PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 122, Number 3, November 1994

COUNTABLE PARACOMPACTNESS OF Σ-PRODUCTS

LECHENG YANG

(Communicated by Franklin D. Tall)

ABSTRACT. It is known that Σ-products of compact spaces always are countably paracompact but not necessarily normal. In the present paper it is proved that a Σ-product of paracompact σ-spaces is normal if and only if it is countably paracompact.

1. INTRODUCTION

The equivalence of normality and countable paracompactness in Cartesian products has been investigated by many authors [5, 8, 9, 17, 22] so that this topic constitutes a very interesting part in the theory of product spaces [7, 14]. In this paper the equivalence of normality and countable paracompactness will be considered for Σ-products.

The concept of Σ-products was introduced by Corson [2] who proved that Σ-products of complete metric spaces are normal. In Gul'ko [4] and Rudin [15] the following is shown:

(i) A Σ-product of metric spaces is normal.

This answers affirmatively a long outstanding question raised by Corson [2]. Kombarov [8] later generalized (i) by obtaining the following result:

(ii) A Σ-product of paracompact p-spaces is (collectionwise) normal if and only if it has countable tightness.

In connection with the above results, the following Questions 1 and 2 are considered by Yajima [18] and Kodama, respectively.

Question 1. Is a Σ-product of paracompact σ-spaces normal if it has countable tightness?

Question 2. Is a Σ-product of Lašnev spaces normal?

Question 1 has been answered positively. In fact Yajima [18] even proved

(iii) A Σ-product of paracompact Σ-spaces is (collectionwise) normal if it has countable tightness.

Since paracompact p-spaces are Σ-spaces, (iii) is also a generalization of the "if" part of (ii). However, the countable tightness is no longer a necessary condition for a Σ-product of paracompact Σ-spaces to be normal, because there

Received by the editors March 3, 1993.
1991 *Mathematics Subject Classification.* Primary 54B10, 54D10, 54D18.
Key words and phrases. Σ-product, σ-space, countably paracompact, normal.

exists a collectionwise normal Σ-product of M_1-spaces which has no countable tightness [18]. Moreover, since there exists a nonnormal Σ-product of M_1-spaces [18], in Question 1 the assumption of countable tightness cannot be dropped. On the other hand, Rudin [16] proved that any Σ-product of metric spaces is shrinking and hence countably paracompact. It is also known from Yajima [19] that any normal Σ-product of σ-spaces is countably paracompact.

The main purpose of this paper is to establish the equivalence of normality and countable paracompactness of Σ-products of paracompact σ-spaces. Namely we prove the following theorem.

Theorem 1. *A Σ-product of paracompact σ-spaces is normal if and only if it is countably paracompact.*

In the rest of the paper, we also consider the subshrinking property of $X \times \kappa$, where X is a semistratifiable space and κ an uncountable regular cardinal with the usual order topology. The subshrinking property was introduced by Yajima [20] which is important for the study of shrinking property (see Yajima [20] and Hoshina [6]). Yajima [21] recently proved that $X \times \kappa$ is subshrinking for any σ-space X, and he asked whether $X \times \kappa$ is subshrinking for a semistratifiable space X. We shall prove

Theorem 2. *Let X be a semistratifiable space with $\chi(X) < \kappa$. Then $X \times \kappa$ is subshrinking.*

All spaces considered here are assumed to be regular T_1. The set of natural numbers is denoted by $\mathbb{N}$ and natural numbers are denoted by i, j, k, and n. κ always denotes an uncountable regular cardinal with the usual order topology.

2. Proof of Theorem 1

Let $X = \prod_{\lambda \in \Lambda} X_\lambda$ be the Cartesian product of spaces X_λ, $\lambda \in \Lambda$, and let $s = (s_\lambda)_{\lambda \in \Lambda}$ be a fixed point of X. The subspace $\Sigma = \{x \in X : x_\lambda = s_\lambda$ for all but countably many $\lambda \in \Lambda\}$ of X is called a *Σ-product* of spaces X_λ, $\lambda \in \Lambda$. Such a point $s \in \Sigma$ is called the *base point* of Σ, which is often omitted.

