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STRASSEN'S THEOREM FOR VECTOR MEASURES

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ABSTRACT. A type of Strassen's Theorem for measures taking values in the positive cone of a Banach lattice is proved. An application is given to metrics for convergence of vector measures.

1. PRELIMINARIES: VECTOR MEASURES

Let $\mathscr{F}$ be a field of subsets of a set X, and let $(B, \|\cdot\|)$ be a Banach space. (All vector spaces we consider are assumed to have real scalars.) Then $\mathrm{a}(\mathscr{F}, B)$ is the set of all additive set functions $V: \mathscr{F} \to B$, i.e., $V(E_1 \cup E_2) = V(E_1) + V(E_2)$ for disjoint E_1 and E_2 in $\mathscr{F}$. The elements of $\mathrm{a}(\mathscr{F}, B)$ we call *charges*. Also, $\mathrm{ca}(\mathscr{F}, B)$ is the set of all countably additive charges in $\mathrm{a}(\mathscr{F}, B)$; we call such charges *vector measures*. Generally, we follow the notation and conventions of Dunford and Schwartz [4] or Diestel and Uhl [2].

Let $(B, \|\cdot\|)$ be a Banach space with dual space B^*. If $V \in \mathrm{ca}(\mathscr{F}, B)$ and $\varphi \in B^*$, then $\varphi(V) = \varphi \circ V$ is a finite signed measure on $\mathscr{F}$ with total variation $|\varphi(V)|$. The *semivariation* of V is the set function $\|V\|: \mathscr{F} \to \mathbb{R}$ defined by

$$\|V\|(E) = \sup\{|\varphi(V)|(E): \varphi \in B^*, \ \|\varphi\| \leq 1\}.$$

We have $\|V(E)\| \leq \|V\|(E) < \infty$. When the Banach space B is equipped with the additional structure of a Banach lattice, then more can be said. We mention [6] and [12] as references for the basic theory of Banach lattices.

1.1. **Lemma.** *Let $(B, \leq)$ be a Banach lattice with positive cone $B^+ = \{x \in B: x \geq 0\}$. Let $(X, \mathscr{F})$ be a measurable space and $V: \mathscr{F} \to B^+$ a vector measure taking values in B^+. Then, for each $E \in \mathscr{F}$, we have $\|V\|(E) = \|V(E)\|$.*

Proof. Let $\varphi \in B^*$ be a functional with $\|\varphi\| \leq 1$. Then $\|\varphi\| = \||\varphi|\| \leq 1$, so that

$$\begin{aligned}|\varphi(V)|(E) &= |\varphi^+(V) - \varphi^-(V)|(E) \leq |\varphi^+(V)|(E) + |\varphi^-(V)|(E) \\ &= \varphi^+(V(E)) + \varphi^-(V(E)) = |\varphi|(V(E)) \leq \||\varphi|\| \, \|V(E)\| \leq \|V(E)\|.\end{aligned}$$

Thus $|\varphi(V)|(E) \leq \|V(E)\|$ for each such φ, so that $\|V\|(E) \leq \|V(E)\|$. The converse inequality always holds. □

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By abuse of notation, we write $\|V\| = \|V\|(X)$.

Let $f: X \to \mathbb{R}$ be an $\mathscr{F}$-measurable simple function $f = \sum a_n I_{E_n}$ for $E_n \in \mathscr{F}$. For $V \in \text{ca}(\mathscr{F}, B)$, we define $\int f\,dV = \sum a_n V(E_n)$. This integral is well defined and linear on simple functions, and $\|\int f\,dV\| \leq \|f\|_\infty \|V\|$. If now $f: X \to \mathbb{R}$ is a uniform limit of $\mathscr{F}$-measurable simple functions f_n, then $\|\int f_n - f_m\,dV\| \leq \|f_n - f_m\|_\infty \|V\|$, so that $(\int f_n\,dV)$ is a Cauchy sequence in B. Define

$$\int f\,dV - \lim \int f_n\,dV.$$

Again, this integral is well defined and linear, and $\|\int f\,dV\| \leq \|f\|_\infty \|V\|$.

