COUNTABLE PARACOMPACTNESS OF $\Sigma$-PRODUCTS

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Abstract. It is known that $\Sigma$-products of compact spaces always are countably paracompact but not necessarily normal. In the present paper it is proved that a $\Sigma$-product of paracompact $\sigma$-spaces is normal if and only if it is countably paracompact.

1. Introduction

The equivalence of normality and countable paracompactness in Cartesian products has been investigated by many authors [5, 8, 9, 17, 22] so that this topic constitutes a very interesting part in the theory of product spaces [7, 14]. In this paper the equivalence of normality and countable paracompactness will be considered for $\Sigma$-products.

The concept of $\Sigma$-products was introduced by Corson [2] who proved that $\Sigma$-products of complete metric spaces are normal. In Gul'ko [4] and Rudin [15] the following is shown:

(i) A $\Sigma$-product of metric spaces is normal.

This answers affirmatively a long outstanding question raised by Corson [2]. Kombarov [8] later generalized (i) by obtaining the following result:

(ii) A $\Sigma$-product of paracompact $p$-spaces is (collectionwise) normal if and only if it has countable tightness.

In connection with the above results, the following Questions 1 and 2 are considered by Yajima [18] and Kodama, respectively.

Question 1. Is a $\Sigma$-product of paracompact $\sigma$-spaces normal if it has countable tightness?

Question 2. Is a $\Sigma$-product of Lašnev spaces normal?

Question 1 has been answered positively. In fact Yajima [18] even proved

(iii) A $\Sigma$-product of paracompact $\Sigma$-spaces is (collectionwise) normal if it has countable tightness.

Since paracompact $p$-spaces are $\Sigma$-spaces, (iii) is also a generalization of the "if" part of (ii). However, the countable tightness is no longer a necessary condition for a $\Sigma$-product of paracompact $\Sigma$-spaces to be normal, because there

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exists a collectionwise normal $\Sigma$-product of $M_1$-spaces which has no countable tightness [18]. Moreover, since there exists a nonnormal $\Sigma$-product of $M_1$-spaces [18], in Question 1 the assumption of countable tightness cannot be dropped. On the other hand, Rudin [16] proved that any $\Sigma$-product of metric spaces is shrinking and hence countably paracompact. It is also known from Yajima [19] that any normal $\Sigma$-product of $\sigma$-spaces is countably paracompact.

The main purpose of this paper is to establish the equivalence of normality and countable paracompactness of $\Sigma$-products of paracompact $\sigma$-spaces. Namely we prove the following theorem.

**Theorem 1.** A $\Sigma$-product of paracompact $\sigma$-spaces is normal if and only if it is countably paracompact.

In the rest of the paper, we also consider the subshrinking property of $X \times \kappa$, where $X$ is a semistratifiable space and $\kappa$ an uncountable regular cardinal with the usual order topology. The subshrinking property was introduced by Yajima [20] which is important for the study of shrinking property (see Yajima [20] and Hoshina [6]). Yajima [21] recently proved that $X \times \kappa$ is subshrinking for any $\sigma$-space $X$, and he asked whether $X \times \kappa$ is subshrinking for a semistratifiable space $X$. We shall prove

**Theorem 2.** Let $X$ be a semistratifiable space with $\chi(X) < \kappa$. Then $X \times \kappa$ is subshrinking.

All spaces considered here are assumed to be regular $T_1$. The set of natural numbers is denoted by $N$ and natural numbers are denoted by $i$, $j$, $k$, and $n$. $\kappa$ always denotes an uncountable regular cardinal with the usual order topology.

2. Proof of Theorem 1

Let $X = \prod_{\lambda \in \Lambda} X_{\lambda}$ be the Cartesian product of spaces $X_{\lambda}$, $\lambda \in \Lambda$, and let $s = (s_\lambda)_{\lambda \in \Lambda}$ be a fixed point of $X$. The subspace $\Sigma = \{ x \in X : x_\lambda = s_\lambda \text{ for all but countably many } \lambda \in \Lambda \}$ of $X$ is called a $\Sigma$-product of spaces $X_\lambda$, $\lambda \in \Lambda$. Such a point $s \in \Sigma$ is called the base point of $\Sigma$, which is often omitted.