Let X be a Σ-product of spaces X_λ, $\lambda \in \Lambda$, with a base point $(s_\lambda)_{\lambda \in \Lambda}$. For a point $x \in X$, denote by $\mathrm{Supp}(x)$ the set $\{\lambda \in \Lambda : x_\lambda \neq s_\lambda\}$. Let Δ be an index set such that for each $\xi \in \Delta$, R_ξ is a subset of Λ. Then we denote by X_ξ the Cartesian product $\prod_{\lambda \in R_\xi} X_\lambda$ and by p_ξ the projection of X onto X_ξ for each $\xi \in \Delta$.

Let $\xi = (\alpha_{ij})_{i,j \le n}$ be an $n \times n$ matrix. By ξ_k we denote the $k \times k$ matrix $(\alpha_{ij})_{i,j \le k}$ for $1 \le k \le n$. In particular, ξ_{n-1} is often abbreviated as ξ_- and ξ_0 denotes the empty set $\varnothing$.

A space is called a *σ-space* if it has a σ-locally finite net [13]. Note that Lašnev spaces (i.e., closed images of metric spaces) are M_1, and M_1-spaces are paracompact σ [13]. It is well known that the countable product of paracompact σ-spaces (Σ-spaces) is paracompact σ (Σ).

The following two lemmas are useful to prove Theorem 1.

Lemma 1 [5, Lemma 2.1]. *Let X be a countably paracompact space and let E and F be a pair of disjoint subsets. Suppose that F is closed and there exists open sets U_n, $n \in \mathbb{N}$, such that $E \subset \bigcap_{n \in \mathbb{N}} U_n$ and $(\bigcap_{n \in \mathbb{N}} \overline{U}_n) \cap F = \varnothing$. Then E and F are separated by open sets.*

Lemma 2 [11, Theorem 1]. *Let X be a σ-space. Then there exists a sequence $\{\mathscr{F}_n:\ n \in \mathbb{N}\}$ of locally finite closed covers of X, satisfying*

(a) $\mathscr{F}_n = \{F(\alpha_1 \cdots \alpha_n) : \alpha_1, \dots, \alpha_n \in \Omega\}$ *for each* $n \in \mathbb{N}$,

(b) $F(\alpha_1 \cdots \alpha_n) = \bigcup\{F(\alpha_1 \cdots \alpha_n \alpha) : \alpha \in \Omega\}$ *for each* $\alpha_1, \dots, \alpha_n \in \Omega$,

(c) *For each* $x \in X$, *there exists a sequence* $\alpha_1, \alpha_2, \dots \in \Omega$ *such that* $x \in \bigcap_{n\in\mathbb{N}} F(\alpha_1 \cdots \alpha_n)$ *and each open nbd of* x *contains some* $F(\alpha_1 \cdots \alpha_n)$.

The above sequence $\{\mathscr{F}_n:\ n \in \mathbb{N}\}$ is called a *spectral σ-net* of X and the sequence $\{F(\alpha_1 \cdots \alpha_n) : n \in \mathbb{N}\}$ in (c) is called a *local σ-net* of X at x.

Our proof of Theorem 1 is based on the idea in Yajima [18, 20] and we shall use the following fact: a space X is normal if and only if for every pair A, B of disjoint closed subsets of X there exists a σ-locally finite open cover $\mathscr{U}$ of X such that either $\overline{U} \cap A = \varnothing$ or $\overline{U} \cap B = \varnothing$ for every $U \in \mathscr{U}$.

Proof of Theorem 1. Let X be a Σ-product of paracompact σ-spaces X_λ, $\lambda \in \Lambda$, with a base point $s = (s_\lambda)_{\lambda\in\Lambda} \in X$, and suppose X is countably paracompact. To prove that X is normal, let A and B be a pair of disjoint closed subsets of X; we shall find a σ-locally finite open cover $\mathscr{G}$ of X such that either $\overline{U} \cap A = \varnothing$ or $\overline{U} \cap B = \varnothing$ for every $U \in \mathscr{U}$. Let $\Delta_0 = \{\xi_0\}$, where $\xi_0 = (\varnothing)$, and take an arbitrary nonempty countable subset $R_{\xi_0} \subset \Lambda$.

Now, for each $n \in \mathbb{N}$ we construct a collection $\mathscr{G}_n$ of open sets in X and an index set Δ_n of $n \times n$ matrices such that for each $\xi \in \Delta_n$, R_ξ, $\Omega(\xi)$, $E(\xi)$, $H(\xi)$, and x_ξ are given satisfying the following conditions:

(1) Each $\mathscr{G}_n$ is locally finite in X such that for each $G \in \mathscr{G}_n$, $\overline{G}$ is disjoint from A or B.