A class $\mathscr{K}$ of subsets of a set X is *compact* if it has the following property: given a sequence (K_n) drawn from $\mathscr{K}$ such that $K_1 \cap K_2 \cap \cdots \cap K_n \neq \varnothing$ for each n, the intersection $K_1 \cap K_2 \cap \cdots$ is nonempty. Let $\mathscr{F}$ be a field of subsets of X, and let $V: \mathscr{F} \to B$ be a charge taking values in a Banach space B. We say that V is a *compact charge* if there is a compact class $\mathscr{K}$ of subsets of X such that for every $F \in \mathscr{F}$ and $\varepsilon > 0$, there are sets $F' \in \mathscr{F}$ and $K \in \mathscr{K}$ with $F' \subseteq K \subseteq F$ and $\|V\|(F - F') < \varepsilon$. In this case we say that the class $\mathscr{K}$ *V-approximates* $\mathscr{F}$. Now suppose that $\mathscr{F}$ is a σ-field. We say that a charge $V: \mathscr{F} \to B$ is *perfect* if the restriction of V to every countably generated sub-σ-field of $\mathscr{F}$ is compact.

Most measures arising in practice are compact; we mention one simple case. A metric space is said to be *absolute Borel* if it is separable and is a Borel subset of its completion. If (X, d) is a metric space with Borel σ-field $\mathscr{B}(X)$ and B is a Banach space, then a charge $V: \mathscr{B}(X) \to B$ is *tight* if for each $\varepsilon > 0$ and set $E \in \mathscr{B}(X)$, there is some compact set $K \subseteq E$ such that $\|V\|(E - K) < \varepsilon$. Clearly, every tight measure is compact.

1.2. **Lemma.** *Let (X, d) be an absolute Borel metric space with Borel σ-field $\mathscr{B}(X)$, and let $(B, \leq)$ be a Banach lattice with positive cone $B^+ = \{x \in B : x \geq 0\}$. Every vector measure $V: \mathscr{B}(X) \to B^+$ is tight and, therefore, compact.*

Indication. This follows from Theorem 3.2 in [9] and [1, p. 99].

2. Strassen's Theorem: finitely additive case

In this section, we state and prove a generalisation of the classical result of Strassen to the context of charges taking values in the positive cone of an order-complete Banach lattice. We shall make use of a recent theorem of Wehrung showing such positive cones to be injective objects in the class of positively ordered monoids.

A *positively ordered monoid* (P.O.M.) is a system $(M, +, 0, \leq)$, where $+$ is a commutative, associative binary operation on a set M, $0 \in M$ is an identity for $+$, and $\leq$ is a transitive relation on M such that

(1) $0 \leq x$ for all $x \in M$; and
(2) $x \leq y$ implies $x + z \leq y + z$ for all $x, y, z \in M$.

A map between P.O.M.'s is a *P.O.M.-homomorphism* if it preserves $+$, 0, and $\leq$.

A Banach lattice $(B, \leq)$ is (*order*) *complete* if every subset $A \subseteq B$ that is bounded above has a supremum $\sup(A)$. We note that every reflexive Banach lattice is complete [12, II.5.11]. Also, a result of Wehrung [15, Theorem 3.11] shows that if $(B, \leq)$ is a complete Banach lattice, then the positive cone $B^+ = \{x \in B : x \geq 0\}$ is a boundedly injective P.O.M.:

2.1. **Lemma.** *Let $(B, \leq)$ be a complete Banach lattice with positive cone B^+, and suppose that M_0 is a sub-P.O.M. of a P.O.M. M. Every bounded P.O.M.-homomorphism $L: M_0 \to B^+$ extends to a P.O.M.-homomorphism $\overline{L}: M \to B^+$.*

We are now ready for a prototype version of our main result (Theorem 3.1) in the finitely additive setting.

2.2. **Theorem.** *Let $\mathscr{A}$ and $\mathscr{B}$ be fields of subsets of nonempty sets X and Y, respectively, and suppose that $V: \mathscr{A} \to B^+$ and $W: \mathscr{B} \to B^+$ are charges taking values in the positive cone of a complete vector lattice $(B, \leq)$. Suppose that $V(X) = W(Y)$. For an arbitrary $S \subseteq X \times Y$ and $u \in B^+$, the following are equivalent:*

(i) *There is a charge $\rho: \mathscr{P}(X \times Y) \to B^+$ such that $\rho(E \times Y) = V(E)$ and $\rho(X \times F) = W(F)$ for all $E \in \mathscr{A}$ and $F \in \mathscr{B}$ and such that $\rho((X \times Y) - S) = u$.*

(ii) *$V(E) \leq W(F) + u$ for all $E \in \mathscr{A}$ and $F \in \mathscr{B}$ such that $(E \times Y) \cap S \subseteq (X \times F) \cap S$.*

Proof. (i) $\Rightarrow$ (ii) We calculate

$$\begin{aligned} V(E) &= \rho(E \times Y) = \rho((E \times Y) \cap S) + \rho((E \times Y) - S) \\ &\leq \rho(X \times F) + \rho((X \times Y) - S) = W(F) + u. \end{aligned}$$

(ii) $\Rightarrow$ (i) Let $S(\mathscr{A})$ denote the set of all $\mathscr{A}$-measurable simple functions $f: X \to \mathbb{R}$; likewise for $S(\mathscr{B})$. Put $D = (X \times Y) - S$.

