STRASSEN'S THEOREM FOR VECTOR MEASURES

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ABSTRACT. A type of Strassen's Theorem for measures taking values in the positive cone of a Banach lattice is proved. An application is given to metrics for convergence of vector measures.

1. Preliminaries: vector measures

Let \( \mathcal{F} \) be a field of subsets of a set \( X \), and let \( (B, \| \cdot \|) \) be a Banach space. (All vector spaces we consider are assumed to have real scalars.) Then \( a(\mathcal{F}, B) \) is the set of all additive set functions \( V: \mathcal{F} \to B \), i.e., \( V(E_1 \cup E_2) = V(E_1) + V(E_2) \) for disjoint \( E_1 \) and \( E_2 \) in \( \mathcal{F} \). The elements of \( a(\mathcal{F}, B) \) we call charges. Also, \( ca(\mathcal{F}, B) \) is the set of all countably additive charges in \( a(\mathcal{F}, B) \); we call such charges vector measures. Generally, we follow the notation and conventions of Dunford and Schwartz [4] or Diestel and Uhl [2].

Let \( (B, \| \cdot \|) \) be a Banach space with dual space \( B^* \). If \( V \in ca(\mathcal{F}, B) \) and \( \varphi \in B^* \), then \( \varphi(V) = \varphi \circ V \) is a finite signed measure on \( \mathcal{F} \) with total variation \( |\varphi(V)| \). The semivariation of \( V \) is the set function \( \|V\|: \mathcal{F} \to \mathbb{R} \) defined by

\[
\|V\|(E) = \sup \{ |\varphi(V)(E)| : \varphi \in B^*, \|\varphi\| \leq 1 \}.
\]

We have \( \|V(E)\| \leq \|V\|(E) < \infty \). When the Banach space \( B \) is equipped with the additional structure of a Banach lattice, then more can be said. We mention [6] and [12] as references for the basic theory of Banach lattices.

1.1. Lemma. Let \( (B, \leq) \) be a Banach lattice with positive cone \( B^+ = \{x \in B : x \geq 0\} \). Let \( (X, \mathcal{F}) \) be a measurable space and \( V: \mathcal{F} \to B^+ \) a vector measure taking values in \( B^+ \). Then, for each \( E \in \mathcal{F} \), we have \( \|V\|(E) = \|V(E)\| \).

Proof. Let \( \varphi \in B^* \) be a functional with \( \|\varphi\| \leq 1 \). Then \( \|\varphi\| = |||\varphi||| \leq 1 \), so that

\[
\|\varphi(V)(E)\| = |\varphi^+(V) - \varphi^-(V)|(E) \leq |\varphi^+(V)|(E) + |\varphi^-(V)|(E) = \varphi^+(V(E)) + \varphi^-(V(E)) = |\varphi|(V(E)) \leq ||\varphi|| |||V(E)|| \leq \|V(E)\|.
\]

Thus \( |\varphi(V)(E)| \leq \|V(E)\| \) for each such \( \varphi \), so that \( \|V\|(E) \leq \|V(E)\| \). The converse inequality always holds. \( \square \)

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By abuse of notation, we write \( \| V \| = \| V \|(X) \).

Let \( f : X \to \mathbb{R} \) be an \( \mathcal{F} \)-measurable simple function \( f = \sum a_n 1_{E_n} \) for \( E_n \in \mathcal{F} \). For \( V \in \text{ca}(\mathcal{F}, B) \), we define \( \int f dV = \sum a_n V(E_n) \). This integral is well defined and linear on simple functions, and \( \| \int f dV \| \leq \| f \|_\infty \| V \| \). If now \( f : X \to \mathbb{R} \) is a uniform limit of \( \mathcal{F} \)-measurable simple functions \( f_n \), then \( \| \int f_n - f_m dV \| \leq \| f_n - f_m \| \| V \| \), so that \( (\int f_n dV) \) is a Cauchy sequence in \( B \). Define

\[
\int f dV = \lim \int f_n dV.
\]

Again, this integral is well defined and linear, and \( \| \int f dV \| \leq \| f \|_\infty \| V \| \).

