscr{E}(R)$. Moreover, $D_\tau^p \theta_T(x, \tau) - D_\tau^p Q(x, \tau, n)$ tends uniformly to zero as $n \to \infty$ for all τ in $K \subset R$ and each x in R. Thus $Q(x, \tau, n)$ tends to $\theta_T(x, \tau)$ in $\mathscr{E}(R)$ so that

$$\begin{aligned} \langle f(\tau), \theta_T(x, \tau)\rangle &= \lim_{n\to\infty} \langle f(\tau), Q(x, \tau, n)\rangle \\ &= \left\langle f(\tau), \tau \sinh(\pi\tau) \int_{1/T}^T \phi(x, y) K_{i\tau}(y)\,dy \right\rangle, \end{aligned} \tag{13}$$

and the limit equality (9) is also valid since the inner integral is absolute and uniformly convergent.

Next, we note that according to the integral 2.16.33.2 from Prudnikov, Brychkov, and Marichev [8]

$$\begin{aligned}\theta(x, \tau, \varepsilon) &= \frac{\tau\sinh(\pi\tau)}{\pi^2}\int_0^\infty y^{\varepsilon-1}K_{ix}(y)K_{i\tau}(y)\,dy \\ &= \frac{2^{\varepsilon-3}\tau\sinh(\pi\tau)}{\pi^2\Gamma(\varepsilon)}\left|\Gamma\left(\frac{\varepsilon+i(x+\tau)}{2}\right)\Gamma\left(\frac{\varepsilon+i(x-\tau)}{2}\right)\right|^2,\end{aligned} \tag{14}$$

where $\Gamma(z)$ denotes the gamma function, see [4]. Hence applying the addition formula of the gamma function, we represent $I(x, \varepsilon)$ by

$$\begin{aligned}I(x, \varepsilon) &= \left\langle f(\tau), \frac{2^{\varepsilon+1}\varepsilon\tau\sinh(\pi\tau)|\Gamma(\frac{\varepsilon+i(x+\tau)}{2}+1)\Gamma(\frac{\varepsilon+i(x-\tau)}{2}+1)|^2}{\pi^2\Gamma(\varepsilon+1)[\varepsilon^2+(x+\tau)^2][\varepsilon^2+(x-\tau)^2]}\right\rangle \\ &= \left\langle f(\tau), \frac{2^{\varepsilon-1}\varepsilon\sinh(\pi\tau)|\Gamma(\frac{\varepsilon+i(x+\tau)}{2}+1)\Gamma(\frac{\varepsilon+i(x-\tau)}{2}+1)|^2}{x\pi^2\Gamma(\varepsilon+1)[\varepsilon^2+(x+\tau)^2]}\right\rangle \\ &\quad - \left\langle f(\tau), \frac{2^{\varepsilon-1}\varepsilon\sinh(\pi\tau)|\Gamma(\frac{\varepsilon+i(x+\tau)}{2}+1)\Gamma(\frac{\varepsilon+i(x-\tau)}{2}+1)|^2}{x\pi^2\Gamma(\varepsilon+1)[\varepsilon^2+(x+\tau)^2]}\right\rangle \\ &= I_1(x, \varepsilon) - I_2(x, \varepsilon).\end{aligned}$$

Now since the function

$$g(x, \tau, \varepsilon) = \frac{2^\varepsilon\sinh(\pi\tau)}{\pi\Gamma(\varepsilon+1)}\left|\Gamma\left(\frac{\varepsilon+i(x+\tau)}{2}+1\right)\Gamma\left(\frac{\varepsilon+i(x-\tau)}{2}+1\right)\right|^2$$

for each x in R and $\varepsilon > 0$ is a C^∞-function, we can represent $I_j(x, \varepsilon)$ for $j = 1, 2$ in the following forms:

$$\begin{aligned}I_{\{{1 \atop 2}\}}(x, \varepsilon) &= \langle f(\tau)g(x, \tau, \varepsilon), \{4\pi x i[\tau - (\pm x + i\varepsilon)]\}^{-1}\rangle \\ &\quad - \langle f(\tau)g(x, \tau, \varepsilon), \{4\pi x i[\tau - (\pm x - i\varepsilon)]\}^{-1}\rangle \\ &= (2x)^{-1}\langle \hat{f}_g(\pm x + i\varepsilon) - \hat{f}_g(\pm x - i\varepsilon)\rangle.\end{aligned} \tag{15}$$