(2) For each $\xi \in \Delta_n$, $\{F(\alpha_1 \cdots \alpha_k) : \alpha_1, \dots, \alpha_k \in \Omega(\xi)\}$, $k \in \mathbb{N}$, is a spectral σ-net of X_ξ.

(3) For each $\xi = (\alpha_{ij})_{1, j\le n} \in \Delta_n$,
  (a) $\xi_- \in \Delta_{n-1}$, $\alpha_{in} \in \Omega((\xi_-)_{i-1})$ for $1 \le i \le n-1$, and $a_{nj} \in \Omega(\xi_-)$ for $1 \le j \le n$, where for $n = 1$, $\alpha_{11} \in \Omega(\xi_0)$;
  (b) $E(\xi) = \bigcap_{i=1}^n p_{\xi_{i-1}}^{-1}(F(\alpha_{i1} \cdots \alpha_{in}))$.

(4) $\{H(\xi):\ \xi \in \Delta_n\}$ is a locally finite collection of open sets of X with $H(\xi) \supset E(\xi)$ for each $\xi \in \Delta_n$.

(5) For each $\xi \in \Delta_{n-1}$, $E(\xi)$ is covered by $\mathscr{G}_n \cup \{E(\eta):\ \eta \in \Delta_n$ with $\eta_- = \xi\}$, and X is covered by $\mathscr{G}_1 \cup \{E(\eta) : \eta \in \Delta_1\}$.

(6) For each $\xi \in \Delta_n$,
  (a) $x_\xi \in A \cap E(\xi)$ if n is odd and $x_\xi \in B \cap E(\xi)$ if n is even;
  (b) $R_\xi = R_{\xi_-} \cup \operatorname{Supp}(x_\xi)$.

Assume that the above construction has already been performed for no greater than n, where, without loss of generality, we may assume that n is odd. Take a $\xi \in \Delta_n$. Put

$$M_\xi = \{\eta = (\alpha_{ij})_{i, j\le n+1} : \eta_- = \xi,\ \alpha_{i, n+1} \in \Omega(\xi_{i-1}) \text{ for } 1 \le i \le n \text{ and } \alpha_{n+1, j} \in \Omega(\xi) \text{ for } 1 \le j \le n+1\}.$$

For each $\eta = (\alpha_{ij})_{i, j\le n+1} \in M_\xi$, we define

$$E(\eta) = \bigcap_{i=1}^{n+1} p_{\xi_{i-1}}^{-1}(F(\alpha_{i,1} \cdots \alpha_{i, n+1})).$$

Moreover, we put

$$\Delta_\xi = \{\eta \in M_\xi : B \cap E(\eta) \neq \varnothing\}.$$

It is easily seen that $\{p_\xi(E(\eta)) : \eta \in M_\xi\}$ is a locally finite collection of closed sets of X_ξ with $p_\xi^{-1}p_\xi(E(\eta)) = E(\eta)$ for each $\eta \in \Delta_\xi$. And so if we define $S(\xi)$ by

$$S(\xi) = \bigcup\{E(\eta) : \eta \in M_\xi \backslash \Delta_\xi\},$$

$P_\xi(S(\xi))$ is closed in X_ξ with $S(\xi) = p_\xi^{-1}p_\xi(S(\xi))$. Note that $S(\xi) \subset E(\xi) \cap (X \backslash B)$. It follows from the perfect normality of X_ξ that

$$S(\xi) = \bigcap_{n=1}^{\infty} p_\xi^{-1}(V_n) \subset \bigcap_{n=1}^{\infty} \overline{p_\xi^{-1}(V_n)} \subset X \backslash B$$

for some countably many open sets V_n, $n \in \mathbb{N}$, in X_ξ. Since X is countably paracompact, Lemma 1 implies that there exists an open set G_ξ in X_ξ contained in $H(\xi)$ such that

$$S(\xi) \subset G_\xi \subset \overline{G}_\xi \subset X \backslash B.$$

Here, keeping $\xi \in \Delta_n$, we let

$$\mathscr{G}_{n+1} = \{G_\xi : \xi \in \Delta_n\} \quad \text{and} \quad \Delta_{n+1} = \bigcup_{\xi \in \Delta_n} \Delta_\xi.$$