A class \( \mathcal{H} \) of subsets of a set \( X \) is compact if it has the following property: given a sequence \( (K_n) \) drawn from \( \mathcal{H} \) such that \( K_1 \cap K_2 \cap \cdots \cap K_n \neq \emptyset \) for each \( n \), the intersection \( K_1 \cap K_2 \cap \cdots \) is nonempty. Let \( \mathcal{F} \) be a field of subsets of \( X \), and let \( V : \mathcal{F} \to B \) be a charge taking values in a Banach space \( B \). We say that \( V \) is a compact charge if there is a compact class \( \mathcal{H} \) of subsets of \( X \) such that for every \( F \in \mathcal{F} \) and \( \varepsilon > 0 \), there are sets \( F' \in \mathcal{F} \) and \( K \in \mathcal{H} \) with \( F' \subseteq K \subseteq F \) and \( \| V \|(F - F') < \varepsilon \). In this case we say that the class \( \mathcal{H} \) \( V \)-approximates \( \mathcal{F} \). Now suppose that \( \mathcal{F} \) is a \( \sigma \)-field. We say that a charge \( V : \mathcal{F} \to B \) is perfect if the restriction of \( V \) to every countably generated sub-\( \sigma \)-field of \( \mathcal{F} \) is compact.

Most measures arising in practice are compact; we mention one simple case. A metric space is said to be absolute Borel if it is separable and is a Borel subset of its completion. If \( (X, d) \) is a metric space with Borel \( \sigma \)-field \( \mathcal{B}(X) \) and \( B \) is a Banach space, then a charge \( V : \mathcal{B}(X) \to B \) is tight if for each \( \varepsilon > 0 \) and set \( E \in \mathcal{B}(X) \), there is some compact set \( K \subseteq E \) such that \( \| V \|(E - K) < \varepsilon \). Clearly, every tight measure is compact.

1.2. Lemma. Let \( (X, d) \) be an absolute Borel metric space with Borel \( \sigma \)-field \( \mathcal{B}(X) \), and let \( (B, \leq) \) be a Banach lattice with positive cone \( B^+ = \{ x \in B : x \geq 0 \} \). Every vector measure \( V : \mathcal{B}(X) \to B^+ \) is tight and, therefore, compact.

Indication. This follows from Theorem 3.2 in [9] and [1, p. 99].

2. Strassen's Theorem: finitely additive case

In this section, we state and prove a generalisation of the classical result of Strassen to the context of charges taking values in the positive cone of an order-complete Banach lattice. We shall make use of a recent theorem of Wehrung showing such positive cones to be injective objects in the class of positively ordered monoids.

A positively ordered monoid (P.O.M.) is a system \((M, +, 0, \leq)\), where + is a commutative, associative binary operation on a set \( M \), \( 0 \in M \) is an identity for +, and \( \leq \) is a transitive relation on \( M \) such that

1. \( 0 \leq x \) for all \( x \in M \); and
2. \( x \leq y \) implies \( x + z \leq y + z \) for all \( x, y, z \in M \).

A map between P.O.M.'s is a P.O.M.-homomorphism if it preserves +, 0, and \( \leq \).

A Banach lattice \((B, \leq)\) is (order) complete if every subset \( A \subseteq B \) that is bounded above has a supremum sup(A). We note that every reflexive Banach lattice is complete [12, II.5.11]. Also, a result of Wehrung [15, Theorem 3.11]
STRASSEN'S THEOREM FOR VECTOR MEASURES

shows that if \((B, \leq)\) is a complete Banach lattice, then the positive cone \(B^+ = \{x \in B : x \geq 0\}\) is a boundedly injective P.O.M.:

2.1. **Lemma.** Let \((B, \leq)\) be a complete Banach lattice with positive cone \(B^+\), and suppose that \(M_0\) is a sub-P.O.M. of a P.O.M. \(M\). Every bounded P.O.M.-homomorphism \(L: M_0 \to B^+\) extends to a P.O.M.-homomorphism \(\overline{L}: M \to B^+\).

We are now ready for a prototype version of our main result (Theorem 3.1) in the finitely additive setting.

2.2. **Theorem.** Let \(\mathcal{A}\) and \(\mathcal{B}\) be fields of subsets of nonempty sets \(X\) and \(Y\), respectively, and suppose that \(V: \mathcal{A} \to B^+\) and \(W: \mathcal{B} \to B^+\) are charges taking values in the positive cone of a complete vector lattice \((B, \leq)\). Suppose that \(V(X) = W(Y)\). For an arbitrary \(S \subseteq X \times Y\) and \(u \in B^+\), the following are equivalent:

(i) There is a charge \(\rho: \mathcal{P}(X \times Y) \to B^+\) such that \(\rho(E \times Y) = V(E)\) and \(\rho(X \times F) = W(F)\) for all \(E \in \mathcal{A}\) and \(F \in \mathcal{B}\) and such that \(\rho((X \times Y) - S) = u\).