Let $X$ be a $\Sigma$-product of spaces $X_\lambda$, $\lambda \in \Lambda$, with a base point $(s_\lambda)_{\lambda \in \Lambda}$. For a point $x \in X$, denote by $\text{Supp}(x)$ the set $\{ \lambda \in \Lambda : x_\lambda \neq s_\lambda \}$. Let $\Delta$ be an index set such that for each $\xi \in \Delta$, $R_\xi$ is a subset of $\Lambda$. Then we denote by $X_{\xi}$ the Cartesian product $\prod_{\lambda \in R_\xi} X_{\lambda}$ and by $p_\xi$ the projection of $X$ onto $X_{\xi}$ for each $\xi \in \Delta$.

Let $\xi = (\alpha_{ij})_{i,j \leq n}$ be an $n \times n$ matrix. By $\xi_k$ we denote the $k \times k$ matrix $(\alpha_{ij})_{i,j \leq k}$ for $1 \leq k \leq n$. In particular, $\xi_{n-1}$ is often abbreviated as $\xi_-$ and $\xi_0$ denotes the empty set $\emptyset$.

A space is called a $\sigma$-space if it has a $\sigma$-locally finite net [13]. Note that Lašnev spaces (i.e., closed images of metric spaces) are $M_1$, and $M_1$-spaces are paracompact $\sigma$ [13]. It is well known that the countable product of paracompact $\sigma$-spaces (\Sigma-spaces) is paracompact $\sigma$ ($\Sigma$).

The following two lemmas are useful to prove Theorem 1.

**Lemma 1** [5, Lemma 2.1]. Let $X$ be a countably paracompact space and let $E$ and $F$ be a pair of disjoint subsets. Suppose that $F$ is closed and there exists open sets $U_n$, $n \in \mathbb{N}$, such that $E \subseteq \bigcap_{n \in \mathbb{N}} U_n$ and $(\bigcap_{n \in \mathbb{N}} U_n) \cap F = \emptyset$. Then $E$ and $F$ are separated by open sets.
Lemma 2 [11, Theorem 1]. Let $X$ be a $\sigma$-space. Then there exists a sequence 
\begin{align*}
\mathcal{F}_n &= \{ F(\alpha_1, \ldots, \alpha_n) : \alpha_1, \ldots, \alpha_n \in \Omega \} \text{ for each } n \in \mathbb{N}, \\
F(\alpha_1, \ldots, \alpha_n) &= \bigcup\{ F(\alpha_1, \ldots, \alpha_n, \alpha) : \alpha \in \Omega \} \text{ for each } \alpha_1, \ldots, \alpha_n \in \Omega, \\
\text{(c) For each } x \in X, \text{ there exists a sequence } \alpha_1, \alpha_2, \ldots \in \Omega \text{ such that } x \in \\
\bigcap_{n \in \mathbb{N}} F(\alpha_1, \ldots, \alpha_n) \text{ and each open nbhd of } x \text{ contains some } F(\alpha_1, \ldots, \alpha_n).
\end{align*}

The above sequence \( \{ \mathcal{F}_n : n \in \mathbb{N} \} \) is called a spectral $\sigma$-net of $X$ and the sequence \( \{ F(\alpha_1, \ldots, \alpha_n) : n \in \mathbb{N} \} \) in (c) is called a local $\sigma$-net of $X$ at $x$.

Our proof of Theorem 1 is based on the idea in Yajima [18, 20] and we shall use the following fact: a space $X$ is normal if and only if for every pair $A, B$ of disjoint closed subsets of $X$ there exists a $\sigma$-locally finite open cover $\mathcal{U}$ of $X$ such that either $\bigcup \mathcal{U} \cap A = \emptyset$ or $\bigcup \mathcal{U} \cap B = \emptyset$ for every $U \in \mathcal{U}$.

Proof of Theorem 1. Let $X$ be a $\Sigma$-product of paracompact $\sigma$-spaces $X_\lambda$, $\lambda \in \Lambda$, with a base point $s = (s_\lambda)_{\lambda \in \Lambda} \in X$, and suppose $X$ is countably paracompact. To prove that $X$ is normal, let $A$ and $B$ be a pair of disjoint closed subsets of $X$; we shall find a $\sigma$-locally finite open cover $\mathcal{G}$ of $X$ such that either $\bigcup \mathcal{G} \cap A = \emptyset$ or $\bigcup \mathcal{G} \cap B = \emptyset$ for every $U \in \mathcal{G}$. Let $\Delta_0 = \{ \xi_0 \}$, where $\xi_0 = (\emptyset)$, and take an arbitrary nonempty countable subset $R_0 \subset \Lambda$.