To define $H(\eta)$ for $\eta \in \Delta_\xi$, note, as mentioned above, that $\{p_\xi(E(\eta)) : \eta \in M_\xi\}$ is a locally finite collection of closed sets of X_ξ with $p_\xi^{-1}p_\xi(E(\eta)) = E(\eta) \subset E(\xi)$ for $\eta \in M_\xi$. By the paracompactness of X_ξ, there exists a locally finite collection $\{W(\eta) : \eta \in M_\xi\}$ of open sets of X_ξ such that $p_\xi(E(\eta)) \subset W(\eta)$ and thus $E(\eta) \subset p_\xi^{-1}(W(\eta))$. Let $H(\eta) = p_\xi^{-1}(W(\eta)) \cap H(\xi)$. It follows from the inductive assumption (4) that $\{H(\eta) : \eta \in \Delta_{n+1}\}$ is locally finite with $H(\eta) \supset E(\eta)$. For each $\eta \in \Delta_{n+1}$, we can choose some $x_\eta \in B \cap E(\eta)$. Let $R_\eta = R_{\eta_-} \cup \operatorname{Supp}(x_\eta)$. Since X_η is a σ-space it follows from Lemma 2 that there exists a spectral σ-net

$$\{F(\alpha_1 \cdots \alpha_k) : \alpha_1, \dots, \alpha_k \in \Omega(\eta)\}, \qquad k \in \mathbb{N},$$

of X_η for each $\eta \in \Delta_{n+1}$. Then the conditions (1)–(6) are satisfied for $n+1$. Here we check only (5). Pick any $\xi = (\alpha_{ij})_{i,j \le n} \in \Delta_n$. Then

$$F(\alpha_{i1} \cdots \alpha_{in}) = \bigcup\{F(\alpha_{i1} \cdots \alpha_{i,n+1}) : \alpha_{i,n+1} \in \Omega(\xi_{i-1})\}$$

for $i = 1, \dots, n$, and

$$X = \bigcup\{p_\xi^{-1}(F(\alpha_{n+1,1} \cdots \alpha_{n+1,n+1})) : \alpha_{n+1,j} \in \Omega(\xi) \text{ for } j = 1, \dots, n+1\}.$$

It follows that

$$E(\xi) = \bigcup\{E(\eta) : \eta \in M_\xi\} \subset \bigcup \mathscr{G}_{n+1} \cup \left(\bigcup\{E(\eta) : \eta \in \Delta_\xi\}\right).$$

We now set $\mathscr{G} = \bigcup_{n=1}^{\infty} \mathscr{G}_n$. By (1), $\mathscr{G}$ is a σ-locally finite collection of open sets of X such that for $G \in \mathscr{G}$, $\overline{G}$ misses A or B. To complete the proof, it suffices to show that $\mathscr{G}$ covers X. Assume the contrary and pick some $x \in X \backslash \bigcup \mathscr{G}$. By (2) and (5), we can inductively choose a sequence $\{\alpha_{ij} : i, j \ge 1\}$ such that for each $n \ge 1$, $\xi^{(n)} = (\alpha_{ij})_{i,j \le n} \in \Delta_n$ and $\{F(\alpha_{n1} \cdots \alpha_{nk}) : k \in \mathbb{N}\}$

is a local σ-net of $X_{\xi^{(n-1)}}$ at point $p_{\xi^{(n-1)}}(x)$, where $\alpha_{nk} \in \Omega(\xi^{(n-1)})$ and $\xi^{(0)} = \xi_0$. Now fix $m \geq 1$. If $n > m$, then

$$x_{\xi^{(n)}} \in E(\xi^{(n)}) \subset p_{\xi^{(m)}}^{-1}(F(\alpha_{m+1,1} \cdots \alpha_{m+1,n})).$$

We thus have $p_{\xi^{(m)}}(x_{\xi^{(n)}}) \in F(\alpha_{m+1,1} \cdots \alpha_{m+1,n})$ for each $n > m$. Since $\{F(\alpha_{m+1,1} \cdots \alpha_{m+1,k}) : k \in \mathbb{N}\}$ is a local σ-net of $X_{\xi^{(m)}}$ at point $p_{\xi^{(m)}}(x)$, the sequence $\{p_{\xi^{(m)}}(x_{\xi^{(n)}})\}_{n>m}$ converges to $p_{\xi^{(m)}}(x)$. Define a point $y = (y_\lambda)_{\lambda \in \Lambda}$ in X by letting $y_\lambda = x_\lambda$ if $\lambda \in \bigcup_{n=1}^{\infty} R_{\xi^{(n)}}$ and $y_\lambda = s_\lambda$ otherwise. Then one can prove that the sequence $\{x_{\xi^{(n)}}\}_{n \in \mathbb{N}}$ converges to y, and thus $y \in A \cap B$. This is a contradiction. The proof of Theorem 1 is complete.

