PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 122, Number 3, November 1994

ON THE HILBERT FUNCTION OF DETERMINANTAL RINGS AND THEIR CANONICAL MODULE

ALDO CONCA AND JÜRGEN HERZOG

(Communicated by Wolmer V. Vasconcelos)

ABSTRACT. We determine the Hilbert function of a determinantal ring and of its canonical module using a combinatorial result of Krattenthaler. This gives a new proof of Abhyankar's formula.

Let K be a field and $X = (X_{ij})$ be an $m \times n$-matrix of indeterminates, with $m \leq n$. We denote by $K[X]$ the polynomial ring over K in the indeterminates X_{ij} and by $I_{r+1}(X)$ the ideal in $K[X]$ generated by the $r+1$-minors of X and set

$$R_{r+1} = K[X]/I_{r+1}(X).$$

The purpose of this note is to derive a compact formula for the Hilbert series of R_{r+1} and its canonical module. The result is actually a simple rewriting of Abhyankar's formula [1]. However, we want to point out that our formula can as well be obtained from a combinatorial result of Krattenthaler [7] and thus gives a new proof of Abhyankar's formula. Of course, the burden of the proof is hidden in the combinatorial part.

Krattenthaler counts the nonintersecting paths with a given number of corners. To be precise consider the set of points $V = \{(i, j): 1 \leq i \leq m, 1 \leq j \leq n\}$. We define a partial order on V by setting $(i, j) \leq (i', j')$ if $i \geq i'$ and $j \leq j'$. Let $P, Q \in V$; a maximal chain C in V with end points P and Q will be called a *path* from P to Q. A *corner* of C is an element $(i, j) \in C$ for which $(i-1, j)$ and $(i, j-1)$ belong to C as well. The path in Figure 1 on the next page has two corners.

Let P_i, Q_i, $i = 1, \dots, r$, be points of V. A subset $W \subset V$ is called an *r-tuple of nonintersecting paths from P_i to Q_i* $(i = 1, \dots, r)$ if $W = C_1 \cup C_2 \cup \cdots \cup C_r$ where each C_i is a path from P_i to Q_i and where $C_i \cap C_j = \varnothing$ if $i \neq j$. The number of corners $c(W)$ of W is the sum of the number of corners of the C_i, $i = 1, \dots, r$.

Received by the editors February 8, 1993.
1991 *Mathematics Subject Classification*. Primary 14M12; Secondary 13D40, 13C40.
The first author's research was supported by a grant from Consiglio Nazionale delle Ricerche, Italy.

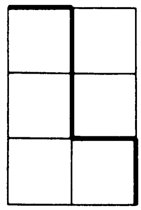

FIGURE 1

Theorem (Kulkarni, Krattenthaler). *Let* $P_i = (a_i, n)$, $1 \le a_1 < \cdots < a_r \le m$, *and* $Q_i = (m, b_i)$, $1 \le b_1 < \cdots < b_r \le n$. *Then the number of nonintersecting paths from* P_i *to* Q_i, $i = 1, \dots, r$, *with exactly* k *corners is given by*

$$\sum \det\left(\binom{m - a_j - i + j}{k_i}\binom{n - b_i + i - j}{k_i + i - j}\right)_{i,j=1,\dots,r},$$

where the sum is taken over all sequences $(k_1, \dots, k_r)$ *such that* $\sum_{i=1}^r k_i = k$.

Kulkarni [8] deduces this theorem from Abhyankar's formula for the Hilbert function of a determinantal ring, while Krattenthaler [7] gives a purely combinatorial proof of it. Actually his result is more general, that is, he allows the end points to be in a more general position.

Let us indicate, as described in [5] and [4], or in Kulkarni's paper [8], how the Hilbert series of determinantal rings is related to the nonintersecting paths. Given integers $1 \le a_1 < \cdots < a_r \le m$ and $1 \le b_1 < \cdots < b_r \le n$, we denote by $[a_1, \dots, a_r | b_1, \dots, b_r]$ the minor with the rows $a_1, \dots, a_r$ and the columns $b_1, \dots, b_r$. The set of all minors $P(X)$ is a poset with the following partial order:

$$[a_1, \dots, a_r | b_1, \dots, b_r] \le [c_1, \dots, c_s | d_1, \dots, d_s]$$

if $r \ge s$ and $a_1 \le c_1, \dots, a_s \le c_s$, $b_1 \le d_1, \dots, b_s \le d_s$. Let $\sigma \in P(X)$; then we denote by $I_\sigma(X)$ the ideal generated by the minors in the set $\{\eta \in P(X)\colon \eta \not\ge \sigma\}$. In case $\sigma = [1, \dots, r | 1, \dots, r]$ one obtains $I_\sigma(X) = I_{r+1}(X)$.