(ii) \(V(E) \leq W(F) + u\) for all \(E \in \mathcal{A}\) and \(F \in \mathcal{B}\) such that \((E \times Y) \cap S \subseteq (X \times F) \cap S\).

**Proof.** (i) \(\Rightarrow\) (ii) We calculate

\[
V(E) = \rho(E \times Y) = \rho((E \times Y) \cap S) + \rho((E \times Y) - S) \\
\leq \rho(X \times F) + \rho((X \times Y) - S) = W(F) + u.
\]

(ii) \(\Rightarrow\) (i) Let \(S(\mathcal{A})\) denote the set of all \(\mathcal{A}\)-measurable simple functions \(f: X \to \mathbb{R}\); likewise for \(S(\mathcal{B})\). Put \(D = (X \times Y) - S\).

**Claim.** If \(f \in S(\mathcal{A})\) and \(g \in S(\mathcal{B})\) are such that \(f(x) - g(y) \leq I_D(x, y)\) for all \((x, y) \in X \times Y\), then \(\int f \, dV - \int g \, dW \leq u\).

**Proof of Claim.** We replace \(f\) and \(g\) with \(f' = f - \min(g)\) and \(g' = g - \min(g)\). Thus it is no loss of generality to assume that \(g \geq 0\) and that there is some \(y\) such that \(g(y) = 0\): this forces \(f \leq 1\). For each \(t \geq 0\), we define \(E_t = \{x \in X : f(x) > t\}\) and \(F_t = \{y \in Y : g(y) > t\}\). Then \((E_t \times Y) \cap S \subseteq (X \times F_t) \cap S\), so \(V(E_t) \leq W(F_t) + u\). We calculate

\[
\int f \, dV - \int g \, dW \leq \int (f \vee 0) \, dV - \int g \, dW = \int_0^1 V(E_t) \, dt - \int g \, dW \\
\leq \int_0^1 W(F_t) \, dt + u - \int g \, dW \leq \int_0^\infty W(F_t) \, dt + u - \int g \, dW = u,
\]

proving the claim.

We define P.O.M.'s \(P\) and \(P_0\) as follows: the elements of \(P\) are all nonnegative functions \(f: X \times Y \to \mathbb{Z}\) taking only finitely many values. The operation \(+\) and the partial order \(\leq\) on \(P\) are defined pointwise. We take \(P_0 \subseteq P\) to be the submonoid comprising all functions on \(X \times Y\) of the form \(f(x) + g(y) + kI_D(x, y)\), where \(f \in S(\mathcal{A})\) and \(g \in S(\mathcal{B})\) are nonnegative functions and \(k\) is a nonnegative integer. Then \(P_0\) is considered as a sub-P.O.M. of \(P\).

We define \(L: P_0 \to B^+\) by putting \(L(f + g + kI_D) = \int f \, dV + \int g \, dW + ku\). We must check that \(L\) is well defined and preserves order. Suppose that \(f + g + kI_D \leq f' + g' + k'I_D\).
Case 1: \( k = k' \). Then \( f(x) - f'(x) \leq g'(y) - g(y) \) for all \( x \) and \( y \). Choose \( x_0 \) and \( y_0 \) so as to maximise the left and minimise the right side of this inequality. Then

\[
\int f - f' \, dV \leq (f(x_0) - f'(x_0))V(X) \leq (g'(y_0) - g(y_0))W(Y) \leq \int g' - g \, dW,
\]

and \( \int f \, dV + \int g \, dW \leq \int f' \, dV + \int g' \, dW \). We have used only that \( V(X) = W(Y) \).

Case 2: \( k \neq k' \). Suppose, for example, that \( k < k' \). Then

\[
\left( \frac{f(x) - f'(x)}{k' - k} \right) - \left( \frac{g'(y) - g(y)}{k' - k} \right) \leq I_D(x, y),
\]

so the Claim applies to show that

\[
\int \left( \frac{f(x) - f'(x)}{k' - k} \right) \, dV - \int \left( \frac{g'(y) - g(y)}{k' - k} \right) \, dW \leq u.
\]

Thus \( L(f + g + kI_D) \leq L(f' + g' + k'I_D) \).