Question 2 is still open. By Theorem 1 we now have

Corollary 1. *A Σ-product of Lašnev (M_1-) spaces is normal if and only if it is countably paracompact.*

By Theorem 1 and [20, Corollary 1] we also have

Corollary 2. *The following are equivalent for a Σ-product X of paracompact σ-spaces.*

(1) X *is collectionwise normal.*
(2) X *is shrinking.*
(3) X *is normal.*
(4) X *is countably paracompact.*

It is not possible to replace σ-spaces by Σ-spaces in Theorem 1. Since, as pointed out in the abstract, Σ-products of compact spaces always are countably paracompact but not necessarily normal [4].

3. Proof of Theorem 2

A space is said to be *semistratifiable* [3] if there exists a function g of $X \times \mathbb{N}$ into the topology of X satisfying

(i) $\bigcap_{n \in \mathbb{N}} g(x, n) = \{x\}$ for each $x \in X$;
(ii) if $\{x_n\}$ is a sequence of points in X with $x \in \bigcap_{n \in \mathbb{N}} g(x_n, n)$ for some $x \in X$, then $\{x_n\}$ converges to x.

A space X is said to be *shrinking* if for every open cover $\{G_\gamma : \gamma \in \Gamma\}$ of X there exists a closed cover $\{F_\gamma : \gamma \in \Gamma\}$ of X such that $F_\gamma \subset G_\gamma$ for each $\gamma \in \Gamma$. If the closed cover can be weakly chosen as a closed cover $\mathscr{F} = \{F_{\gamma,n} : \gamma \in \Gamma$ and $n \in \mathbb{N}\}$ with $F_{\gamma,n} \subset G_\gamma$ for each $\gamma \in \Gamma$ and $n \in \mathbb{N}$, then the space is said to be *subshrinking*. Such a cover $\mathscr{F}$ is called a *subshrinking* of $\{G_\gamma : \gamma \in \Gamma\}$.

It follows from Bešlagić [1] that a space is shrinking if and only if it is normal and subshrinking. Note that subparacompact spaces are subshrinking.

Proof of Theorem 2. Let X be a semistratifiable space and $\mathscr{G} = \{G_\gamma : \gamma \in \Gamma\}$ an open cover of $X \times \kappa$. We shall find a subshrinking for $\mathscr{G}$.

For a set $F \subset X$, set

$$\mathscr{W}(F) = \{W : W \text{ is open in } \kappa \text{ such that } F \times W \subset G_\gamma \text{ for some } \gamma \in \Gamma\}$$

and

$$\mathscr{V} = \{V : V \text{ is open in } X \text{ such that } \kappa = \bigcup \mathscr{W}(V)\}.$$

Now for each $n \in \mathbb{N}$ we construct inductively two σ-locally finite collections $\mathscr{G}_n$ and $\mathscr{F}_n$ of closed sets of X satisfying the following conditions (1)–(3):

(1) $\mathscr{F}_{n+1}$ can be expressed as $\mathscr{F}_{n+1} = \bigcup\{\mathscr{F}_F \in \mathscr{F}_n\}$.
(2) For each $C \in \mathscr{C}_n$, $\kappa = \bigcup \mathscr{W}(C)$.
(3) For each $F \in \mathscr{F}_n$,
 (a) $F \subset g(x_F, n)$ for some $x_F \in X \backslash \bigcup \mathscr{V}$;
 (b) F is covered by $\mathscr{C}_{n+1} \cup \mathscr{F}_F$; and X is covered by $\mathscr{C}_1 \cup \mathscr{F}_1$.