Using the properties of the analytic representations

$$\hat{f}(z) = (2\pi i)^{-1}\langle f_\tau, (\tau - z)^{-1}\rangle$$

for distributions from Bremermann [1], we see that $I(x, \varepsilon) = O(|x|^{-2})$ as $|x| \to \infty$, so according to the conditions of the theorem, $\langle I(x, \varepsilon), \phi(x)\rangle$ is a regular distribution.

Finally, applying the theorem from §5.6 in Bremermann [1], we have

$$\begin{aligned}
\lim_{\varepsilon\to0+}\langle I(x,\varepsilon),\phi(x)\rangle &= \lim_{\varepsilon\to0+}\langle I_1(x,\varepsilon),\phi(x)\rangle - \lim_{\varepsilon\to0+}\langle I_2(x,\varepsilon),\phi(x)\rangle \\
&= \lim_{\varepsilon\to0+}\frac{1}{2}\int_R[\hat{f}_g(x+i\varepsilon)-\hat{f}_g(x-i\varepsilon)]x^{-1}\phi(x)\,dx \\
&\quad - \lim_{\varepsilon\to0+}\frac{1}{2}\int_R[\hat{f}_g(-x+i\varepsilon)-\hat{f}_g(-x-i\varepsilon)]x^{-1}\phi(x)\,dx \\
&= \frac{1}{2\pi}\left\langle f_\tau\sinh(\pi\tau)|\Gamma(i\tau+1)|^2,\frac{\phi(\tau)}{\tau}\right\rangle \\
&\quad + \frac{1}{2\pi}\left\langle f_{-\tau}\sinh(\pi\tau)|\Gamma(i\tau+1)|^2,\frac{\phi(-\tau)}{\tau}\right\rangle \\
&= \frac{\langle f_\tau,\phi(\tau)\rangle+\langle f_{-\tau},\phi(-\tau)\rangle}{2}.
\end{aligned}$$

The theorem is now completely proved.

REFERENCES

1. H. Bremermann, *Distributions, complex variables and Fourier transforms*, Addison Wesley, Reading, MA, 1965.
2. Yu. A. Brychkov and A. P. Prudnikov, *Integral transforms of generalized functions*, M. Nauka, 1977.
3. G. Buggle, *Die Kontorovich-Lebedev-Transformation und die Mehler-Fok-Transformation für Klassen verallgemeinerter Functionen*, Dissertation, TH Darmstadt, 1977.
4. A. Erdelyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher transcendental functions*, Vol. 2, McGraw-Hill, New York, 1953.
5. G. Z. Forristall and J. D. Ingram, *Evaluation of distributions useful in Kontorovich-Lebedev transform theory*, SIAM J. Math. Anal. **3** (1972), 561–566.
6. H. J. Glaeske and A. A. Heß, *Convolution connected with Kontorovich-Lebedev transform*, Math. Z. **193** (1986), 67–78.
7. N. N. Lebedev, *Sur une formule d'inversion*, Dokl. Akad. Nauk SSSR **52** (1946), 655–658.
8. A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series*, Vol. 2, Gordon and Breach, New York, London, Paris, Montreux, and Tokyo, 1986.
9. S. B. Yakubovich, *The general integral transforms of the Kontorovich-Lebedev type and some of their applications*, Ph.D. Dissertation, Minsk, 1987. (Russian)
10. A. H. Zemanian, *The Kontorovich-Lebedev transformation on distributions of compact support and its inversion*, Math. Proc. Cambridge Philos. Soc. **77** (1975), 139–143.

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