Now, for each $n \in \mathbb{N}$ we construct a collection $\mathcal{G}_n$ of open sets in $X$ and an index set $\Delta_n$ of $n \times n$ matrices such that for each $\xi \in \Delta_n$, $R_\xi$, $\Omega(\xi)$, $E(\xi)$, $H(\xi)$, and $x_\xi$ are given satisfying the following conditions:

1. Each $\mathcal{G}_n$ is locally finite in $X$ such that for each $G \in \mathcal{G}_n$, $\bigcup G$ is disjoint from $A$ or $B$.
2. For each $\xi \in \Delta_n$, \( \{ F(\alpha_1, \ldots, \alpha_k) : \alpha_1, \ldots, \alpha_k \in \Omega(\xi) \} \), $k \in \mathbb{N}$, is a spectral $\sigma$-net of $X_\xi$.
3. For each $\xi = (\alpha_{ij})_{j \leq n} \in \Delta_n$,
   \( \xi_- \in \Delta_{n-1} \), $\alpha_{in} \in \Omega(\xi_-)$ for $1 \leq i \leq n - 1$, and $\alpha_{nj} \in \Omega(\xi_-)$ for $1 \leq j \leq n$, where for $n = 1$, $\alpha_{11} \in \Omega(\xi_0)$;
   \( E(\xi_0) = \bigcap_{i=1}^n p_{\xi_0,i}^{-1} (F(\alpha_{i1}, \ldots, \alpha_{in})) \).
4. $\{ H(\xi) : \xi \in \Delta_n \}$ is a locally finite collection of open sets of $X$ with $H(\xi) \supset E(\xi)$ for each $\xi \in \Delta_n$.
5. For each $\xi \in \Delta_{n-1}$, $E(\xi)$ is covered by $\mathcal{G}_n \cup \{ E(\eta) : \eta \in \Delta_n \}$ with $\eta_- = \xi$, and $X$ is covered by $\mathcal{G}_1 \cup \{ E(\eta) : \eta \in \Delta_1 \}$.
6. For each $\xi \in \Delta_n$,
   \( x_\xi \in A \cap E(\xi) \) if $n$ is odd and $x_\xi \in B \cap E(\xi)$ if $n$ is even;
   \( R_\xi = R_\xi_- \cup \text{Supp}(x_\xi) \).

Assume that the above construction has already been performed for no greater than $n$, where, without loss of generality, we may assume that $n$ is odd. Take a $\xi \in \Delta_n$. Put
\[ M_\xi = \{ \eta = (\alpha_{ij})_{i,j \leq n+1} : \eta_- = \xi, \alpha_{i,n+1} \in \Omega(\xi_{i-1}) \text{ for } 1 \leq i \leq n \text{ and } \alpha_{n+1,j} \in \Omega(\xi) \text{ for } 1 \leq j \leq n + 1 \}. \]

For each $\eta = (\alpha_{ij})_{i,j \leq n+1} \in M_\xi$, we define
\[ E(\eta) = \bigcap_{i=1}^{n+1} p_{\xi_{i-1}}^{-1} (F(\alpha_{i,1}, \ldots, \alpha_{i,n+1})). \]
Moreover, we put
\[ \Delta_\xi = \{ \eta \in M_\xi : B \cap E(\eta) \neq \emptyset \}. \]