Claim. If $f \in S(\mathscr{A})$ and $g \in S(\mathscr{B})$ are such that $f(x) - g(y) \leq I_D(x, y)$ for all $(x, y) \in X \times Y$, then $\int f\,dV - \int g\,dW \leq u$.

Proof of Claim. We replace f and g with $f' = f - \min(g)$ and $g' = g - \min(g)$. Thus it is no loss of generality to assume that $g \geq 0$ and that there is some y such that $g(y) = 0$: this forces $f \leq 1$. For each $t \geq 0$, we define $E_t = \{x \in X : f(x) > t\}$ and $F_t = \{y \in Y : g(y) > t\}$. Then $(E_t \times Y) \cap S \subseteq (X \times F_t) \cap S$, so $V(E_t) \leq W(F_t) + u$. We calculate

$$\begin{aligned} \int f\,dV - \int g\,dW &\leq \int (f \vee 0)\,dV - \int g\,dW = \int_0^1 V(E_t)\,dt - \int g\,dW \\ &\leq \int_0^1 W(F_t)\,dt + u - \int g\,dW \leq \int_0^\infty W(F_t)\,dt + u - \int g\,dW = u, \end{aligned}$$

proving the claim.

We define P.O.M.'s P and P_0 as follows: the elements of P are all nonnegative functions $F: X \times Y \to \mathbb{Z}$ taking only finitely many values. The operation $+$ and the partial order $\leq$ on P are defined pointwise. We take $P_0 \subseteq P$ to be the submonoid comprising all functions on $X \times Y$ of the form $(x, y) \to f(x) + g(y) + kI_D(x, y)$, where $f \in S(\mathscr{A})$ and $g \in S(\mathscr{B})$ are nonnegative functions and k is a nonnegative integer. Then P_0 is considered as a sub-P.O.M. of P.

We define $L: P_0 \to B^+$ by putting $L(f + g + kI_D) = \int f\,dV + \int g\,dW + ku$. We must check that L is well defined and preserves order. Suppose that $f + g + kI_D \leq f' + g' + k'I_D$.

Case 1: $k = k'$. Then $f(x) - f'(x) \le g'(y) - g(y)$ for all x and y. Choose x_0 and y_0 so as to maximise the left and minimise the right side of this inequality. Then

$$\int f - f'\,dV \le (f(x_0) - f'(x_0))V(X) \le (g'(y_0) - g(y_0))W(Y) \le \int g' - g\,dW,$$

and $\int f\,dV + \int g\,dW \le \int f'\,dV + \int g'\,dW$. We have used only that $V(X) = W(Y)$.

Case 2: $k \neq k'$. Suppose, for example, that $k < k'$. Then

$$\left(\frac{f(x) - f'(x)}{k' - k}\right) - \left(\frac{g'(y) - g(y)}{k' - k}\right) \le I_D(x, y),$$

so the Claim applies to show that

$$\int \left(\frac{f(x) - f'(x)}{k' - k}\right) dV - \int \left(\frac{g'(y) - g(y)}{k' - k}\right) dW \le u.$$

Thus $L(f + g + kI_D) \le L(f' + g' + k'I_D)$.

In all cases, L preserves order and thus is well defined; L is a bounded P.O.M.-homomorphism from P_0 to B^+. Lemma 2.1 now shows that L extends to a P.O.M.-homomorphism $\overline{L}: P \to B^+$. The desired charge ρ may now be defined by $\rho(c) = \overline{L}(I_C)$. That ρ has the desired properties may easily be checked. □

3. Strassen's Theorem: countably additive case

We prove Strassen's Theorem for a pair of countably additive vector measures, one of which is perfect. The original source for Strassen's Theorem is [14]. Other versions and improvements are to be found in [3, 5, 9, 10, 13].

We begin with a generalisation to vector measures of a theorem of Marczewski and Ryll-Nardzewski [8]. Let $\mathscr{A}$ and $\mathscr{B}$ be fields of subsets of sets X and Y, respectively. Then $\mathscr{A} \times \mathscr{B}$ is the field on $X \times Y$ generated by all rectangles $E \times F$ for $E \in \mathscr{A}$ and $F \in \mathscr{B}$. If $\mathscr{A}$ and $\mathscr{B}$ are σ-fields, then $\mathscr{A} \otimes \mathscr{B}$ is the σ-field generated by such rectangles.

