RIEMANNIAN METRICS WITH LARGE $\lambda_1$

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(Communicated by Peter Li)

Abstract. We show that every compact smooth manifold of three or more dimensions carries a Riemannian metric of volume one and arbitrarily large first eigenvalue of the Laplacian.

Let $(M^n, g)$ be a compact, connected Riemannian manifold of $n$ dimensions. The Laplacian $\Delta_g$ acting on functions on $M$ has discrete spectrum. Let $\lambda_1(g)$ denote the smallest positive eigenvalue of $\Delta_g$. Hersch [5] proved that

$$\lambda_1(g) \cdot \text{vol}(S^2, g) \leq 8\pi$$

for every Riemannian metric $g$ on the 2-sphere $S^2$.

In connection with this result, Berger [2] asked whether there exists a constant $k(M)$ such that

$$(1) \quad \lambda_1(g) \cdot \text{vol}(M^n, g)^{2/n} \leq k(M)$$

for any Riemannian metric on $M$. Yang and Yau [8] proved that the inequality above holds for a compact surface $S$ of genus $\gamma$ with $k(S) = 2\pi(\gamma + 1)$.

Subsequently, numerous examples of manifolds were constructed for which (1) is false (cf. [3] for a discussion and references). In particular, for every $n \geq 3$, the sphere $S^n$ admits metrics of volume one with $\lambda_1$ arbitrarily large [3, 6]. Bleecker conjectured in [3] that such metrics exist on every manifold $M^n$ if $n \geq 3$. In this note we give a very simple proof of Bleecker's conjecture using known examples and quite general principles. The same result has been proved independently by Xu [7] by a construction similar to ours. His argument, however, is much harder than our proof.

Theorem 1. Every compact manifold $M^n$ with $n \geq 3$ admits metrics $g$ of volume one with arbitrarily large $\lambda_1(g)$.

Proof. The idea of the proof is very simple. We take a metric $g_0$ on $S^n$ with $\text{vol}(S^n, g_0) = 1$ and $\lambda_1(g_0) \geq k + 1$, where $k$ is a large constant. We excise from $S^n$ a very small ball $B(p, \eta) = B_\eta$ and form the connected sum of $S^n$ with $M$. The resulting manifold is diffeomorphic to $M$ and has a submanifold $\Omega$, with smooth boundary, naturally identified with $S^n \setminus B_\eta$. Let $g_1$ be an
arbitrary metric on $M$ whose restriction to $\Omega$ is equal to $g_0|\Omega$. We modify the metric $g_1$ making it very small on “most of” $M \setminus \Omega$ without altering it on $\Omega$. With the new metric, $M$ looks practically like $(S^n, g_0)$ in the sense that all of the topology of $M$ is contained in a part which is metrically very small. In particular, the smallest positive eigenvalue of this metric is very close to $\lambda_1(g_0)$.

To make this into a rigorous proof we use results of Colin de Verdière [4, Theorem III.1] and Anné [1]. Thus, by [1, Theorem 2], if $\eta$ is chosen sufficiently small, the first positive eigenvalue $\mu_1$ of the Laplacian of $(\Omega, g_0)$ for the Neumann boundary conditions is a very good approximation of $\lambda_1(g_0)$ so that $\mu_1 \geq k + \frac{1}{2}$. Let $\epsilon$ be a small positive number. Take a sequence of smooth functions $F_i, \epsilon$ such that $F_i, \epsilon|\Omega \equiv 1$, $1 \geq F_i, \epsilon \geq \epsilon$, and $\lim_{i \to \infty} F_i, \epsilon(x) = \epsilon$ for every $x \in M \setminus \Omega$, and consider metrics $g_i, \epsilon = F_i, \epsilon g_1$. Colin de Verdière showed in the course of proof of Theorem III.1 of [4] that for every positive integer $J$ the eigenvalues $\mu_j$, $j \leq J$, of the Neumann problem for $\Omega$ can be approximated to arbitrary accuracy by $\lambda_j(g_i, \epsilon)$ by first choosing $\epsilon$ sufficiently small and then $i$ sufficiently large (condition $\ast$ appearing in [4, Theorem III.1] is satisfied for some choice of indices and constants since the spectrum of $(\Omega, g_0)$ is discrete). It follows that $\lambda_1(g_i, \epsilon) \geq k + \frac{1}{4}$ for appropriate choices of $\eta$, $\epsilon$, and $i$. Finally, we multiply the metric $g_i, \epsilon$ by a constant to make the volume equal to one and call the resulting metric $g$. If the choices of $\eta$ and $\epsilon$ were sufficiently small and $i$ is sufficiently large then the rescaling factor is practically equal to one so that $\lambda_1(g) \geq k$. \qed

References