It is shown in [5] that for a suitable term order (order of the monomials) the ideal of initial forms $I_\sigma(X)^*$ of $I_\sigma(X)$ is generated by squarefree monomials. Thus $K[X]/I_\sigma(X)^*$ may be viewed as a Stanley-Reisner ring of a certain simplicial complex Δ_σ. Further it is shown in [5] that Δ_σ is shellable and that its facets may be identified with the nonintersecting paths connecting $P_i = (a_i, n)$ with $Q_i = (m, b_i)$ for $i = 1, \dots, r$. Using the fact that $R_\sigma = K[X]/I_\sigma(X)$ and $K[X]/I_\sigma(X)^*$ have the same Hilbert series and that Δ_σ is shellable, one deduces as in [4] from the McMullen-Walkup formula that the Hilbert series $H_{R_\sigma}(t)$ of R_σ is of the form

$$H_{R_\sigma}(t) = \frac{\sum_k h_k t^k}{(1-t)^d}, \tag{1}$$

where $d = \dim R_\sigma(X)$ and where h_k is the number of the nonintersecting paths from P_i to Q_i $(i = 1, \dots, r)$ with exactly k corners. Thus applying the theorem we get

(2)
$$H_{R_\sigma}(t) = \frac{\det\left(\sum_k \binom{m-a_j-i+j}{k}\binom{n-b_i+i-j}{k+i-j} t^k\right)_{i,j=1,\dots,r}}{(1-t)^d},$$

where $d = (m+n+1)r - \sum_{i=1}^r (a_i + b_i)$ is the dimension of R_σ.

From now on we concentrate our attention to the Hilbert function of the ring R_{r+1} which is defined by the ideal of minors $I_{r+1}(X)$ of size $r+1$. In this case we have to consider the nonintersecting paths from $P_i = (i, n)$ to $Q_i = (m, i)$. Let us denote by $\mathscr{C}(P_i, Q_j)_k$ the number of paths from P_i to Q_j with exactly k corners and by $\mathscr{C}(P_i, Q_j)$ the polynomial $\sum_k \mathscr{C}(P_i, Q_j)_k t^k$. Then we have

Corollary 1.

$$H_{R_{r+1}}(t) = \frac{\det(\mathscr{C}(P_i, Q_j))_{i,j=1,\dots,r}}{t^{\binom{r}{2}}(1-t)^d} = \frac{\det\left(\sum_k \binom{m-i}{k}\binom{n-j}{k}\right)_{i,j=1,\dots,r}}{t^{\binom{r}{2}}(1-t)^d}.$$

Proof. For $P = (a, b)$ we set $|P| = a^2 + b^2$. One proves easily by induction on $|P_i - Q_j|$ that $\mathscr{C}(P_i, Q_j)_k = \binom{m-i}{k}\binom{n-j}{k}$. Thus it remains to show that

$$\det\left(\sum_k \binom{m-i}{k}\binom{n-j}{k+i-j} t^k\right)_{i,j=1,\dots,r}$$
$$= \frac{1}{t^{\binom{r}{2}}} \det\left(\sum_k \binom{m-i}{k}\binom{n-j}{k} t^k\right)_{i,j=1,\dots,r}.$$

This identity is a special case $(u = 0)$ of the next

Lemma. *Let $u \geq 0$ an integer, and consider the following $r \times r$ matrices of polynomials*

$$H_u = \left(\sum_k \binom{m-i}{k}\binom{n-j}{k+u+i-j} t^k\right),$$
$$H'_u = \left(\frac{1}{t^{i-1}} \sum_k \binom{m-i}{k}\binom{n-j}{k+u} t^k\right),$$
$$A = \left((-1)^{i-j}\binom{i-1}{j-1}\frac{1}{t^{i-j}}\right), \quad \text{and} \quad B = \left((-1)^{j-i}\binom{j-1}{i-1}\right).$$

Then $H'_u = AH_uB$. In particular, since A and B are triangular matrices whose diagonal elements are all 1, it follows that $\det H'_u = \det H_u$.

The proof follows by straightforward calculation using the identity

$$\sum_{q \geq 1} (-1)^{p-q}\binom{p-1}{q-1}\binom{a-q}{b-q} = (-1)^{p-1}\binom{a-p}{b-1}.$$

In order to compute the Hilbert series of the graded canonical module ω_{r+1} of R_{r+1} we use the equation

(3)
$$H_{\omega_{r+1}}(t) = (-1)^d H_{R_{r+1}}(t^{-1}),$$

where $d = (n+m-r)r$ is the dimension of R_{r+1}, and obtain

Corollary 2.

$$\begin{aligned} H_{\omega_{r+1}}(t) &= \frac{t^{nr}\det(\sum_k \binom{m-i}{k}\binom{n-j}{n-m+k+i-j}t^k)}{(1-t)^d} \\ &= \frac{t^{nr}\det(\sum_k \binom{m-i}{k}\binom{n-j}{n-m+k}t^k)}{t^{\binom{r}{2}}(1-t)^d}. \end{aligned}$$

Proof. The first equation follows directly from the substitution of t by t^{-1}, while for the second equation we use our lemma with $u = n - m$.