In all cases, \( L \) preserves order and thus is well defined; \( L \) is a bounded P.O.M.-homomorphism from \( P_0 \) to \( B^+ \). Lemma 2.1 now shows that \( L \) extends to a P.O.M.-homomorphism \( \overline{L} : P \to B^+ \). The desired charge \( \rho \) may now be defined by \( \rho(c) = \overline{L}(I_C) \). That \( \rho \) has the desired properties may easily be checked. □

3. Strassen's Theorem: countably additive case

We prove Strassen's Theorem for a pair of countably additive vector measures, one of which is perfect. The original source for Strassen's Theorem is [14]. Other versions and improvements are to be found in [3, 5, 9, 10, 13]. We begin with a generalisation to vector measures of a theorem of Marczewski and Ryll-Nardzewski [8]. Let \( \mathcal{A} \) and \( \mathcal{B} \) be fields of subsets of sets \( X \) and \( Y \), respectively. Then \( \mathcal{A} \times \mathcal{B} \) is the field on \( X \times Y \) generated by all rectangles \( E \times F \) for \( E \in \mathcal{A} \) and \( F \in \mathcal{B} \). If \( \mathcal{A} \) and \( \mathcal{B} \) are \( \sigma \)-fields, then \( \mathcal{A} \otimes \mathcal{B} \) is the \( \sigma \)-field generated by such rectangles.

3.1. Theorem. Let \( \mathcal{A} \) and \( \mathcal{B} \) be \( \sigma \)-fields of subsets of sets \( X \) and \( Y \), respectively, and let \( \rho : \mathcal{A} \times \mathcal{B} \to B^+ \) be a charge taking values in the positive cone of a Banach lattice \( (B, \leq) \). Define charges \( V : \mathcal{A} \to B^+ \) and \( W : \mathcal{B} \to B^+ \) by the rule \( V(E) = \rho(E \times Y) \) and \( W(F) = \rho(X \times F) \).

If \( V \) is countably additive and \( W \) is perfect, then \( \rho \) is countably additive on \( \mathcal{A} \times \mathcal{B} \).

Note. In the case \( B = \mathbb{R} \), this result is due to Marczewski and Ryll-Nardzewski [8].

Proof. We prove the theorem first in the case where \( W \) is compact. Let \( \mathcal{H} \) be a compact class \( W \)-approximating \( \mathcal{B} \), and let \( \mathcal{L} \) be the field generated by all measurable rectangles \( E \times F \) for \( E \in \mathcal{A} \) and \( F \in \mathcal{B} \). Suppose that \( C_1 \supseteq C_2 \supseteq \cdots \) is a decreasing sequence drawn from \( \mathcal{L} \) such that \( \|\rho(C_i)\| > \alpha > 0 \) for all \( i \). We prove that \( \bigcap C_i \neq \emptyset \).

Given any set \( F \in \mathcal{B} \) and \( \varepsilon > 0 \), there are sets \( F' \in \mathcal{B} \) and \( K \in \mathcal{H} \) such that \( F' \subseteq K \subseteq F \) and \( \|W(F - F')\| = \|W\|(F - F') < \varepsilon \). Then, for each
Let $E \in \mathcal{A}$, we have $E \times F' \subseteq E \times K \subseteq E \times F$ and
\[
\|p((E \times F) - (E \times F'))\| = \|p(E \times (F - F'))\| \leq \|p((X \times (F - F'))\|
\leq \|W(F - F')\| < \varepsilon.
\]

Let $\mathcal{M}$ be the class of all subsets of $X \times Y$ that are unions of finitely many rectangles $E \times K$, where $E \in \mathcal{A}$ and $K \in \mathcal{H}$. It follows easily that for each $C \in \mathcal{L}$ and $\varepsilon > 0$, there are sets $M \in \mathcal{M}$ and $C' \in \mathcal{L}$ such that $C' \subseteq M \subseteq C$ and $\|p(C - C')\| < \varepsilon$.

For each $n$, we choose $C'_n \in \mathcal{L}$ and $M_n \in \mathcal{M}$ such that $C'_n \subseteq M_n \subseteq C_n$ so that $\|p(C_n - C'_n)\| < \alpha/2^{n+1}$. Then
\[
\rho(C_{n+1}) = \rho(C_{n+1} \cap C'_1 \cap C'_2 \cap \cdots \cap C'_n) + \rho(C_{n+1} \cap (C'_1 \cup \cdots \cup C'_n)^c)
\leq \rho(C_{n+1} \cap C'_1 \cap \cdots \cap C'_n) + \sum_{i=1}^{n} \rho(C_{n+1} - C_i')
\leq \rho(C'_1 \cap \cdots \cap C'_n) + \sum_{i=1}^{n} \rho(C_i - C'_i),
\]
so that $\|\rho(C_{n+1})\| \leq \|\rho(C'_1 \cap \cdots \cap C'_n)\| + \alpha/2$. It follows that $\|\rho(C'_1 \cap \cdots \cap C'_n)\| > \alpha/2$ for each $n$.