3.1. **Theorem.** *Let $\mathscr{A}$ and $\mathscr{B}$ be σ-fields of subsets of sets X and Y, respectively, and let $\rho: \mathscr{A} \times \mathscr{B} \to B^+$ be a charge taking values in the positive cone of a Banach lattice $(B, \le)$. Define charges $V: \mathscr{A} \to B^+$ and $W: \mathscr{B} \to B^+$ by the rule $V(E) = \rho(E \times Y)$ and $W(F) = \rho(X \times F)$.*

If V is countably additive and W is perfect, then ρ is countably additive on $\mathscr{A} \times \mathscr{B}$.

Note. In the case $B = \mathbb{R}$, this result is due to Marczewski and Ryll-Nardzewski [8].

Proof. We prove the theorem first in the case where W is compact. Let $\mathscr{K}$ be a compact class W-approximating $\mathscr{B}$, and let $\mathscr{L}$ be the field generated by all measurable rectangles $E \times F$ for $E \in \mathscr{A}$ and $F \in \mathscr{B}$. Suppose that $C_1 \supseteq C_2 \supseteq \cdots$ is a decreasing sequence drawn from $\mathscr{L}$ such that $\|\rho(C_i)\| > \alpha > 0$ for all i. We prove that $\bigcap C_i \neq \varnothing$.

Given any set $F \in \mathscr{B}$ and $\varepsilon > 0$, there are sets $F' \in \mathscr{B}$ and $K \in \mathscr{K}$ such that $F' \subseteq K \subseteq F$ and $\|W(F - F')\| = \|W\|(F - F') < \varepsilon$. Then, for each

$E \in \mathscr{A}$, we have $E \times F' \subseteq E \times K \subseteq E \times F$ and

$$\begin{aligned}\|\rho((E \times F) - (E \times F'))\| = \|\rho\|(E \times (F - F')) \le \|\rho\|(X \times (F - F')) \\ = \|W(F - F')\| < \varepsilon.\end{aligned}$$

Let $\mathscr{M}$ be the class of all subsets of $X \times Y$ that are unions of finitely many rectangles $E \times K$, where $E \in \mathscr{A}$ and $K \in \mathscr{K}$. It follows easily that for each $C \in \mathscr{L}$ and $\varepsilon > 0$, there are sets $M \in \mathscr{M}$ and $C' \in \mathscr{L}$ such that $C' \subseteq M \subseteq C$ and $\|\rho(C - C')\| < \varepsilon$.

For each n, we choose $C'_n \in \mathscr{L}$ and $M_n \in \mathscr{M}$ such that $C'_n \subseteq M_n \subseteq C_n$ so that $\|\rho(C_n - C'_n)\| < \alpha/2^{n+1}$. Then

$$\begin{aligned}\rho(C_{n+1}) &= \rho(C_{n+1} \cap C'_1 \cap C'_2 \cap \cdots \cap C'_n) + \rho(C_{n+1} \cap (C'_1 \cup \cdots \cup C'_n)^c) \\ &\le \rho(C_{n+1} \cap C'_1 \cap \cdots \cap C'_n) + \sum_{i=1}^{n} \rho(C_{n+1} - C'_i) \\ &\le \rho(C'_1 \cap \cdots \cap C'_n) + \sum_{i=1}^{n} \rho(C_i - C'_i),\end{aligned}$$

so that $\|\rho(C_{n+1})\| \le \|\rho(C'_1 \cap \cdots \cap C'_n)\| + \alpha/2$. It follows that $\|\rho(C'_1 \cap \cdots \cap C'_n)\| > \alpha/2$ for each n.

We now use the lemma on projections proved in [7]. The vertical sections of the sets in $\mathscr{M}$ form a compact class, so

$$\begin{aligned}\pi(C_1 \cap C_2 \cap \cdots) &\supseteq \pi(M_1 \cap M_2 \cap \cdots) \\ &= \bigcap_{n=1}^{\infty} \pi(M_1 \cap \cdots \cap M_n) \supseteq \bigcap_{n=1}^{\infty} \pi(C'_1 \cap \cdots \cap C'_n),\end{aligned}$$

where $\pi: X \times Y \to X$ is a projection to the first factor. Now $\pi(C) \in \mathscr{A}$ whenever $C \in \mathscr{L}$, so we may estimate

$$\|V(\pi(C'_1 \cap \cdots \cap C'_n))\| \ge \|\rho(C'_1 \cap \cdots \cap C'_n)\| > \alpha/2.$$

Since V is countably additive, it follows that $\bigcap C_n \supseteq \bigcap C'_n$ is nonempty.