It is easily seen that \( \{ p_\xi(E(\eta)) : \eta \in M_\xi \} \) is a locally finite collection of closed sets of \( X_\xi \) with \( p_\xi^{-1}p_\xi(E(\eta)) = E(\eta) \) for each \( \eta \in \Delta_\xi \). And so if we define \( S(\xi) \) by
\[ S(\xi) = \bigcup \{ E(\eta) : \eta \in M_\xi \setminus \Delta_\xi \}, \]
then \( p_\xi(S(\xi)) \) is closed in \( X_\xi \) with \( S(\xi) = p_\xi^{-1}p_\xi(S(\xi)) \). Note that \( S(\xi) \subset E(\xi) \cap (X \setminus B) \). It follows from the perfect normality of \( X_\xi \) that
\[ S(\xi) = \bigcap_{n=1}^{\infty} p_\xi^{-1}(V_n) \subset \bigcap_{n=1}^{\infty} \overline{p_\xi^{-1}(V_n)} \subset X \setminus B \]
for some countably many open sets \( V_n, n \in \mathbb{N} \), in \( X_\xi \). Since \( X \) is countably paracompact, Lemma 1 implies that there exists an open set \( G_\xi \) in \( X_\xi \) contained in \( H(\xi) \) such that
\[ S(\xi) \subset G_\xi \subset \overline{G_\xi} \subset X \setminus B. \]

Here, keeping \( \xi \in \Delta_\xi \), we let
\[ \mathcal{G}_{n+1} = \{ G_\xi : \xi \in \Delta_n \} \quad \text{and} \quad \Delta_{n+1} = \bigcup_{\xi \in \Delta_n} \Delta_\xi. \]

To define \( H(\eta) \) for \( \eta \in \Delta_\xi \), note, as mentioned above, that \( \{ p_\xi(E(\eta)) : \eta \in M_\xi \} \) is a locally finite collection of closed sets of \( X_\xi \) with \( p_\xi^{-1}p_\xi(E(\eta)) = E(\eta) \subset E(\xi) \) for \( \eta \in M_\xi \). By the paracompactness of \( X_\xi \), there exists a locally finite collection \( \{ W(\eta) : \eta \in M_\xi \} \) of open sets of \( X_\xi \) such that \( p_\xi(E(\eta)) \subset W(\eta) \) and thus \( E(\eta) \subset p_\xi^{-1}(W(\eta)) \). Let \( H(\eta) = p_\xi^{-1}(W(\eta)) \cap H(\xi) \). It follows from the inductive assumption (4) that \( \{ H(\eta) : \eta \in \Delta_{n+1} \} \) is locally finite with \( H(\eta) \supset E(\eta) \). For each \( \eta \in \Delta_{n+1}, \) we can choose some \( x_\eta \in B \cap E(\eta) \). Let \( R_\eta = R_{\eta-} \cup \text{Supp}(x_\eta) \). Since \( X_\eta \) is a \( \sigma \)-space it follows from Lemma 2 that there exists a spectral \( \sigma \)-net
\[ \{ F(\alpha_1 \cdots \alpha_k) : \alpha_1, \ldots, \alpha_k \in \Omega(\eta) \}, \quad k \in \mathbb{N}, \]
of \( X_\eta \) for each \( \eta \in \Delta_{n+1} \). Then the conditions (1)–(6) are satisfied for \( n+1 \). Here we check only (5). Pick any \( \xi = (\alpha_{ij})_{i,j \leq n} \in \Delta_n \). Then
\[ F(\alpha_{i1} \cdots \alpha_{in}) = \bigcup \{ F(\alpha_{i1} \cdots \alpha_{i,n+1}) : \alpha_{i,n+1} \in \Omega(\xi_{i-1}) \} \]
for \( i = 1, \ldots, n \), and
\[ X = \bigcup \{ p_\xi^{-1}(F(\alpha_{n+1,1} \cdots \alpha_{n+1,n+1})) : \alpha_{n+1,j} \in \Omega(\xi) \text{ for } j = 1, \ldots, n+1 \}. \]

It follows that
\[ E(\xi) = \bigcup \{ E(\eta) : \eta \in M_\xi \} \subset \bigcup \mathcal{G}_{n+1} \cup \bigcup \{ E(\eta) : \eta \in \Delta_\xi \}. \]