We now use the lemma on projections proved in [7]. The vertical sections of the sets in $\mathcal{M}$ form a compact class, so
\[
\pi(C_1 \cap C_2 \cap \cdots) \supseteq \pi(M_1 \cap M_2 \cap \cdots)
= \bigcap_{n=1}^{\infty} \pi(M_1 \cap \cdots \cap M_n) \supseteq \bigcap_{n=1}^{\infty} \pi(C'_1 \cap \cdots \cap C'_n),
\]
where $\pi: X \times Y \to X$ is a projection to the first factor. Now $\pi(C) \in \mathcal{A}$ whenever $C \in \mathcal{L}$, so we may estimate
\[
\|V(\pi(C'_1 \cap \cdots \cap C'_n))\| \geq \|\rho(C'_1 \cap \cdots \cap C'_n)\| > \alpha/2.
\]
Since $V$ is countably additive, it follows that $\bigcap C_n \supseteq \bigcap C'_n$ is nonempty.

We now consider the case where $W$ is a perfect measure. But we know that $\rho$ is countably additive on each $\sigma$-field $\mathcal{A} \otimes \mathcal{B}$, where $\mathcal{B}_0 \subseteq \mathcal{B}$ is countably generated. (We recall our definition of perfection.) It follows easily that $\rho$ is countably additive on $\mathcal{A} \otimes \mathcal{B}$. \(\square\)

We are now ready for the principal result.

3.2. Theorem. Let $\mathcal{A}$ and $\mathcal{B}$ be $\sigma$-fields of subsets of sets $X$ and $Y$, respectively, and let $V: \mathcal{A} \to B^+$ and $W: \mathcal{B} \to B^+$ be vector measures taking values in the positive cone of a complete Banach lattice $(B, \leq)$. Suppose that $W$ is a perfect measure and that $S \in \mathcal{A} \otimes \mathcal{B}$ is a countable intersection of sets in $\mathcal{A} \times \mathcal{B}$ for any $u \in B^+$, the following are equivalent:

(i) There is a vector measure $\rho: \mathcal{A} \otimes \mathcal{B} \to B^+$ with margins $V$ and $W$ such that $\rho((X \times Y) - S) \leq u$.

(ii) $V(E) \leq W(F) + u$ for all $E \in \mathcal{A}$ and $F \in \mathcal{B}$ such that $(E \times Y) \cap S \subseteq (X \times F) \cap S$.

Proof. (i) $\Rightarrow$ (ii) The calculation in the proof of Theorem 2.2 applies.
(ii) ⇒ (i) From Theorem 2.2, we see that there exists a charge \( \rho_0 : \mathcal{A} \times \mathcal{B} \to B^+ \) with margins \( V \) and \( W \) such that \( \rho_0((X \times Y) - S) = u \). It follows from Theorem 3.1 that \( \rho_0 \) is countably additive on \( \mathcal{A} \times \mathcal{B} \) and so extends to a countably additive charge \( \rho : \mathcal{A} \otimes \mathcal{B} \to B^+ \) [2, I.5.2]. Now let \( (C_n) \) be a sequence of sets in \( \mathcal{A} \otimes \mathcal{B} \) such that \( (X \times Y) - S = \bigcup C_n \). It is no loss of generality to suppose that the sequence \( C_n \) is increasing. Then
\[
\rho((X \times Y) - S) = \lim \rho(C_n) = \lim \rho_0(C_n) \leq \rho_0((X \times Y) - S) = u. \quad \square
\]

Let \( (X, d) \) be a metric space. If \( E \subseteq X \) and \( \varepsilon > 0 \), we define \( E_\varepsilon = \{x: d(x, E) \leq \varepsilon\} \).

3.3. **Corollary.** Let \( (X, d) \) be a complete, separable metric space with Borel \( \sigma \)-field \( \mathcal{B}(X) \), and let \( (B, \leq) \) be a complete Banach lattice. Suppose that \( V, W \in \text{ca}(\mathcal{B}(X), B) \) are vector measures taking values in the positive cone \( B^+ \). If \( u \in B^+ \) and \( \varepsilon \geq 0 \), then the following conditions are equivalent:

(i) \( V(E) \leq W(E_\varepsilon) + u \) for all \( E \in \mathcal{B}(X) \).

(ii) \( \text{There is a countably additive vector measure } \rho : \mathcal{B}(X) \otimes B(X) \to B^+ \) with margins \( V \) and \( W \) such that \( \rho\{(x, y): d(x, y) > \varepsilon\} \leq u \).