We now consider the case where W is a perfect measure. But we know that ρ is countably additive on each σ-field $\mathscr{A} \otimes \mathscr{B}$, where $\mathscr{B}_0 \subseteq \mathscr{B}$ is countably generated. (We recall our definition of perfection.) It follows easily that ρ is countably additive on $\mathscr{A} \times \mathscr{B}$. □

We are now ready for the principal result.

3.2. **Theorem.** *Let $\mathscr{A}$ and $\mathscr{B}$ be σ-fields of subsets of sets X and Y, respectively, and let $V: \mathscr{A} \to B^+$ and $W: \mathscr{B} \to B^+$ be vector measures taking values in the positive cone of a complete Banach lattice $(B, \le)$. Suppose that W is a perfect measure and that $S \in \mathscr{A} \otimes \mathscr{B}$ is a countable intersection of sets in $\mathscr{A} \times B$. For any $u \in B^+$, the following are equivalent:*

(i) *There is a vector measure $\rho: \mathscr{A} \otimes \mathscr{B} \to B^+$ with margins V and W such that $\rho((X \times Y) - S) \le u$.*

(ii) *$V(E) \le W(F) + u$ for all $E \in \mathscr{A}$ and $F \in \mathscr{B}$ such that $(E \times Y) \cap S \subseteq (X \times F) \cap S$.*

Proof. (i) ⇒ (ii) The calculation in the proof of Theorem 2.2 applies.

(ii) $\Rightarrow$ (i) From Theorem 2.2, we see that there exists a charge $\rho_0: \mathscr{A} \times \mathscr{B} \to B^+$ with margins V and W such that $\rho_0((X \times Y) - S) = u$. It follows from Theorem 3.1 that ρ_0 is countably additive on $\mathscr{A} \times \mathscr{B}$ and so extends to a countably additive charge $\rho: \mathscr{A} \otimes \mathscr{B} \to B^+$ [2, I.5.2]. Now let (C_n) be a sequence of sets in $\mathscr{A} \otimes \mathscr{B}$ such that $(X \times Y) - S = \bigcup C_n$. It is no loss of generality to suppose that the sequence C_n is increasing. Then

$$\rho((X \times Y) - S) = \lim \rho(C_n) = \lim \rho_0(C_n) \le \rho_0((X \times Y) - S) = u. \quad \square$$

Let (X, d) be a metric space. If $E \subseteq X$ and $\varepsilon > 0$, we define $E_\varepsilon = \{x: d(x, E) \le \varepsilon\}$.

3.3. **Corollary.** *Let (X, d) be a complete, separable metric space with Borel σ-field $\mathscr{B}(X)$, and let $(B, \le)$ be a complete Banach lattice. Suppose that $V, W \in$ ca$(\mathscr{B}(X), B)$ are vector measures taking values in the positive cone B^+. If $u \in B^+$ and $\varepsilon \ge 0$, then the following conditions are equivalent:*

(i) *$V(E) \le W(E_\varepsilon) + u$ for all $E \in \mathscr{B}(X)$.*

(ii) *There is a countably additive vector measure $\rho: \mathscr{B}(X) \otimes B(X) \to B^+$ with margins V and W such that $\rho\{(x, y): d(x, y) > \varepsilon\} \le u$.*

Proof. Put $S = \{(x, y): d(x, y) \le \varepsilon\}$, a closed subset of $X \times X$. We apply Theorem 3.2, noting that the condition $V(E) \le W(E_\varepsilon) + u$ is equivalent to $V(E) \le W(F) + u$ for all E and F in $\mathscr{B}(X)$ such that $(E \times X) \cap S \subseteq (X \times F) \cap S$. $\square$

4. Metrics for strong convergence

We turn our attention to an application of the preceding results to a form of measure convergence. Let (X, d) be a metric space with Borel σ-field $\mathscr{B}(X)$, and suppose $V, V_n \in$ ca$(\mathscr{B}(X), B)$, where B is a Banach space. We say that the sequence (V_n) *converges strongly to* V if $\int f\,dV_n \to \int f\,dV$ strongly (i.e., in norm) as $n \to \infty$ for all bounded, uniformly continuous $f: X \to \mathbb{R}$. When X is compact, strong convergence of (V_n) corresponds to convergence of the operators $T_n = \int \cdot\, dV_n$ for the strong operator topology.