Recall that, by a result of Stanley, a Cohen-Macaulay domain homogeneous K-algebra R is Gorenstein if and only if

$$H_R(t) = (-1)^d t^a H_R(t^{-1}) \tag{4}$$

for some a. If this is the case, then a is the degree of the rational function $H_R(t)$, the so-called a-invariant of R.

Comparing the formulas in Corollaries 1 and 2 we deduce the well-known fact that R_{r+1} is Gorenstein if and only if $m = n$.

As a last application we compute the Cohen-Macaulay type $r(R_{r+1})$ of R_{r+1}, that is, the minimal number of generators of ω_{r+1}. For this we use the fact, proved by Bruns [2] (see also [3]), that R_{r+1} is a level ring, which means that all generators of ω_{r+1} have the same degree. Therefore, by (3) and Corollary 1, the type of R_{r+1} is the leading coefficient of the polynomial $\det(\sum_k \binom{m-i}{k}\binom{n-j}{k}t^k)$. The (i, j)th polynomial in the matrix has degree $m - i$ for $j \le n - m + i$ and degree $n - j$ for $j > n - m + i$. Hence we see that

$$r(R_{r+1}) = \det\left(\binom{n-j}{m-i}\right)_{i,j=1,\dots,r}.$$

Using the Vandermonde determinant we get

$$r(R_{r+1}) = \prod_{i=1}^{r} \binom{n-i}{m-r} \Big/ \binom{m-i}{m-r}.$$

In [3] Bruns and Vetter quote the formula $r(R_{r+1}) = \prod_{i=1}^{m-r}\binom{n-i}{r}/\binom{m-i}{r}$. This formula was obtained by J. Brennan from the explicit computation of the Hilbert series of Schubert varieties due to Hodge and Pedoe [6, Theorem 3, p. 387]. It can be shown directly, by induction on r, that these formulas agree. In particular, one sees that

$$r(R_{r+1}) = r(R_{m-r+1}).$$

We conclude with one observation. The formula in Corollary 1 has the following combinatorial interpretation. Let σ be an element of S_r, the group of permutations of r elements. Let us denote by $\mathscr{C}(P_\sigma, Q)_k$ the number of the families of paths from $P_{\sigma(i)} = (\sigma(i), n)$ to $Q_i = (m, i)$, $i = 1, \dots, r$, with k corners, and by $\mathscr{C}(P, Q)_k^+$ the number of the nonintersecting paths from (i, n) to (m, i), $i = 1, \dots, r$, with k corners. Expanding the determinant

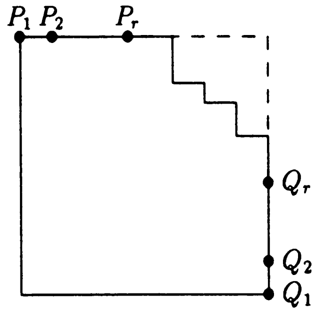

FIGURE 2

in the formula in Corollary 1, one gets

$$\mathscr{C}(P, Q)^+_{k-\binom{r}{2}} = \sum_{\sigma \in S_r} (-1)^\sigma \mathscr{C}(P_\sigma, Q)_k$$

for all $k \geq 0$. It would be nice to have a direct proof of this identity. We believe that it is true even when we consider paths in a restricted region having the shape of a one-sided ladder, as indicated in Figure 2. We can give a proof for the case when $r = 2$.

REFERENCES

1. S. S. Abhyankar, *Enumerative combinatorics of Young tableaux*, Marcel Dekker, New York, 1988.
2. W. Bruns, *The canonical module of a determinantal ring*, Commutative Algebra, Durham 1981 (R. Y. Sharp, ed.), London Math. Soc. Lecture Note Ser., vol. 72, Cambridge Univ. Press, Cambridge, 1982, pp. 109–120.
3. W. Bruns and U. Vetter, *Determinantal rings*, Lecture Notes in Math., vol. 1327, Springer, Heidelberg, 1988.
4. W. Bruns and J. Herzog, *On the computation of a-invariants*, Manuscripta Math. **77** (1992), 201–213.
5. J. Herzog and N. V. Trung, *Gröbner bases and multiplicity of determinantal and pfaffian ideals*, Adv. Math. **96** (1992), 1–37.
6. W. V. D. Hodge and D. Pedoe, *Methods of algebraic geometry*, Vol. II, Cambridge Univ. Press, Cambridge, 1952.
7. C. Krattenthaler, *Non-Crossing two-rowed arrays and summations for Schur functions*, preprint, 1992.
8. D. M. Kulkarni, *Counting of paths and coefficients of Hilbert polynomial of a determinantal ideal*, preprint.

FB6 MATHEMATIK UND INFORMATIK, UNIVERSITÄT GESAMTHOCHSCHULE ESSEN, UNIVERSITÄTSSTR. 3, 45117 ESSEN, GERMANY

E-mail address: `mat302@vm.hrz.uni-essen.de`

E-mail address: `mat300@vm.hrz.uni-essen.de`