**Proof.** Put \( S = \{(x, y): d(x, y) < \varepsilon\} \), a closed subset of \( X \times X \). We apply Theorem 3.2, noting that the condition \( V(E) \leq W(E_\varepsilon) + u \) is equivalent to \( V(E) \leq W(F) + u \) for all \( E \) and \( F \) in \( \mathcal{B}(X) \) such that \( (E \times X) \cap S \subseteq (X \times F) \cap S \). \( \square \)

4. **Metrics for strong convergence**

We turn our attention to an application of the preceding results to a form of measure convergence. Let \( (X, d) \) be a metric space with Borel \( \sigma \)-field \( \mathcal{B}(X) \), and suppose \( V, V_n \in \text{ca}(\mathcal{B}(X), B) \), where \( B \) is a Banach space. We say that the sequence \( (V_n) \) converges strongly to \( V \) if \( \int f dV_n \to \int f dV \) strongly (i.e., in norm) as \( n \to \infty \) for all bounded, uniformly continuous \( f : X \to \mathbb{R} \). When \( X \) is compact, strong convergence of \( (V_n) \) corresponds to convergence of the operators \( T_n = \int f dV_n \) for the strong operator topology.

Let \( (B, \leq) \) be an order-complete Banach lattice with positive cone \( B^+ \), and let \( V, W \in \text{ca}(\mathcal{B}(X), B) \) be vector measures taking values in \( B^+ \). Given \( f : X \to \mathbb{R} \), let \( \|f\|_{BL} = \|f\|_L + \|f\|_{\infty} \), where \( \|f\|_L \) is the Lipschitz constant for \( f \) (see [3, 11.2]). Define
\[
\beta_0(V, W) = \sup \left\{ \left| \int f d(V - W) \right| : \|f\|_{BL} \leq 1 \right\}.
\]

This supremum exists by completeness of \( (B, \leq) \): the quantities in question are bounded above by \( \int \|f\| dV + \int \|f\| dW \leq V(X) + W(X) \). Then define
\[
\beta(V, W) = \|\beta_0(V, W)\|.
\]

The function \( \beta \) is analogous to the classical metric of Fortet and Mourier [3, 11.3]. The following is easily checked:

4.1. **Lemma.** The function \( \beta \) is a metric on the set of vector measures \( V : \mathcal{B}(X) \to B^+ \).
We also define

\[ \rho(V, W) = \inf \{ \varepsilon + \| u \| : \varepsilon \geq 0, \ u \in B^+, \text{ and for all } E \in \mathcal{B}(X) \}
\]

\[ V(E) \leq W(E_{\varepsilon}) + u \}, \]

\[ \sigma(V, W) = \rho(V, W) + \rho(W, V). \]

The functions \( \rho \) and \( \sigma \) are analogues to the classical Prokhoroff metric [3, 11.3; 11].

4.2. Lemma. The function \( \sigma \) is a metric on the set of vector measures \( V : \mathcal{B}(X) \to B^+ \).

Proof. Obviously, we have \( \sigma \geq 0 \) and \( \sigma(V, V) = 0 \); also \( \sigma(V, W) = \sigma(W, V) \).

Suppose now that \( \sigma(V, W) = 0 \). For each \( n \geq 1 \), choose \( \varepsilon(n) \geq 0 \) and \( u(n) \in B^+ \) such that \( \varepsilon(n) + \| u(n) \| < \frac{1}{n} \) and \( V(E) \leq W(E_{\varepsilon(n)}) + u(n) \) for all \( E \in \mathcal{B}(X) \). This is true in particular when \( E \) is closed. Let \( \varphi \) be a positive functional in \( B^* \) with \( \| \varphi \| \leq 1 \). Then \( \varphi(V(E)) \leq \varphi(W(E_{\varepsilon(n)})) + \varphi(u(n)) \). Taking a limit as \( n \to \infty \) and noting \( E \subseteq E_{\varepsilon(n)} \subseteq E_{1/n} \downarrow E \), we obtain \( \varphi(V(E)) \leq \varphi(W(E)) \). Since \( \varphi \) was arbitrary, \( V(E) \leq W(E) \). A similar argument using the equation \( \rho(W, V) = 0 \) shows that \( W(E) \leq V(E) \) for \( E \) closed. Thus \( V = W \).