Let $(B, \le)$ be an order-complete Banach lattice with positive cone B^+, and let $V, W \in$ ca$(\mathscr{B}(X), B)$ be vector measures taking values in B^+. Given $f: X \to \mathbb{R}$, let $\|f\|_{BL} = \|f\|_L + \|f\|_\infty$, where $\|f\|_L$ is the Lipschitz constant for f (see [3, 11.2]). Define

$$\beta_0(V, W) = \sup\left\{\left|\int f\,d(V - W)\right| : \|f\|_{BL} \le 1\right\}.$$

This supremum exists by completeness of $(B, \le)$: the quantities in question are bounded above by $\int |f|\,dV + \int |f|\,dW \le V(X) + W(X)$. Then define

$$\beta(V, W) = \|\beta_0(V, W)\|.$$

The function β is analogous to the classical metric of Fortet and Mourier [3, 11.3]. The following is easily checked:

4.1. **Lemma.** *The function β is a metric on the set of vector measures $V: \mathscr{B}(X) \to B^+$.*

We also define

$$\rho(V, W) = \inf\{\varepsilon + \|u\| : \varepsilon \geq 0,\ u \in B^+,\ \text{and for all } E \in \mathscr{B}(X)\ \ V(E) \leq W(E_\varepsilon) + u\},$$

$$\sigma(V, W) = \rho(V, W) + \rho(W, V).$$

The functions ρ and σ are analogues to the classical Prokhoroff metric [3, 11.3; 11].

4.2. **Lemma.** *The function σ is a metric on the set of vector measures $V: \mathscr{B}(X) \to B^+$.*

Proof. Obviously, we have $\sigma \geq 0$ and $\sigma(V, V) = 0$; also $\sigma(V, W) = \sigma(W, V)$.

Suppose now that $\sigma(V, W) = 0$. For each $n \geq 1$, choose $\varepsilon(n) \geq 0$ and $u(n) \in B^+$ such that $\varepsilon(n) + \|u(n)\| < \frac{1}{n}$ and $V(E) \leq W(E_{\varepsilon(n)}) + u(n)$ for all $E \in \mathscr{B}(X)$. This is true in particular when E is closed. Let φ be a positive functional in B^* with $\|\varphi\| \leq 1$. Then $\varphi(V(E)) \leq \varphi(W(E_{\varepsilon(n)})) + \varphi(u(n))$. Taking a limit as $n \to \infty$ and noting $E \subseteq E_{\varepsilon(n)} \subseteq E_{(1/n)} \downarrow E$, we obtain $\varphi(V(E)) \leq \varphi(W(E))$. Since φ was arbitrary, $V(E) \leq W(e)$. A similar argument using the equation $\rho(W, V) = 0$ shows that $W(E) \leq V(E)$ for E closed. Thus $V = W$.

Now let U, V, W be measures with $\rho(U, V) < x$ and $\rho(V, W) < y$. Then there are $\varepsilon \geq 0$ and $u \in B^+$ such that $\varepsilon + \|u\| < x$ and $u(E) \leq V(E_\varepsilon) + u$ for all Borel $E \subseteq X$. Also, there are $\eta \geq 0$ and $v \in B^+$ such that $\eta + \|v\| < y$ and $V(E) \leq W(E_\eta) + v$. In particular, this inequality holds for sets of the form E_ε, so

$$U(E) \leq V(E_\varepsilon) + u \leq W((E_\varepsilon)_\eta) + u + v \leq W(E_{\varepsilon+\eta}) + u + v$$

for all $E \in \mathscr{B}(X)$. Since $\varepsilon + \eta + \|u + v\| \leq \varepsilon + \eta + \|u\| + \|v\| < x + y$, we have $\rho(U, W) \leq x + y$. Letting x and y decrease to $\rho(U, V)$ and $\rho(V, W)$ yields $\rho(U, W) \leq \rho(U, V) + \rho(V, W)$. A slight extension of this argument shows that $\sigma(U, W) \leq \sigma(U, V) + \sigma(V, W)$. □

4.3. **Lemma.** *Let (X, d) be a metric space with Borel structure $\mathscr{B}(X)$, and suppose that $(B, \leq)$ is a Banach lattice. If $V_n, V \in \mathrm{ca}(\mathscr{B}(X), B)$ are measures taking values in B^+ such that $V_n \to V$ strongly, then $\|V(F)\| \geq \limsup_n \|V_n(F)\|$ for every closed set $F \subseteq X$.*