We now set \( \mathcal{G} = \bigcup_{n=1}^{\infty} \mathcal{G}_n \). By (1), \( \mathcal{G} \) is a \( \sigma \)-locally finite collection of open sets of \( X \) such that for \( G \in \mathcal{G} \), \( \overline{G} \) misses \( A \) or \( B \). To complete the proof, it suffices to show that \( \mathcal{G} \) covers \( X \). Assume the contrary and pick some \( x \in X \setminus \bigcup \mathcal{G} \). By (2) and (5), we can inductively choose a sequence \( \{ \alpha_{ij} : i, j \geq 1 \} \) such that for each \( n \geq 1, \xi(n) = (\alpha_{ij})_{i,j \leq n} \in \Delta_n \) and \( \{ F(\alpha_{n1} \cdots \alpha_{nk}) : k \in \mathbb{N} \} \)
is a local \( \sigma \)-net of \( X_{\xi(n-1)} \) at point \( p_{\xi(n-1)}(x) \), where \( \alpha_{nk} \in \Omega(\xi(n-1)) \) and \( \xi(0) = \xi_0 \). Now fix \( m \geq 1 \). If \( n > m \), then
\[
x_{\xi(n)} \in E(\xi(n)) \subset p_{\xi(m)}^{-1}(F(\alpha_{m+1,1} \cdots \alpha_{m+1,n})).
\]
We thus have \( p_{\xi(m)}(x_{\xi(n)}) \in F(\alpha_{m+1,1} \cdots \alpha_{m+1,n}) \) for each \( n > m \). Since \( \{ F(\alpha_{m+1,1} \cdots \alpha_{m+1,k}) : k \in \mathbb{N} \} \) is a local \( \sigma \)-net of \( X_{\xi(m)} \) at point \( p_{\xi(m)}(x) \), the sequence \( \{ p_{\xi(n)}(x_{\xi(n)}) \}_{n > m} \) converges to \( p_{\xi(m)}(x) \). Define a point \( y = (y_\lambda)_{\lambda \in \Lambda} \) in \( X \) by letting \( y_\lambda = x_\lambda \) if \( \lambda \in \bigcup_{n=1}^{\infty} R_{\xi(n)} \) and \( y_\lambda = s_\lambda \) otherwise. Then one can prove that the sequence \( \{ x_{\xi(n)} \}_{n \in \mathbb{N}} \) converges to \( y \), and thus \( y \in A \cap B \). This is a contradiction. The proof of Theorem 1 is complete.

Question 2 is still open. By Theorem 1 we now have

**Corollary 1.** A \( \Sigma \)-product of Lašnev \((M_1-)\) spaces is normal if and only if it is countably paracompact.

By Theorem 1 and [20, Corollary 1] we also have

**Corollary 2.** The following are equivalent for a \( \Sigma \)-product \( X \) of paracompact \( \sigma \)-spaces.

1. \( X \) is collectionwise normal.
2. \( X \) is shrinking.
3. \( X \) is normal.
4. \( X \) is countably paracompact.

It is not possible to replace \( \sigma \)-spaces by \( \Sigma \)-spaces in Theorem 1. Since, as pointed out in the abstract, \( \Sigma \)-products of compact spaces are always countably paracompact but not necessarily normal [4].

### 3. Proof of Theorem 2

A space is said to be semistratifiable [3] if there exists a function \( g \) of \( X \times \mathbb{N} \) into the topology of \( X \) satisfying

(i) \( \bigcap_{n \in \mathbb{N}} g(x, n) = \{ x \} \) for each \( x \in X \);
(ii) if \( \{ x_n \} \) is a sequence of points in \( X \) with \( x \in \bigcap_{n \in \mathbb{N}} g(x_n, n) \) for some \( x \in X \), then \( \{ x_n \} \) converges to \( x \).

A space \( X \) is said to be shrinking if for every open cover \( \{ G_\gamma : \gamma \in \Gamma \} \) of \( X \) there exists a closed cover \( \{ F_\gamma : \gamma \in \Gamma \} \) of \( X \) such that \( F_\gamma \subset G_\gamma \) for each \( \gamma \in \Gamma \).

If the closed cover can be weakly chosen as a closed cover \( \mathcal{F} = \{ F_\gamma, n : \gamma \in \Gamma \} \) and \( n \in \mathbb{N} \) with \( F_\gamma, n \subset G_\gamma \) for each \( \gamma \in \Gamma \) and \( n \in \mathbb{N} \), then the space is said to be subshrinking. Such a cover \( \mathcal{F} \) is called a subshrinking of \( \{ G_\gamma : \gamma \in \Gamma \} \).

It follows from Bešlagić [1] that a space is shrinking if and only if it is normal and subshrinking. Note that subparacompact spaces are subshrinking.