Now let \( U, V, W \) be measures with \( \rho(U, V) < x \) and \( \rho(V, W) < y \). Then there are \( \varepsilon \geq 0 \) and \( u \in B^+ \) such that \( \varepsilon + \| u \| < x \) and \( u(E) \leq V(E_{\varepsilon}) + u \) for all Borel \( E \subseteq X \). Also, there are \( \eta \geq 0 \) and \( v \in B^+ \) such that \( \eta + \| v \| < y \) and \( V(E) \leq W(E_{\eta}) + v \). In particular, this inequality holds for sets of the form \( E_{\varepsilon} \), so

\[ U(E) \leq V(E_{\varepsilon}) + u \leq W((E_{\varepsilon})_{\eta}) + u + v \leq W(E_{\varepsilon + \eta}) + u + v \]

for all \( E \in \mathcal{B}(X) \). Since \( \varepsilon + \eta + \| u + v \| \leq \varepsilon + \eta + \| u \| + \| v \| < x + y \), we have \( \rho(U, W) \leq x + y \). Letting \( x \) and \( y \) decrease to \( \rho(U, V) \) and \( \rho(V, W) \) yields \( \rho(U, W) \leq \rho(U, V) + \rho(V, W) \). A slight extension of this argument shows that \( \sigma(U, W) \leq \sigma(U, V) + \sigma(V, W) \).

4.3. Lemma. Let \( (X, d) \) be a metric space with Borel structure \( \mathcal{B}(X) \), and suppose that \( (B, \leq) \) is a Banach lattice. If \( V_n, V \in \text{ca} \mathcal{B}(X), B \) are measures taking values in \( B^+ \) such that \( V_n \to V \) strongly, then \( \| V(F) \| \geq \lim \sup_n \| V_n(F) \| \)

for every closed set \( F \subseteq X \).

Proof. For each positive integer \( m \), define \( h_m : X \to \mathbb{R} \) by

\[ h_m(x) = (1 - md(x, F)) \vee 0. \]

Then \( I_F \leq h_m \leq I_{U_m} \), where \( U_m = \{ x : d(x, F) < \frac{1}{m} \} \). So

\[ V_n(F) \leq \int g_m dV_n \text{ and } \int g_m dV \leq V(U_m), \]

or

\[ \| V_n(F) \| \leq \| \int g_m dV_n \| \text{ and } \| \int g_m dV \| \leq \| V(U_m) \|. \]

Since \( h_m \) is uniformly continuous, this yields \( \lim \sup_n \| V_n(F) \| \leq \| V(U_m) \|. \)

Letting \( m \to \infty \) yields the desired result. \( \square \)

The metrics \( \beta \) and \( \sigma \) metrise the strong convergence; the proof makes use of Strassen's Theorem.
4.4. **Theorem.** Let \((X, d)\) be a complete, separable metric space with Borel \(\sigma\)-field \(\mathcal{B}(X)\), and let \((B, \leq)\) be an order-complete Banach lattice with positive cone \(B^+\). If \(V_n, V : \mathcal{B}(X) \to B^+\) are vector measures with \(V_n(X) = V(X)\) for all \(n\), then the following conditions are equivalent:

(i) The sequence \((V_n)\) converges strongly to \(V\).

(ii) \(\beta(V_n, V) \to 0\) as \(n \to \infty\).

(iii) \(\sigma(V_n, V) \to 0\) as \(n \to \infty\).

**Proof.** (i) \(\Rightarrow\) (ii) Given \(\varepsilon > 0\), we use the tightness of \(V\) (Lemma 1.2) to take \(K \subseteq X\) compact such that \(\|V\|(X - K) < \varepsilon\). Define \(B_0\) to be the set of functions \(f : X \to \mathbb{R}\) with \(\|f\|_{BL} \leq 1\). The Arzelà-Ascoli Theorem implies that the restrictions of these functions to \(K\) form a compact (hence totally bounded) subset of \(C(K)\). So there is a finite set \(\{f_1, \ldots, f_k\} \subseteq B_0\) such that for any \(f \in B_0\), there is some \(f_i\) with \(\sup\{|f(x) - f_i(x)| : x \in K\} < \varepsilon\).

Claim 1. \(\sup\{|f(x) - f_j(x)| : x \in K^\varepsilon\} < 3\varepsilon\).

**Proof of Claim.** Given \(y \in K^\varepsilon\) and \(d(x, y) < \varepsilon\),

\[
|f(x) - f_j(x)| \leq |f(x) - f(y)| + |f(y) - f_j(y)| + |f_j(y) - f_j(x)| \leq \|f\|_{Ld}(x, y) + \varepsilon + \|f_j\|_{Ld}(x, y) < 3\varepsilon.
\]

The claim is established.