Proof. For each positive integer m, define $h_m: X \to \mathbb{R}$ by

$$h_m(x) = (1 - md(x, F)) \vee 0.$$

Then $I_F \leq h_m \leq I_{U_m}$, where $U_m = \{x : d(x, F) < \frac{1}{m}\}$. So

$$V_n(F) \leq \int g_m \, dV_n \quad \text{and} \quad \int g_m \, dV \leq V(U_m),$$

or

$$\|V_n(F)\| \leq \left\| \int g_m \, dV_n \right\| \quad \text{and} \quad \left\| \int g_m \, dV \right\| \leq \|V(U_m)\|.$$

Since h_m is uniformly continuous, this yields $\limsup_n \|V_n(F)\| \leq \|V(U_m)\|$. Letting $m \to \infty$ yields the desired result. □

The metrics β and σ metrise the strong convergence; the proof makes use of Strassen's Theorem.

4.4. Theorem. *Let (X, d) be a complete, separable metric space with Borel σ-field $\mathscr{B}(X)$, and let $(B, \leq)$ be an order-complete Banach lattice with positive cone B^+. If $V_n, V: \mathscr{B}(X) \to B^+$ are vector measures with $V_n(X) = V(X)$ for all n, then the following conditions are equivalent:*

(i) *The sequence (V_n) converges strongly to V.*
(ii) $\beta(V_n, V) \to 0$ *as* $n \to \infty$.
(iii) $\sigma(V_n, V) \to 0$ *as* $n \to \infty$.

Proof. (i) $\Rightarrow$ (ii) Given $\varepsilon > 0$, we use the tightness of V (Lemma 1.2) to take $K \subseteq X$ compact such that $\|V\|(X - K) < \varepsilon$. Define B_0 to be the set of functions $f: X \to \mathbb{R}$ with $\|f\|_{BL} \leq 1$. The Arzelà-Ascoli Theorem implies that the restrictions of these functions to K form a compact (hence totally bounded) subset of $C(K)$. So there is a finite set $\{f_1, \ldots, f_k\} \subseteq B_0$ such that for any $f \in B_0$, there is some f_j with $\sup\{|f(x) - f_j(x)|: x \in K\} < \varepsilon$.

Claim 1. $\sup\{|f(x) - f_j(x)|: x \in K^\varepsilon\} < 3\varepsilon$.
Proof of Claim. Given $y \in K$ and $d(x, y) < \varepsilon$,

$$\begin{aligned}|f(x) - f_j(x)| &\leq |f(x) - f(y)| + |f(y) - f_j(y)| + |f_j(y) - f_j(x)| \\ &\leq \|f\|_L d(x, y) + \varepsilon + \|f_j\|_L d(x, y) < 3\varepsilon.\end{aligned}$$

The claim is established.

Now, using Lemma 4.3, we see that for all sufficiently large n,

$$\|V_n(X - K^\varepsilon)\| \leq \|V(X - K^\varepsilon)\| \leq \|V(X - K)\| < \varepsilon.$$

Thus, for each $f \in B$ and f_j as above,

$$\begin{aligned}\left|\int f\,d(V_n - V)\right| &\leq \int |f - f_j|\,d(V_n + V) + \left|\int f_j\,d(V_n - V)\right| \\ &= \int_{X-K^\varepsilon} 2 \cdot d(V_n + V) + \int_{K^\varepsilon} |f - f_j|\,d(V_n + V) \\ &\quad + \left|\int f_j\,d(V_n - V)\right| \leq 2(V_n + V)(X - K^\varepsilon) \\ &\quad + 3\varepsilon(V_n + V)(X) + \left|\int f_j\,d(V_n - V)\right|.\end{aligned}$$

Taking norms yields

$$\left\|\int f\,d(V_n - V)\right\| \leq 2(2\varepsilon) + 2\varepsilon\|V(X)\| + \left\|\int f_j\,d(V_n - V)\right\|.$$

This last term tends to zero as $n \to \infty$.

Given $\eta > 0$, choose $\varepsilon > 0$ small enough and n large enough so that $\|\int f\,d(V_n - V)\| < \eta$ independent of $f \in B_0$. Thus, for n large, we have $\beta(V_n - V) < \eta$, and $\beta(V_n, V) \to 0$ as $n \to \infty$.