**Proof of Theorem 2.** Let \( X \) be a semistratifiable space and \( \mathcal{G} = \{ G_\gamma : \gamma \in \Gamma \} \) an open cover of \( X \times \kappa \). We shall find a subshrinking for \( \mathcal{G} \).

For a set \( F \subset X \), set
\[
\mathcal{W}(F) = \{ W : W \text{ is open in } \kappa \text{ such that } F \times W \subset G_\gamma \text{ for some } \gamma \in \Gamma \}
\]
and
\[
\mathcal{V} = \{ V : V \text{ is open in } X \text{ such that } \kappa = \bigcup \mathcal{W}(V) \}.
\]
Now for each \( n \in \mathbb{N} \) we construct inductively two \( \sigma \)-locally finite collections \( \mathcal{G}_n \) and \( \mathcal{F}_n \) of closed sets of \( X \) satisfying the following conditions (1)-(3):

(1) \( \mathcal{F}_{n+1} \) can be expressed as \( \mathcal{F}_{n+1} = \bigcup \{ \mathcal{F}_F \in \mathcal{F}_n \} \).

(2) For each \( C \in \mathcal{G}_n \), \( \kappa = \bigcup \mathcal{W}(C) \).

(3) For each \( F \in \mathcal{F}_n \),
   (a) \( F \subset g(x_F, n) \) for some \( x_F \in X \setminus \bigcup \mathcal{V} \);
   (b) \( F \) is covered by \( \mathcal{G}_{n+1} \cup \mathcal{F}_F \); and \( X \) is covered by \( \mathcal{G}_1 \cup \mathcal{F} \).

Assume \( n \in \mathbb{N} \), and \( \mathcal{G}_i \) and \( \mathcal{F}_i \) for \( i \leq n \) have already been defined satisfying the conditions. Take an \( F \in \mathcal{F}_n \) and fix it. Put

\[
\mathcal{W} = \{ F \cap V : V \in \mathcal{V} \} \cup \{ F \cap g(x, n + 1) : x \in F \setminus \bigcup \mathcal{V} \}.
\]

By the subparacompactness of \( X \), there exists a \( \sigma \)-locally finite closed cover \( \mathcal{F} \) of \( F \) refining \( \mathcal{W} \). Let \( \mathcal{G}_F = \{ F \in \mathcal{F} : F \subset V \) for some \( V \in \mathcal{V} \} \) and \( \mathcal{F}_F = \mathcal{F} \setminus \mathcal{G}_F \). Here running \( F \in \mathcal{F}_n \) we put

\[
\mathcal{G}_{n+1} = \bigcup \{ \mathcal{G}_F : F \in \mathcal{F}_n \} \quad \text{and} \quad \mathcal{F}_{n+1} = \bigcup \{ \mathcal{F}_F : F \in \mathcal{F}_n \}.
\]

Then both \( \mathcal{G}_{n+1} \) and \( \mathcal{F}_{n+1} \) are locally finite satisfying conditions (1)-(3).

Claim. \( \mathcal{G} = \bigcup_{n=1}^{\infty} \mathcal{G}_n \) covers \( X \).

Assume the contrary and pick some \( x \in X \setminus \bigcup \mathcal{G} \). Then one can easily find a sequence \( \{ x_n \}_{n \in \mathbb{N}} \) in \( X \setminus \bigcup \mathcal{V} \) such that \( x \in g(x_n, n) \) for each \( n \in \mathbb{N} \). It follows from the definition of semistratifiable spaces above Definition 3 that the sequence \( \{ x_n \}_{n \in \mathbb{N}} \) converges to \( x \). Since \( \chi(X) < \kappa \), we may take a \( \lambda < \kappa \) and find a nbd base \( \{ V(x_n, \alpha) : \alpha < \lambda \} \) for \( x_n \), \( n \in \mathbb{N} \). By the definition of \( x_n \), for each \( n \in \mathbb{N} \) there exists a point

\[
\xi(n, \alpha) \in \kappa \setminus \bigcup \mathcal{W}(V(x_n, \alpha))
\]

for each \( \alpha < \lambda \). Let \( \xi(n) \) be a cluster point of the net \( \{ \xi(n, \alpha) : \alpha < \kappa \} \), \( n \in \mathbb{N} \), and let \( \xi \) be a cluster point of the sequence \( \{ \xi(n) \}_{n \in \mathbb{N}} \). Then \( (x, \xi) \in G_\gamma \) for some \( \gamma \in \Gamma \). It is not hard to find an \( n \), an \( \alpha < \lambda \), and a nbd of \( O_\xi \) of \( \xi \) such that

\[
V(x_n, \alpha) \times O_\xi \subset G_\gamma.
\]

It follows that \( \xi(n, \alpha) \in \bigcup \mathcal{W}(V(x_n, \alpha)) \).