Now, using Lemma 4.3, we see that for all sufficiently large \(n\),

\[
\|V_n(X - K^\varepsilon)\| \leq \|V(X - K^\varepsilon)\| \leq \|V(X - K)\| \leq \varepsilon.
\]

Thus, for each \(f \in B\) and \(f_j\) as above,

\[
\left|\int f \, d(V_n - V)\right| \leq \int |f - f_j| \, d(V_n + V) + \int f_j \, d(V_n - V) = \int_{X - K^\varepsilon} 2 \cdot d(V_n + V) + \int_{K^\varepsilon} |f - f_j| \, d(V_n + V) + \int f_j \, d(V_n - V) \leq 2(V_n + V)(X - K^\varepsilon) + 3\varepsilon(V_n + V)(X) + \int f_j \, d(V_n - V).
\]

Taking norms yields

\[
\left\|\int f \, d(V_n - V)\right\| \leq 2(2\varepsilon) + 2\varepsilon\|V(X)\| + \left\|\int f_j \, d(V_n - V)\right\|.
\]

This last term tends to zero as \(n \to \infty\).

Given \(\eta > 0\), choose \(\varepsilon > 0\) small enough and \(n\) large enough so that \(\|\int f \, d(V_n - V)\| < \eta\) independent of \(f \in B_0\). Thus, for \(n\) large, we have \(\beta(V_n - V) < \eta\) and \(\beta(V_n, V) \to 0\) as \(n \to \infty\).

(ii) \(\Rightarrow\) (iii) Given a Borel set \(E \subseteq X\) and \(\varepsilon > 0\), define \(h_\varepsilon : X \to \mathbb{R}\) by \(h_\varepsilon(x) = 0 \lor (1 - d(x, E)/\varepsilon)\). Then \(\|h_\varepsilon\|_{BL} \leq 1 + \frac{1}{\varepsilon}\). For any positive vector measures \(V\) and \(W\), we have

\[
V(E) \leq \int h_\varepsilon \, dV \quad \text{and} \quad \int h_\varepsilon \, dW \leq W(E_\varepsilon),
\]
so that

\[ V(E) \leq W(E_{\varepsilon}) + \int h_{\varepsilon}d(W - V) \leq W(E_{\varepsilon}) + \left(1 + \frac{1}{\varepsilon}\right)\beta_0(V, W). \]

Thus \( \rho(V, W) \leq \varepsilon + (1 + \frac{1}{\varepsilon})\beta(V, W) \). Suppose now that \( \beta(V_n, V) \to 0 \). Given \( \eta > 0 \), choose \( \varepsilon = \frac{\eta}{2} \) and \( n \) large enough so that \( \beta(V_n, V) < \frac{\eta}{2}(1 + \frac{1}{\varepsilon}) \). Then \( \rho(V_n, V) < \eta \). This shows that \( \rho(V_n, V) \to 0 \) and similarly \( \sigma(V_n, V) \to 0 \).

(iii) \( \Rightarrow \) (i) Let \( f : X \to \mathbb{R} \) be bounded and uniformly continuous and \( \varepsilon > 0 \). Choose \( \delta > 0 \) so that \( |f(x) - f(y)| < \varepsilon \) whenever \( |x - y| \leq \delta \). Suppose that \( V \) and \( W \) are positive vector measures with \( V(X) = W(X) \) and \( \rho(V, W) < \delta \); then there are \( \eta > 0 \) and \( u \in B^+ \) with \( \eta + \|u\| \leq \delta \) and \( V(E) \leq W(E_{\eta}) + u \) for all Borel \( E \subseteq X \). We now apply Corollary 3.2 to find a measure \( \rho \) on \( X \times Y \) with margins \( V \) and \( W \) such that \( \rho((x, y) : d(X, y) > \eta) \leq u \). Then

\[
\int f d(V - W) = \int f(x) - f(y) \, d\rho(x, y)
= \int_{d(x, y) \leq \eta} f(x) - f(y) \, d\rho(x, y) + \int_{d(x, y) > \eta} f(x) - f(y) \, d\rho(x, y)
\]

and

\[
(*) \quad \left\| \int f d(V - W) \right\| \leq \varepsilon\|V(X)\| + 2\|f\|_{\infty}\|u\|.
\]

Given an arbitrary \( \varepsilon > 0 \), choose \( \delta > 0 \) such that \( |f(x) - f(y)| < \frac{\delta}{2}\|V(X)\| \) for \( |x - y| \leq \delta \) and \( \delta < \frac{\varepsilon}{2}\|f\|_{\infty} \). Then from (*), \( \|\int f d(V - W)\| < \varepsilon \) whenever \( \rho(V, W) < \delta \). It follows that \( \int f d(V_n - V) \to 0 \) whenever \( \sigma(V_n, V) \to 0 \). \( \square \)

REFERENCES


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