(ii) $\Rightarrow$ (iii) Given a Borel set $E \subseteq X$ and $\varepsilon > 0$, define $h_\varepsilon: X \to \mathbb{R}$ by $h_\varepsilon(x) = 0 \vee (1 - d(x, E)/\varepsilon)$. Then $\|h_\varepsilon\|_{BL} \leq 1 + \frac{1}{\varepsilon}$. For any positive vector measures V and W, we have

$$V(E) \leq \int h_\varepsilon\,dV \quad \text{and} \quad \int h_\varepsilon\,dW \leq W(E_\varepsilon),$$

so that

$$V(E) \le W(E_\varepsilon) + \int h_\varepsilon d(W - V) \le W(E_\varepsilon) + \left(1 + \frac{1}{\varepsilon}\right) \beta_0(V, W).$$

Thus $\rho(V, W) \le \varepsilon + (1+\frac{1}{\varepsilon})\beta(V, W)$. Suppose now that $\beta(V_n, V) \to 0$. Given $\eta > 0$, choose $\varepsilon = \frac{\eta}{2}$ and n large enough so that $\beta(V_n, V) < \frac{\eta}{2}(1 + \frac{1}{\varepsilon})$. Then $\rho(V_n, V) < \eta$. This shows that $\rho(V_n, V) \to 0$ and similarly $\sigma(V_n, V) \to 0$.

(iii) $\Rightarrow$ (i) Let $f: X \to \mathbb{R}$ be bounded and uniformly continuous and $\varepsilon > 0$. Choose $\delta > 0$ so that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| \le \delta$. Suppose that V and W are positive vector measures with $V(X) = W(X)$ and $\rho(V, W) < \delta$; then there are $\eta > 0$ and $u \in B^+$ with $\eta + \|u\| \le \delta$ and $V(E) \le W(E_\eta) + u$ for all Borel $E \subseteq X$. We now apply Corollary 3.2 to find a measure ρ on $X \times Y$ with margins V and W such that $\rho\{(x, y): d(X, y) > \eta\} \le u$. Then

$$\int f\, d(V - W) = \int f(x) - f(y)\, d\rho(x, y)$$
$$= \int_{d(x,y) \le \eta} f(x) - f(y)\, d\rho(x, y) + \int_{d(x,y) > \eta} f(x) - f(y)\, d\rho(x, y)$$

and

$$(*) \qquad \left\| \int f\, d(V - W) \right\| \le \varepsilon \|V(X)\| + 2\|f\|_\infty \|u\|.$$

Given an arbitrary $\varepsilon > 0$, choose $\delta > 0$ such that $|f(x) - f(y)| < \frac{\varepsilon}{2}\|V(X)\|$ for $|x - y| \le \delta$ and $\delta < \frac{\varepsilon}{4}\|f\|_\infty$. Then from $(*)$, $\|\int f\, d(V - W)\| < \varepsilon$ whenever $\rho(V, W) < \delta$. It follows that $\int f\, d(V_n - V) \to 0$ whenever $\sigma(V_n, V) \to 0$. □

References

1. M. Dekiert, *Kompaktheit, Fortsetzbarkeit und Konvergenz von Vectormassen*, Dissertation, University of Essen, 1991.
2. J. Diestel and J. J. Uhl, *Vector measures*, Amer. Math. Soc. Surveys, vol. 15, Amer. Math. Soc., Providence, RI, 1977.
3. R. M. Dudley, *Real analysis and probability*, Wadsworth and Brooks/Cole, Pacific Grove, 1989.
4. N. Dunford and J. T. Schwartz, *Linear operators,* Part I, Wiley, New York, 1957.
5. H. G. Kellerer, *Duality theorems for marginal problems*, Z. Wahrsch. Verw. Gebiete **67** (1984), 399–432.
6. J. L. Kelley and I. Namioka, *Linear topological spaces*, Springer-Verlag, New York, 1963.
7. E. Marczewski and C. Ryll-Nardzewski, *Projections in abstract sets*, Fund. Math. **40** (1953), 160–164.
8. ——, *Remarks on the compactness and nondirect products of measures*, Fund. Math. **40** (1953), 165–170.
9. M. März and R. M. Shortt, *Weak convergence of vector measures*, Publ. Math. (Debrecen) (to appear).
10. G. Plebanek, *Measures on two-dimensional products*, Mathematika **36** (1989), 253–258.
11. U. V. Prokhoroff, *Convergence of random processes and limit theorems in probability theory*, Theory Probab. Appl. **1** (1958), 157–214.
12. H. H. Schaefer, *Banach lattices and positive operators*, Springer-Verlag, New York, 1974.
13. R. M. Shortt, *Strassen's marginal problem in two or more dimensions*, Z. Wahrsch. Verw. Gebiete **64** (1983), 313–325.

14. V. Strassen, *Probability measures with given marginals*, Ann. of Math. Statist. **36** (1965), 423–439.

15. F. Wehrung, *Injective positively ordered monoids*, J. Pure Appl. Algebra **83** (1992), 43–100.

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