Assume $n \in \mathbb{N}$, and $\mathscr{C}_i$ and $\mathscr{F}_i$ for $i \leq n$ have already been defined satisfying the conditions. Take an $F \in \mathscr{F}_n$ and fix it. Put

$$\mathscr{W} = \{F \cap V : V \in \mathscr{V}\} \cup \{F \cap g(x, n+1) : x \in F \backslash \bigcup \mathscr{V}\}.$$

By the subparacompactness of X, there exists a σ-locally finite closed cover $\mathscr{F}$ of F refining $\mathscr{W}$. Let $\mathscr{C}_F = \{F \in \mathscr{F} : F \subset V$ for some $V \in \mathscr{V}\}$ and $\mathscr{F}_F = \mathscr{F} \backslash \mathscr{C}_F$. Here running $F \in \mathscr{F}_n$ we put

$$\mathscr{C}_{n+1} = \bigcup\{\mathscr{C}_F : F \in \mathscr{F}_n\} \quad \text{and} \quad \mathscr{F}_{n+1} = \bigcup\{\mathscr{F}_F : F \in \mathscr{F}_n\}.$$

Then both $\mathscr{C}_{n+1}$ and $\mathscr{F}_{n+1}$ are locally finite satisfying conditions (1)–(3).

Claim. $\mathscr{C} = \bigcup_{n=1}^{\infty} \mathscr{C}$ covers X.

Assume the contrary and pick some $x \in X \backslash \bigcup \mathscr{C}$. Then one can easily find a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $X \backslash \bigcup \mathscr{V}$ such that $x \in g(x_n, n)$ for each $n \in \mathbb{N}$. It follows from the definition of semistratifiable spaces above Definition 3 that the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x. Since $\chi(X) < \kappa$, we may take a $\lambda < \kappa$ and find a nbd base $\{V(x_n, \alpha) : \alpha < \lambda\}$ for x_n, $n \in \mathbb{N}$. By the definition of x_n, for each $n \in \mathbb{N}$ there exists a point

$$\xi(n, \alpha) \in \kappa \backslash \bigcup \mathscr{W}(V(x_n, \alpha))$$

for each $\alpha < \lambda$. Let $\xi(n)$ be a cluster point of the net $\{\xi(n, \alpha) : \alpha < \kappa\}$, $n \in \mathbb{N}$, and let ξ be a cluster point of the sequence $\{\xi(n)\}_{n\in\mathbb{N}}$. Then $(x, \xi) \in G_\gamma$ for some $\gamma \in \Gamma$. It is not hard to find an n, an $\alpha < \lambda$, and a nbd of O_ξ of ξ such that

$$V(x_n, \alpha) \times O_\xi \subset G_\gamma.$$

It follows that $\xi(n, \alpha) \in \bigcup \mathscr{W}(V(x_n, \alpha))$.

Now decompose $\mathscr{C}$ as $\mathscr{C} = \bigcup_{n=1}^{\infty} \mathscr{C}'_n$ so that $\mathscr{C}'_n$ is locally finite. Pick a $C \in \mathscr{C}$ and fix it. For each $\alpha \in \kappa$, there exists an $f(\alpha) < \alpha$ such that $C \times (f(\alpha), \alpha] \subset G_\gamma$ for some $\gamma \in \Gamma$; let $\gamma(\alpha, C)$ denote this γ. By the Pressing Down Lemma, there exists a $\beta \in \kappa$ and a stationary set $S \subset \kappa$ such that $f(\alpha) = \beta$ for all $\alpha \in S$. Therefore we have $C \times (\beta, \alpha] \subset G_{\gamma(\alpha, C)}$ for all $\alpha \in S$. Without loss of generality we can assume that either all $\gamma(\alpha, C)$, $\alpha \in S$, are the same or different. If all $\gamma(\alpha, C)$, $\alpha \in S$, are the same, we may put $\gamma(C) = \gamma(\alpha, C)$ for all $\alpha \in S$. We let β_C denote the chosen β and index the chosen stationary set S as $S = \{\alpha(C, \mu) : \mu \in \kappa\}$. Here keeping $C \in \mathscr{C}$, decompose $\mathscr{C}'_n$ for each $n \in \mathbb{N}$ as

$$\mathscr{C}'_n(1) = \{C \in \mathscr{C}'_n : \text{ all } \gamma(\alpha(C, \mu), C),\ \mu < \kappa, \text{ are the same}\}$$

and $\mathscr{C}'_n(2) = \mathscr{C}'_n \backslash \mathscr{C}'_n(1)$. We now put

$$H_{n\gamma} = \left(\bigcup\{C \times (\beta_C, \kappa) : C \in \mathscr{C}'_n(1) \text{ with } \gamma(C) = \gamma\}\right) \cup \left(\bigcup\{C \times (\beta_C, \alpha(C, \mu)] : C \in \mathscr{C}'_n(2) \text{ and } \mu < \kappa \text{ with } \gamma(\alpha(C, \mu), C) = \gamma\}\right)$$

for each $\gamma \in \Gamma$ and $n \geq 1$. Then $H_{n\gamma}$ is a closed set in $X \times \kappa$ with $H_{n\gamma} \subset G_\gamma$ for each $\gamma \in \Gamma$ and $n \geq 1$.