Now decompose \( \mathcal{G} \) as \( \mathcal{G} = \bigcup_{n=1}^{\infty} \mathcal{G}_n \) so that \( \mathcal{G}_n \) is locally finite. Pick a \( C \in \mathcal{G} \) and fix it. For each \( \alpha \in \kappa \), there exists an \( f(\alpha) < \alpha \) such that \( C \times (f(\alpha), \alpha] \subset G_\gamma \) for some \( \gamma \in \Gamma \); let \( \gamma(\alpha, C) \) denote this \( \gamma \). By the Pressing Down Lemma, there exists a \( \beta \in \kappa \) and a stationary set \( S \subset \kappa \) such that \( f(\alpha) = \beta \) for all \( \alpha \in S \). Therefore we have \( C \times (\beta, \alpha] \subset G_{\gamma(\alpha, C)} \) for all \( \alpha \in S \). Without loss of generality we can assume that either all \( \gamma(\alpha, C) \), \( \alpha \in S \), are the same or different. If all \( \gamma(\alpha, C) \), \( \alpha \in S \), are the same, we may put \( \gamma(C) = \gamma(\alpha, C) \) for all \( \alpha \in S \). We let \( \beta_C \) denote the chosen \( \beta \) and index the chosen stationary set \( S \) as \( S = \{ \alpha(C, \mu) : \mu \in \kappa \} \). Here keeping \( C \in \mathcal{G} \), decompose \( \mathcal{G}_n \) for each \( n \in \mathbb{N} \) as

\[
\mathcal{G}_n(1) = \{ C \in \mathcal{G}_n : \text{all } \gamma(\alpha(C, \mu), C), \mu < \kappa, \text{ are the same} \}.
\]
and $E'_n(2) = E'_n \setminus E'_n(1)$. We now put

$$H_{n\gamma} = \left( \bigcup \{ C \times (\beta C, \kappa) : C \in E'_n(1) \text{ with } \gamma(C) = \gamma \} \right) \cup \left( \bigcup \{ C \times (\beta C, \alpha(C, \mu)) : C \in E'_n(2) \text{ and } \mu < \kappa \right)$$

with $\gamma(\alpha(C, \mu), C) = \gamma$)

for each $\gamma \in \Gamma$ and $n \geq 1$. Then $H_{n\gamma}$ is a closed set in $X \times \kappa$ with $H_{n\gamma} \subset G_\gamma$ for each $\gamma \in \Gamma$ and $n \geq 1$.

Moreover, for each $C \in E$, since the subspace $C \times [0, \beta C]$ is subparacompact, there exists a closed cover $\{Z_{n,C,\gamma} : n \in \mathbb{N} \text{ and } \gamma \in \Gamma\}$ of it such that $Z_{n,C,\gamma} \subset G_\gamma$ for each $n \in \mathbb{N}$ and $\gamma \in \Gamma$. Let us set

$$H_{n,m,\gamma} = \bigcup \{Z_{n,C,\gamma} : C \in E_m'\}$$

for each $n, m \in \mathbb{N}$ and $\gamma \in \Gamma$. It is easy to see that $H_{n,m,\gamma}$ is closed with $H_{n,m,\gamma} \subset G_\gamma$ for each $n, m \in \mathbb{N}$ and $\gamma \in \Gamma$. So we find a subshrinking

$$\{H_{n,m,\gamma} : n, m \in \mathbb{N} \text{ and } \gamma \in \Gamma\} \cup \{H_{n\gamma} : n \in \mathbb{N} \text{ and } \gamma \in \Gamma\}$$

for the open cover $\mathcal{G}$ which completes the proof.

Notice that for any subparacompact space $X$ Yajima [21] gives a sufficient condition for $X \times \kappa$ to be subshrinking.

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