Moreover, for each $C \in \mathscr{C}$, since the subspace $C \times [0, \beta_C]$ is subparacompact, there exists a closed cover $\{Z_{n,C,\gamma} : n \in \mathbb{N} \text{ and } \gamma \in \Gamma\}$ of it such that $Z_{n,C,\gamma} \subset G_\gamma$ for each $n \in \mathbb{N}$ and $\gamma \in \Gamma$. Let us set

$$H_{n,m,\gamma} = \bigcup\{Z_{n,C,\gamma} : C \in \mathscr{C}'_m\}$$

for each $n, m \in \mathbb{N}$ and $\gamma \in \Gamma$. It is easy to see that $H_{n,m,\gamma}$ is closed with $H_{n,m,\gamma} \subset G_\gamma$ for each $n, m \in \mathbb{N}$ and $\gamma \in \Gamma$. So we find a subshrinking

$$\{H_{n,m,\gamma} : n, m \in \mathbb{N} \text{ and } \gamma \in \Gamma\} \cup \{H_{n\gamma} : n \in \mathbb{N} \text{ and } \gamma \in \Gamma\}$$

for the open cover $\mathscr{G}$ which completes the proof.

Notice that for any subparacompact space X Yajima [21] gives a sufficient condition for $X \times \kappa$ to be subshrinking.

Acknowledgment

The author would like to thank Professor T. Hoshina for his useful help with this paper.

References

1. A. Bešlagić, *Normality in products*, Topology Appl. **22** (1986), 71–88.
2. H. H. Corson, *Normality in subsets of product spaces*, Amer. J. Math. **81** (1959), 785–796.
3. G. D. Creede, *Concerning semi-stratifiable spaces*, Pacific J. Math. **32** (1970), 47–54.
4. S. P. Gul′ko, *On the properties of Σ-products*, Soviet Math. Dokl. **18** (1977), 1438–1442.
5. T. Hoshina, *Products of normal spaces with Lasnev spaces*, Fund. Math. **124** (1984), 143–153.
6. ——, *Shrinking and normal products*, Questions Answers Gen. Topology **2** (1984), 83–91.
7. ——, *Normality of product spaces*. II, Topics in General Topology (K. Morita and J. Nagata, eds.), North-Holland, Amsterdam, 1989.
8. A. P. Kombarov, *On tightness and normality of Σ-products*, Soviet Math. Dokl. **19** (1978), 403–407.
9. K. Morita, *Note on products of normal spaces with metric spaces*, unpublished.
10. K. Nagami, *σ-spaces and product spaces*, Math. Ann. **181** (1969), 109–118.
11. ——, *Countable paracompactness of inverse limits and products*, Fund. Math. **73** (1972), 261–270.
12. ——, *Σ-spaces*, Fund. Math. **65** (1969), 169–192.
13. J. Nagata, *Modern general topology*, North-Holland, Amsterdam, 1985.
14. T. C. Przymusinski, *Products of normal spaces*, Handbook of Set Theoretic Topology (K. Kunen and J. Vaughan, eds.), North-Holland, Amsterdam, 1984.

15. M. E. Rudin, Book Review, Bull. Amer. Math. Soc. **84** (1978), 271–272.

16. ———, *The shrinking property*, Canad. Math. Bull. **26** (1983), 385–388.

17. M. E. Rudin and M. Starbird, *Products with a metric factor*, Gen. Topology Appl. **5** (1975), 235–248.

18. Y. Yajima, *On Σ-products of Σ-spaces*, Fund. Math. **123** (1984), 29–37.

19. ———, *On Σ-products of semi-stratifiable spaces*, Topology Appl. **25** (1987), 1–11.

20. ———, *The shrinking property of Σ-products*, Tsukuba J. Math. **13** (1989), 83–98.

21. ———, *Subnormality of $X \times \kappa$ and Σ-products*, Topology Appl. (to appear).

22. P. Zenor, *Countable paracompactness in product spaces*, Proc. Amer. Math. Soc. **30** (1971), 199–201.

INSTITUTE OF MATHEMATICS, UNIVERSITY OF TSUKUBA, TSUKUBA-SHI 305, JAPAN